

NORM AND ESSENTIAL NORM ESTIMATES OF TOEPLITZ OPERATORS ON THE BERGMAN SPACE

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ABSTRACT. On the setting of product of balls we consider Toeplitz operators, with symbols satisfying a certain condition, on the Bergman space. Norms and essential norms of such operators are estimated by means of certain integral quantities.

1. Introduction

Throughout the paper Ω will be a fixed domain which is a product of balls in the complex n -space \mathbb{C}^n . More precisely, Ω is a domain of the form

$$\Omega = \prod_{j=1}^m B_{n_j}$$

where each B_{n_j} is the unit ball of \mathbb{C}^{n_j} and $n_1 + \cdots + n_m = n$. Let L^p denote the usual Lebesgue space on Ω with respect to the Lebesgue volume measure V on Ω normalized to have total mass 1. The Bergman space A^2 is the space of holomorphic functions in L^2 . By the mean value property for holomorphic functions it is easy to see that the Bergman space A^2 is a closed subspace of L^2 , so there is a unique Hilbert space orthogonal projection P — called the Bergman projection — from L^2 onto A^2 . The Bergman projection P extends to an integral operator taking L^1 into the space of holomorphic functions and P is L^p bounded for $1 < p < \infty$ (see Section 2 and Appendix). For a function $u \in L^2$, the Toeplitz operator T_u with symbol u is defined by

$$T_u f = P(uf)$$

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for functions $f \in A^2$. The densely-defined operator $T_u : A^2 \rightarrow A^2$ is clearly bounded for $u \in L^\infty$, but not necessarily bounded in general.

For $u \in L^\infty$, compactness of T_u is characterized by Zheng [Z] in terms of certain integral vanishing property of u on the ball or polydisk. Recently, the authors [CL] have generalized Zheng's characterization to product of balls and found another characterization by using a different argument from Zheng's argument. In this paper we consider a certain class of symbols containing bounded symbols and estimate norms and essential norms of corresponding Toeplitz operators by modifying the argument of [CL].

To introduce the symbol class which will be concerned in this paper, we first present a version of [Ru, Proposition 1.4.10] (see, for example, [BCZ, Lemma 9]): There are constants $\mu > 0$ and $q > 1$, depending only on Ω , such that

$$(1) \quad \sup_{a \in \Omega} \int_{\Omega} |K(a, z)|^{q(1-2\mu)} K(z, z)^{q\mu} dV(z) < \infty$$

where the notation $K(z, w)$ denotes the Bergman kernel on Ω . For example, one may take $\mu = (2n+2)^{-1}$, $1 < q < (2n-2)/(2n+1)$ on the ball and $\mu = 1/4$, $1 < q < 4/3$ on the polydisk to insure (1).

In the present paper, we will consider symbols u for which the Berzin transforms of $|u|^p$ are bounded for a sufficiently large p . More explicitly, we will consider symbols u which have the following boundedness property:

$$(*) \quad \sup_{a \in \Omega} \int_{\Omega} |u \circ \varphi_a|^p dV < \infty$$

where $p/2$ is the conjugate exponent of q appeared in (1). Here, φ_a denotes a biholomorphic automorphism of Ω with the properties that

$$\varphi_a(a) = 0, \quad \varphi_a \circ \varphi_a = \text{the identity map.}$$

The automorphisms φ_a are explicitly described in [Ru, Chapter 2] in the case of the ball, and hence can be defined in an obvious way for general Ω . Note that every function satisfying condition $(*)$ belongs to L^2 by

taking $a = 0$ and bounded functions clearly satisfy condition (*). For the symbols u satisfying condition (*), we have, in particular,

$$\sup_{a \in \Omega} \int_{\Omega} |u \circ \varphi_a| dV < \infty$$

and therefore the multiplication operator $f \mapsto uf$ is a bounded operator from A^2 into L^2 by Theorem A of [Z3]. It follows that T_u is bounded on A^2 .

In Section 2 we collect some basic facts about the Bergman kernel and some preliminary results on Toeplitz operators which will be used later. In Sections 3, we estimate norms (Theorem 6) and essential norms (Theorem 9) of T_u with u satisfying (*). In Section 4, some remarks related to Hankel operators are indicated. Finally in Appendix, we provide a proof of L^p boundedness of the Bergman projection P for $1 < p < \infty$. This may be already known and it is included here for the sake of completeness.

2. Preliminaries

We collect in this section some notations and preliminary results which will be used in the sequel. Most of those are well-known and necessary verifications can be found, for example, in [BCZ], [CL], [Kr] or [Ra].

