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DUAL OPERATOR ALGEBRAS AND HEREDITARY PROPERTIES OF ISOMETRIC DILATIONS AND COISOMETRIC EXTENSIONS

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Abstract. We discuss contraction operators T in the class $\mathbb{A}$, where $\mathbb{A}$ is the class of absolutely continuus contractions for which the Sz.-Nagy-Foias funtional calculus is isometry. We obtain relationship between the class $\mathbb{A}_{n,\aleph_0}$ and the hereditary propety $(\tilde{\mathbf{P}}_n)$.

1. Introduction

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. A *dual algebra* is a subalgebra of $\mathcal{L}(\mathcal{H})$ that contains the identity operator $I_{\mathcal{H}}$ and is closed in the ultraweak operator topology on $\mathcal{L}(\mathcal{H})$. This notion of dual algebras was introduced by S. Brown in [5], where he proved that every subnormal operator has a nontrivial invariant subspace. The theory of dual algebras is deeply related to the study of the problem of solving systems of simultaneous equations in the predual of a singly generated dual algebra (see [1], [3] and [4]). This theory is applied to the study of invariant subspaces and compression theory. In particular, in [6] Chevreau-Exner-Pearcy obtained some characterizations of the class $\mathbb{A}_{1,\aleph_0}$. In addition, Exner-Jung [11] defined certain hereditary properties concerning a minimal isometric dilation of T in $\mathbb{A}$ and obtained some characterizations for membership in the the class $\mathbb{A}_{1,\aleph_0}$.

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In section 2 we recall some notation and terminology concerning dual algebras. In section 3 we define certain hereditary properties concerning minimal isometric dilations or minimal coisometric extensions of a contraction operator $T \in \mathcal{L}(\mathcal{H})$ and apply those notion to the class $\mathbb{A}_{1,\aleph_0}$. In section we apply the hereditary property to the classes $\mathbb{A}_{m,n}$.

2. Notation and preliminaries

The notation and terminology employed here agree with those in [2], [4] and [19]. We recall nonetheless them for the convenience of the reader.

Suppose that $\mathcal{A}$ is a dual algebra in $\mathcal{L}(\mathcal{H})$. Let $\mathcal{C}_1 = \mathcal{C}_1(\mathcal{H})$ be the von Neumann-Schatten ideal of trace class operators in $\mathcal{L}(\mathcal{H})$ under the trace norm and let ${}^\perp\mathcal{A}$ denote the preannihilator of $\mathcal{A}$ in $\mathcal{C}_1$. Let $\mathcal{Q}_\mathcal{A}$ denote the quotient space $\mathcal{C}_1/{}^\perp\mathcal{A}$. One knows that $\mathcal{A}$ is the dual space of $\mathcal{Q}_\mathcal{A}$ and that the duality is given by

$$(2.1) \qquad < T, [L] > = \ trace(TL), \ T \in \mathcal{A}, \ [L] \in \mathcal{Q}_\mathcal{A}.$$

Furthermore, the weak* topology that accrues to $\mathcal{A}$ by virtue of this duality coincides with the ultraweak operator topology on $\mathcal{A}$ (cf. [9]). For $T \in \mathcal{L}(\mathcal{H})$, let $\mathcal{A}_T$ denote the dual algebra generated by T. For vectors x and y in $\mathcal{H}$, we write, as usual, $x \otimes y$ for the rank one operator in $\mathcal{C}_1$ defined by $(x \otimes y)(u) \ = \ (u, y)x, \ \ u \in \mathcal{H}$.

We shall denote by $\mathbb{D}$ the open unit disc in the complex plane $\mathbb{C}$ and we write $\mathbb{T}$ for the boundary of $\mathbb{D}$. For $1 \leq p \leq \infty$ we denote the usual Lebesgue function space by $L^p = L^p(\mathbb{T})$. For $1 \leq p \leq \infty$ we denote by $H^p = H^p(\mathbb{T})$ the subspace of L^p consisting of those functions whose negative Fourier coefficients vanish. One knows that the preannihilator ${}^\perp(H^\infty)$ of H^∞ in L^1 is the subspace H_0^1 consisting of those functions g in H^1 for which analytic extension $\widetilde{g}$ to $\mathbb{D}$ satisfies $\widetilde{g}(0) = 0$ (cf. [14]). It is well known that H^∞ is the dual space of L^1/H_0^1.