As is well known, the Bergman kernel $K(z, w)$ has the following reproducing property:

$$(2) \quad f(z) = \langle f, K(\cdot, z) \rangle \quad \text{for all } f \in A^2$$

where the notation $\langle \cdot, \cdot \rangle$ denotes the usual inner product in L^2 with respect to the measure V . It is easy to show (see, for example, [CL]) that the Bergman kernel $K(z, w)$ on Ω has the following explicit formula:

$$(3) \quad K(z, w) = \prod_{j=1}^m \frac{1}{(1 - z^j \cdot \bar{w}^j)^{n_j+1}} \quad (z, w \in \Omega).$$

Here, we use the notation $z = (z^1, \dots, z^m)$ with each $z^j = (z_1^j, \dots, z_{n_j}^j)$ $\in B_{n_j}$ for a point $z \in \Omega$ and

$$z^j \cdot \bar{w}^j = z_1^j \bar{w}_1^j + \dots + z_{n_j}^j \bar{w}_{n_j}^j$$

for the Hermitian inner product of $z^j, w^j \in \mathbb{C}^{n_j}$. Then the Bergman projection P can be easily written as an integral operator by (2) and (3) as follows:

$$\begin{aligned} (P\psi)(z) &= \langle P\psi, K(\cdot, z) \rangle \\ &= \langle \psi, K(\cdot, z) \rangle \\ &= \int_{\Omega} \prod_{j=1}^m \frac{\psi(w)}{(1 - z^j \cdot \bar{w}^j)^{n_j+1}} dV(w) \quad (z \in \Omega) \end{aligned}$$

for functions $\psi \in L^2$. In fact, the above integral representation shows that the operator P can be well defined for functions $\psi \in L^1$.

It is often very convenient to use normalized kernels. So, we let

$$k_a(z) = \frac{K(z, a)}{\sqrt{K(a, a)}} \quad (a, z \in \Omega)$$

Then the real Jacobian determinant of φ_a turns out to be the same as $|J\varphi_a|^2$ for which we have the identity

$$(4) \quad |J\varphi_a|^2 = |k_a|^2$$

on Ω . Since φ_a is an involution, another straightforward calculation shows

$$(5) \quad k_a(\varphi_a(z))k_a(z) = 1 \quad (a, z \in \Omega)$$

By a manipulation of the above formulas, we have a useful change-of-variable formula (see Section 2 of [CL]):

$$\begin{aligned} (6) \quad & \int_{\Omega} h(w) |K(z, w)|^{\alpha} K(w, w)^{\beta} dV(w) \\ &= K(z, z)^{\alpha+\beta-1} \int_{\Omega} h(\varphi_z(w)) |K(z, w)|^{2-\alpha-2\beta} K(w, w)^{\beta} dV(w) \end{aligned}$$

for all α, β real whenever the integrals make sense.

The following two propositions are proved in [CL] with $u \in L^{\infty}$ and exactly the same proof works with u satisfying the given weaker conditions. The following shows how Toeplitz operators act on the Bergman kernels.

PROPOSITION 1. *Let $u \in L^2$. Then we have*

$$T_u k_a = [P(u \circ \varphi_a) \circ \varphi_a] k_a$$

for every $a \in \Omega$. \square

As is well-known, norm and essential norm of a bounded operator on a Hilbert space are equal to those of its adjoint operator. We have a convenient way to represent the adjoint operators of Toeplitz operators.

PROPOSITION 2. *Let u satisfy condition $(*)$. Then the adjoint operator T_u^* of T_u is given by*

$$(T_u^* h)(a) = \int_{\Omega} h(w) \overline{P(u \circ \varphi_a)(\varphi_a(w))} K(a, w) dV(w)$$

for every $a \in \Omega$ and $h \in A^2$. \square

Some of our characterizations will be in terms of certain quantities over the balls induced by the Bergman metric. The Bergman distance function between two points $z, w \in \Omega$ will be denoted by $\beta(z, w)$ and the corresponding Bergman metric ball with center at $a \in \Omega$ and radius $r > 0$ will be denoted by $E_r(a)$:

$$E_r(a) = \{z \in \Omega : \beta(a, z) < r\}.$$

Also, we use the notation $|A|$ for the measure of a Borel subset A of Ω with respect to the measure V and the same letter C for various positive constants which may change from one occurrence to the next.

The following lemma gives some information about the size of the volume of the Bergman metric balls $E_r(a)$.

LEMMA 3. *For $r > 0$, there are constants $C(r), c(r)$ so that*

$$c(r) \leq \frac{|k_a(w)|^2}{|E_r(a)|} \leq C(r)$$

for all $a \in \Omega$ and $w \in E_r(0)$.

PROOF. See [CL, Lemma 6]. \square

3. Norms and essential norms

In this section we estimate the norms and essential norms of the Toeplitz operators under consideration. First we have a couple of simple lemmas.

LEMMA 4. Let $\mu > 0$ and $q > 1$ be the constants as in (1) and $2/p + 1/q = 1$. Then there exists a constant $C = C(\Omega)$ such that

$$\int_{\Omega} |h(\varphi_z(w))|^2 |K(z, w)| K(w, w)^{\mu} dV(w) \leq C K(z, z)^{\mu} \left(\int_{\Omega} |h|^p dV \right)^{2/p}$$

for every $z \in \Omega$ and $h \in L^p$.

PROOF. Let $h \in L^p$ and pick a point $a \in \Omega$. Apply the change-of-variable formula (6) and then use Hölder's inequality to obtain

$$\begin{aligned} & \int_{\Omega} |h(\varphi_z(w))|^2 |K(z, w)| K(w, w)^{\mu} dV(w) \\ &= K(z, z)^{\mu} \int_{\Omega} |h(w)|^2 |K(z, w)|^{1-2\mu} K(w, w)^{\mu} dV(w) \\ &\leq K(z, z)^{\mu} \left(\int_{\Omega} |h|^p dV \right)^{2/p} \left(\int_{\Omega} |K(z, w)|^{q(1-2\mu)} K(w, w)^{q\mu} dV(w) \right)^{1/q}. \end{aligned}$$

Now the lemma follows from (1). The proof is complete. \square

LEMMA 5. For a given $r > 0$, there is a constant $C = C(r)$ such that

$$\frac{1}{|E_r(a)|} \int_{E_r(a)} |h(\varphi_a)|^s dV \leq C \int_{\Omega} |h|^s dV$$

for every $s > 0$, $h \in L^s$ and $a \in \Omega$.

PROOF. Let $s > 0$, $h \in L^s$. We first note that $\varphi_a E_r(a) = E_r(0)$ for $a \in \Omega$ because the Bergman distance is invariant under automorphisms (see, for example [Kr]). By a change of variables, together with (4) and Lemma 3, we have

$$\frac{1}{|E_r(a)|} \int_{E_r(a)} |h(\varphi_a)|^s dV = \frac{1}{|E_r(0)|} \int_{E_r(0)} |h|^s |k_a|^2 dV \leq C \int_{\Omega} |h|^s dV$$

for some constant $C = C(r)$. This completes the proof. \square

We are now ready to estimate norms of Toeplitz operators.

THEOREM 6. *Let u satisfy condition (*). Then the following inequalities hold.*

$$(a) \sup_{a \in \Omega} \int_{\Omega} |P(u \circ \varphi_a)|^2 dV \leq \|T_u\|^2.$$

(b) *For each $r > 0$, there exists a constant $C = C(r)$ such that*

$$\sup_{a \in \Omega} \frac{1}{|E_r(a)|} \int_{E_r(a)} |P(u \circ \varphi_a)(\varphi_a)|^2 dV \leq C \sup_{a \in \Omega} \int_{\Omega} |P(u \circ \varphi_a)|^2 dV.$$

(c) *For a given $r = r(u, \Omega)$ sufficiently large, there exists a constant $C = C(r)$ such that*

$$\|T_u\|^2 \leq C \sup_{a \in \Omega} \frac{1}{|E_r(a)|} \int_{E_r(a)} |P(u \circ \varphi_a)(\varphi_a)|^2 dV.$$

PROOF. We first prove (a). Since the kernel k_a has L^2 -norm 1, we have

$$(7) \quad \int_{\Omega} |T_u k_a|^2 dV \leq \|T_u\|^2 \int_{\Omega} |k_a|^2 dV = \|T_u\|^2$$

for every $a \in \Omega$. On the other hand, using Proposition 1, we can easily see by a change of variables that

$$(8) \quad \int_{\Omega} |T_u k_a|^2 dV = \int_{\Omega} |P(u \circ \varphi_a)(\varphi_a)|^2 |k_a|^2 dV = \int_{\Omega} |P(u \circ \varphi_a)|^2 dV$$

for every $a \in \Omega$. Combining the above with (7), we obtain (a).

Part (b) is a consequence of Lemma 5 with $s = 2$.