Let us recall that any contraction T can be written as a direct sum $T = T_1 \oplus T_2$, where T_1 is a completely nonunitary contraction and T_2 is a unitary operator (cf. [19]). If T_2 is absolutely continuous or acts on the space (0), T will be called an *absolutely continuous contraction.*

The following provides a good relationship between the function space H^∞ and a singly generated dual algebra $\mathcal{A}_T$.

FOIAŞ-NAGY FUNCTIONAL CALCULUS [5, Theorem 4.1]. *Let T be an absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$. Then there is an algebra homomorphism $\Phi_T : H^\infty \longrightarrow \mathcal{A}_T$ defined by $\Phi_T(f) = f(T)$ such that*
(a) $\Phi_T(1) = 1_{\mathcal{H}}$, $\Phi_T(\xi) = T$,
(b) $\|\Phi_T(f)\| \leq \|f\|_\infty$, $f \in H^\infty$,
(c) *Φ_T is continuous if both H^∞ and $\mathcal{A}_T$ are given their weak* topologies,*
(d) *the range of Φ_T is weak* dense in $\mathcal{A}_T$,*
(e) *there exists a bounded, linear, one-to-one map $\phi_T : \mathcal{Q}_T \longrightarrow L^1/H^1_0$ such that $\phi_T^* = \Phi_T$, and*
(f) if Φ_T is an isometry, then Φ_T is a weak* homeomorphism of H^∞ onto $\mathcal{A}_T$ and ϕ_T is an isometry of $\mathcal{Q}_T$ onto L^1/H^1_0.

Suppose that m and n are any cardinal numbers such that $1 \leq m, n \leq \aleph_0$. A dual algebra $\mathcal{A}$ will be said to have property $(\mathbb{A}_{m,n})$ if every $m \times n$ system of simultaneous equations of the form $[x_i \otimes y_j] = [L_{ij}]$, $0 \leq i < m$, $0 \leq j < n$, where $\{[L_{ij}]\}_{\substack{0\leq i<m \\ 0\leq j<n}}$ is an arbitrary $m \times n$ array from $\mathcal{Q}_A$, has a solution $\{x_i\}_{0\leq i<m}$, $\{y_j\}_{0\leq j<n}$ consisting of a pair of sequences of vectors from $\mathcal{H}$. For brevity, we shall denote $(\mathbb{A}_{n,n})$ by $(\mathbb{A}_n)$. The class $\mathbb{A}(\mathcal{H})$ consists of all those absolutely continuous contractions T in $\mathcal{L}(\mathcal{H})$ for which the functional calculus $\Phi_T : H^\infty \longrightarrow \mathcal{A}_T$ is an isometry. Furthermore, we denote by $\mathbb{A}_{m,n}(\mathcal{H})$ the set of all T in $\mathbb{A}(\mathcal{H})$ such that the algebra $\mathcal{A}_T$ has property $(\mathbb{A}_{m,n})$. We write simply $\mathbb{A}_{m,n}$ for $\mathbb{A}_{m,n}(\mathcal{H})$ unless we mention otherwise.

If $\mathcal{M}$ is a semi-invariant subspace for $T \in \mathcal{L}(\mathcal{H})$ (i.e., there exist invariant subspaces $\mathcal{N}_1$ and $\mathcal{N}_2$ for T with $\mathcal{N}_1 \supset \mathcal{N}_2$ such that $\mathcal{M} = \mathcal{N}_1 \ominus \mathcal{N}_2$, $T_{\mathcal{M}}$ denotes the *compression* of T to $\mathcal{M}$. In other words, $T_{\mathcal{M}} = P_{\mathcal{M}} T|\mathcal{M}$, where $P_{\mathcal{M}}$ is the orthogonal projection whose range is $\mathcal{M}$.