Finally we show (c). Let $h \in A^2$ and fix a point $a \in \Omega$. Then, by Proposition 2, we have

$$T_u^* h(a) = \int_{\Omega} h(w) \overline{P(u \circ \varphi_a)(\varphi_a(w))} K(a, w) dV(w).$$

Given $r > 0$, decompose

$$(9) \quad T_u^* = U_r + V_r$$

where

$$U_r h(a) = \int_{E_r(a)} h(w) \overline{P(u \circ \varphi_a)(\varphi_a(w))} K(a, w) dV(w)$$

and

$$V_r h(a) = \int_{\Omega \setminus E_r(a)} h(w) \overline{P(u \circ \varphi_a)(\varphi_a(w))} K(a, w) dV(w).$$

We first estimate the operator U_r . Put

$$I(a, r) = \frac{1}{|E_r(a)|} \int_{E_r(a)} |P(u \circ \varphi_a)(\varphi_a)|^2 dV$$

for notational simplicity. By the Cauchy-Schwarz inequality

$$|U_r h(a)|^2 \leq I(a, r) \int_{E_r(a)} |h(w)|^2 |E_r(a)| |K(a, w)|^2 dV(w).$$

An application of Fubini's theorem therefore yields

$$\begin{aligned} & \int_{\Omega} |U_r h|^2 dV \\ & \leq \left(\sup_{a \in \Omega} I(a, r) \right) \int_{\Omega} |h(w)|^2 \int_{E_r(w)} |E_r(a)| |K(a, w)|^2 dV(a) dV(w). \end{aligned}$$

By the proof of [CL, Theorem 8], we have

$$(10) \quad \sup_{w \in \Omega} \int_{E_r(w)} |E_r(a)| |K(a, w)|^2 dV(a) < \infty$$

and thus obtain the following estimate for the operator norm of U_r :

$$(11) \quad \|U_r\|^2 \leq C \sup_{a \in \Omega} \frac{1}{|E_r(a)|} \int_{E_r(a)} |P(u \circ \varphi_a)(\varphi_a)|^2 dV \quad (r > 0)$$

for some constant $C = C(r)$.

Now we estimate the operator V_r . Let $\mu > 0$ and $q > 1$ be the constants as in (1), and let $2/p + 1/q = 1$. We note by the boundedness of P (see Corollary 16 of Appendix) that

$$(12) \quad \int_{\Omega} |P(u \circ \varphi_a)|^p dV \leq C \int_{\Omega} |u \circ \varphi_a|^p dV \quad (a \in \Omega)$$

for some constant $C = C(\Omega)$ independent of $a \in \Omega$. By the Cauchy-Schwarz inequality again,

$$\begin{aligned} |V_r h(a)|^2 &\leq \left(\int_{\Omega \setminus E_r(a)} |P(u \circ \varphi_a)(\varphi_a(w))|^2 |K(a, w)| K(w, w)^{\mu} dV(w) \right) \\ &\quad \times \left(\int_{\Omega \setminus E_r(a)} |h(w)|^2 |K(a, w)| K(w, w)^{-\mu} dV(w) \right) \\ &\leq C \left(\int_{\Omega} |P(u \circ \varphi_a)|^p dV \right)^{2/p} K(a, a)^{\mu} \\ &\quad \times \int_{\Omega \setminus E_r(a)} |h(w)|^2 |K(a, w)| K(w, w)^{-\mu} dV(w) \end{aligned}$$

where $C = C(\Omega)$ is the constant provided by Lemma 4. Thus, by (12) and Fubini's theorem, we have

$$\begin{aligned} \int_{\Omega} |V_r h|^2 dV &\leq C \left(\sup_{a \in \Omega} \int_{\Omega} |u \circ \varphi_a|^p dV \right)^{2/p} \\ &\quad \times \int_{\Omega} |h(w)|^2 K(w, w)^{-\mu} \int_{\Omega \setminus E_r(w)} K(a, a)^{\mu} |K(a, w)| dV(a) dV(w) \end{aligned}$$

for some constant $C = C(\Omega)$. Let

$$J(w, r) = K(w, w)^{-\mu} \int_{\Omega \setminus E_r(w)} K(a, a)^\mu |K(a, w)| dV(a)$$

for simplicity. Then we have

$$\int_{\Omega} |V_r h|^2 dV \leq C \left(\sup_{a \in \Omega} \int_{\Omega} |u \circ \varphi_a|^p dV \right)^{2/p} \left(\sup_{w \in \Omega} J(w, r) \right) \int_{\Omega} |h|^2 dV.$$

In other words,

$$(13) \quad \|V_r\|^2 \leq C \left(\sup_{a \in \Omega} \int_{\Omega} |u \circ \varphi_a|^p dV \right)^{2/p} \left(\sup_{w \in \Omega} J(w, r) \right)$$

for some constant $C = C(\Omega)$. On the other hand, by Lemma 4 we obtain

$$\sup_{w \in \Omega} J(w, r) \leq C |\Omega \setminus E_r(0)|^{2/p}$$

for some constant C independent of r , which implies that $\sup_{w \in \Omega} J(w, r) \rightarrow 0$ as $r \rightarrow \infty$. Thus we have $\|V_r\| \leq \|T_u^*\|/2$ for all sufficiently large $r = r(u, \Omega)$. Consequently, combining this with (9)–(11) and using the fact $\|T_u\| = \|T_u^*\|$, we have (c). The proof is complete. \square

Since we have the condition (*), Theorem 6 can be slightly generalized. First we prove a simple lemma.