Throughout this paper, we write $\mathbb{N}$ for the set of natural numbers. For a Hilbert space $\mathcal{K}$ and any operators $T_i \in \mathcal{L}(\mathcal{K})$, $i = 1, 2$, we write $T_1 \cong T_2$ if T_1 is unitarily equivalent to T_2.

Recall that $T \in C_{\cdot 0}$ if $\|T^{*n}x\| \longrightarrow 0$ for any $x \in \mathcal{H}$. We say $T \in C_{0\cdot}$ if $T^* \in C_{\cdot 0}$. And we denote that $C_{00} = C_{0\cdot} \cap C_{\cdot 0}$.

3. Some hereditary properties

Let T be a contraction operator in $\mathcal{L}(\mathcal{H})$ and let $B_T \in \mathcal{L}(\mathcal{K}'_+)$ be a minimal isometric dilation of T with the Wold decomposition

$$B_T = S^{(n)} \oplus R_T, \tag{3.1}$$

where

$$\mathcal{K}_+ = \bigvee_{n=0}^{\infty} B_T^n \mathcal{H}. \tag{2}$$

It follows from Wold decomposition theorem that

$$B_T = S_T \oplus R_T, \tag{3}$$

where $S_T \in \mathcal{L}(\mathcal{U}_T)$ is the unilateral shift part and $R_T \in \mathcal{L}(\mathcal{R}_T)$ is the residual part. Now suppose that $T \in \mathcal{L}(\mathcal{H})$ has a non-zero semi-invariant subspace $\mathcal{M}$ (i.e, $\mathcal{M} \neq (0)$). For $\widetilde{T} = T_{\mathcal{M}}$, we write a minimal isometric dilation of $\widetilde{T}$ by

$$B_{\widetilde{T}} = S^{(m)} \oplus R_{\widetilde{T}}. \tag{3.2}$$

Let us denote

$$B'_T = S^{*(n')} \oplus R'_T \tag{3.3}$$

a minimal coisometric extension of T and denote

$$B'_{\widetilde{T}} = S^{*(m')} \oplus R'_{\widetilde{T}} \tag{3.4}$$

a minimal coisometric extension of $\widetilde{T}$. With the notation of (4.1)-(4.4), we construct the following definition and discuss this section.

DEFINITION 3.1. Suppose that T is a contraction operator on $\mathcal{H}$.
(a) T has property (**P**) if $n \geq m$ for any nonzero semi-invariant subspace $\mathcal{M}$ for T.
(b) T has property (**P***) if $n' \geq m'$ for any nonzero semi-invariant subspace $\mathcal{M}$ for T.

The following examples are slight modifications of those in [9].

EXAMPLE 3.2. If $T \in C_{\cdot 0}$, then T has property (**P**).

EXAMPLE 3.3. If $S \in \mathcal{L}(\mathcal{H})$ is a unilateral shift operator of multiplicity one, then S^* can not have property (**P**).

REMARK 3.4. We can exchange the notions made by semi-invariant subspace with those of invariant subspaces in Definition 3.1. For example, T has property (**P**) if and only if $n \geq m$ for for any nonzero invariant subspace $\mathcal{M}$ for T, which is said to be property (**P′**) for being time. To establish the validity, we assume that T has property (**P′**) and suppose that $B_T = S^{(n)} \oplus R_T$ is the minimal isometric dilation of T. Let $\mathcal{M}$ be a non-zero semi-invariant subspace for T. Then there exist $\mathcal{N}_1, \mathcal{N}_2 \in \mathrm{Lat}(T)$ with $\mathcal{N}_1 \supset \mathcal{N}_2$ such that $\mathcal{M} = \mathcal{N}_1 \ominus \mathcal{N}_2$. Let

$$\mathcal{K}_1 = \bigvee_{n \geq 0} B_T^n \mathcal{N}_1 \tag{3.5}$$

and

$$\mathcal{L} = \bigvee_{n \geq 0} B_T^n \mathcal{M}. \tag{3.6}$$

Then the restrictions $C = B_T|\mathcal{K}_1 = S^{(m)} \oplus R_{T|\mathcal{N}_1}$ and $D = B_T|\mathcal{L} = S^{(l)} \oplus R_{T_{\mathcal{M}}}$ are the minimal isometric dilations of $T|\mathcal{N}_1$ and $T_{\mathcal{M}}$, respectively. Let $x \in \mathcal{K}_1 \ominus \mathcal{L}$ and let $a \in \mathcal{M}$. Note that

$$(Cx, B_T^n a) = \begin{cases} (B_T x, a) & \text{if} \quad n = 0 \\ (x, B_T^{n-1} a) & \text{if} \quad n \geq 1. \end{cases} \tag{3.7}$$

By the structure of B_T it is easy to show that $B_T x \in \mathcal{M}^{\perp}$. Hence it follows by (3.7) $(Cx, B_T^n a) = 0$ for any $n \geq 0$. Hence $\mathcal{L}$ is reducing for C, which implies that $m \geq l$. Moreover, by the assumption, it is obvious that $m \leq n$. Hence $l \leq n$ and T has property (**P**). Since the converse implication is obvious, property (**P**) is equivalent to property (**P′**). The imitative cases of (b), (c) and (d) can be proved by the similar method. Hence two notions are equivalent. □

We use any one of two notations in the above remark without confusion.

PROPOSITION 3.5. *Every contraction operator T on $\mathcal{H}$ has property* ($\mathbf{P}^*$).

PROOF. With the notation (3.1)-(3.4), we prove this theorem. Let $\mathcal{M}$ be a non-zero invariant subspace for T and let $B'_{\widetilde{T}} = S^{*(n')} \oplus R'_{\widetilde{T}}$ be a minimal coisometric extension of $\widetilde{T}$ on $\mathcal{S}'_{\widetilde{T}} \oplus \mathcal{R}'_{\widetilde{T}} = \mathcal{K}''_+$. Then B'_T is a coisometric extension of $\widetilde{T}$ on $\mathcal{K}'_+$. Hence there exists $\widetilde{\mathcal{K}}'' \in \mathrm{Lat}(B'_T)$ such that $B'_T|\widetilde{\mathcal{K}}'' \cong B'_{\widetilde{T}}$. Note that $B'_T|\widetilde{\mathcal{K}}'' \cong S' \oplus R' \in \mathcal{L}(\mathcal{S}' \oplus \mathcal{R}')$, where S'' is the shift part and R'' is the unitary part, and that

$$B'_T|\widetilde{\mathcal{K}}'' \cong B'_{\widetilde{T}} = S^{(m')}_{\widetilde{T}} \oplus R'_{\widetilde{T}} \in \mathcal{L}(\mathcal{S}'_{\widetilde{T}} \oplus \mathcal{R}'_{\widetilde{T}})$$

and $S'' \cong S^{(m')}_{\widetilde{T}}$. To show that $\mathcal{S}'' \subset \mathcal{S}'_T$, let $x \in \mathcal{S}' \subset \widetilde{\mathcal{K}}'' \subset \mathcal{K}'_+$ and let $x = s \oplus r \in \mathcal{S}'_T \oplus \mathcal{R}'_T$. Then

$$\begin{aligned} \|S''^{*n} x\|^2 &= \|B'^{n}_T x\|^2 = \|S'^{*n}_T s\|^2 + \|R'^{n}_T r\|^2 \\ &= \|S'^{*n}_T s\|^2 + \|r\|^2. \end{aligned} \tag{3.8}$$

Letting $n \to \infty$, we have that $r = 0$ and $x \in \mathcal{S}'_T$. Hence the multiplicity of $S'_T \geq m'$ which implies that T has property ($\mathbf{P}^*$). Therefore the proof is complete. □

REMARK 3.6. We apply the rule of properties ($\mathbf{P}$) and ($\mathbf{P}^*$) to the unitary part of B_T and B'_T, respectively. For example, if we define as follows:

(a) T has property ($\mathbf{Q}$) if there exists $\mathcal{U} \in \mathrm{Lat}(R_T)$ such that $R_T|\mathcal{U} \cong R_{\widetilde{T}}$ for any nonzero semi-invariant subspace $\mathcal{M}$ for T;

(b) T has property ($\mathbf{Q}^*$) if there exists $\mathcal{U}' \in \mathrm{Lat}(R'_T)$ such that $R'_T|\mathcal{U}' \cong R'_{\widetilde{T}}$ for any nonzero semi-invariant subspace $\mathcal{M}$ for T, according to the method of the proof of Proposition 3.1 and Proposition 4.5 we have that every contraction operator T has both property ($\mathbf{Q}^*$) and property ($\mathbf{Q}$).

4. Application to the classes $\mathbb{A}_{m,n}$

Let T be a contraction on $\mathcal{H}$. Recall that if $d_T < \infty$ and $d_{T^*} < \infty$, then T is a Fredholm operator and the Fredholm index $\text{ind}(T)$ is equal to $d_T - d_{T^*}$ (cf. [12]).

DEFINITION 4.1. A contraction operator $T \in \mathbb{A}$ has property $(\widetilde{\mathbf{P}})$ if there exists $\mathcal{M} \in \text{Lat}(T)$ such that
(a) $T|\mathcal{M} \in \mathbb{A}(\mathcal{M})$ and
(b) $T|\mathcal{M}$ has property $(\mathbf{P})$.
The contraction operator $T \in \mathbb{A}(\mathcal{H}) \setminus \mathbb{A}_{\aleph_0}$ has property $(\widetilde{\mathbf{P}}_n)$ if it satisfies additionary condition:
(c) $\text{ind}(T|\mathcal{M}) \leq -n$.

THEOREM 4.2. *Suppose $T \in \mathbb{A}(\mathcal{H})$. Then $T \in \mathbb{A}_{1,\aleph_0}$ if and only if T has property $(\widetilde{\mathbf{P}})$.*

PROOF. According to [9, Theorem 3.4], it is easy to show that property $(\widetilde{\mathbf{H}})$ is equivalent to $(\widetilde{\mathbf{P}})$, where property $(\widetilde{\mathbf{H}})$ appeared in [9]. □

THEOREM 4.3. *Suppose $T \in \mathbb{A} \cap C_{\cdot 0}$ with $d_T < \infty$. Then $T \in \mathbb{A}_{n,\aleph_0}$ if and only if T has property $(\widetilde{\mathbf{P}}_n)$*

PROOF. $(\Longrightarrow)$: obvious.
$(\Longleftarrow)$: Let us denote $\widetilde{T} = T|\mathcal{M}$ for a proper invariant subspace $\mathcal{M}$. Then $d_{\widetilde{T}} < \infty$ and $\text{ind}(\widetilde{T}) \leq -n$. Since $\widetilde{T} \in C_{\cdot 0}$ by [8], $\widetilde{T} \in \mathbb{A}_{n,\aleph_0}$ and $T \in \mathbb{A}_{n,\aleph_0}$. □

COROLLARY 4.4. *Suppose $T \in \mathbb{A}$ with $d_T < \infty$. Then T has property $(\widetilde{\mathbf{P}})$ if and only if T has property $(\widetilde{\mathbf{P}}_1)$.*

REMARK 4.5. If $T \in \mathbb{A}$ and $\mathcal{R}_T = (0)$, T has property $(\widetilde{\mathbf{P}})$ and $T \in \mathbb{A}_{1,\aleph_0}$. But if $\mathcal{S}_T = (0)$, in general $T \in \mathbb{A}$ can not belongs to the $\mathbb{A}_{1,\aleph_0}$ (for a counterexample, consider the backward unilateral shift operator of multiplicity one). Moreover, assume that $\mathcal{R}_T \neq (0)$. Then according to [6, Theorem 2.5], we that $R_T \in \mathbb{A}$ implies $T \in \mathbb{A}_{1,\aleph_0}$. And if $\mathcal{S}_T \neq (0)$, T can not belong to the $\mathbb{A}_{1,\aleph_0}$ (for a counterexample, consider $T = S^* \oplus S(\theta)$, where $S(\theta)$ is a Jordan block. Then it follows from [11, Corollary 5.4] that $T \notin \mathbb{A}_{1,\aleph_0}$).

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