LEMMA 7. (a) Given positive numbers r, s, t with $4 - t \leq s \leq 2$, there exists a constant $C = C(r, s, t)$ such that

$$\begin{aligned} & \frac{1}{|E_r(a)|} \int_{E_r(a)} |Ph(\varphi_a)|^2 dV \\ & \leq C \left(\frac{1}{|E_r(a)|} \int_{E_r(a)} |Ph(\varphi_a)|^s dV \right)^{1/2} \left(\int_{\Omega} |h|^t dV \right)^{(4-s)/2t} \end{aligned}$$

for all $a \in \Omega$, $h \in L^t$.

(b) Given s, t with $2 \leq s \leq 1 + t/2$, there exists a constant $C = C(s, t)$ such that

$$\int_{\Omega} |Ph|^s dV \leq C \left(\int_{\Omega} |Ph|^2 dV \right)^{1/2} \left(\int_{\Omega} |h|^t dV \right)^{(s-1)/t}$$

for all $h \in L^1$.

PROOF. To prove part (a), let $r > 0$. By the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \frac{1}{|E_r(a)|} \int_{E_r(a)} |Ph(\varphi_a)|^2 dV &\leq \left(\frac{1}{|E_r(a)|} \int_{E_r(a)} |Ph(\varphi_a)|^s dV \right)^{1/2} \\ &\quad \times \left(\frac{1}{|E_r(a)|} \int_{E_r(a)} |Ph(\varphi_a)|^{4-s} dV \right)^{1/2} \end{aligned}$$

for every $a \in \Omega$. Now use Lemma 5, Jensen's inequality and the boundedness of P , to get the following estimate for the second integral of the right side of the above:

$$\begin{aligned} \frac{1}{|E_r(a)|} \int_{E_r(a)} |Ph(\varphi_a)|^{4-s} dV &\leq C \int_{\Omega} |Ph|^{4-s} dV \\ &\leq C \left(\int_{\Omega} |Ph|^t dV \right)^{(4-s)/t} \\ &\leq C \left(\int_{\Omega} |h|^t dV \right)^{(4-s)/t} \end{aligned}$$

for some constant $C = C(r, \Omega)$. This proves (a). Part (b) is again an easy consequence of Jensen's inequality, the Cauchy-Schwarz inequality and the boundedness of P . The proof is complete. \square

Using Lemma 7, we have a little bit more general version of Theorem 6 as follows. For simplicity we use the notation

$$M = \sup_{a \in \Omega} \int_{\Omega} |u \circ \varphi_a|^p dV.$$

COROLLARY 8. Assume u satisfies condition (*) and let p be the exponent as in (*). Let $s > 0$, $4 - p \leq s \leq 1 + p/2$. Then the following inequalities hold.

- (a) There exist constants $C = C(p, s)$, $\delta = \delta(s)$ and $\varepsilon = \varepsilon(p, s)$ such that

$$\sup_{a \in \Omega} \int_{\Omega} |P(u \circ \varphi_a)|^s dV \leq CM^\varepsilon \|T_u\|^\delta.$$

(b) For each $r > 0$, there exists a constant $C = C(r)$ such that

$$\sup_{a \in \Omega} \frac{1}{|E_r(a)|} \int_{E_r(a)} |P(u \circ \varphi_a)(\varphi_a)|^s dV \leq C \sup_{a \in \Omega} \int_{\Omega} |P(u \circ \varphi_a)|^s dV.$$

(c) For each $r = r(u, \Omega)$ sufficiently large, there exist constants $C = C(r, s)$, $\delta = \delta(s)$ and $\varepsilon = \varepsilon(p, s)$ such that

$$\|T_u\|^\delta \leq CM^\varepsilon \sup_{a \in \Omega} \frac{1}{|E_r(a)|} \int_{E_r(a)} |P(u \circ \varphi_a)(\varphi_a)|^s dV.$$

PROOF. (a) and (c) are consequences of Jensen's inequality, Lemma 7 and Theorem 6. Part (b) is a consequence of Lemma 5. \square

We now turn to essential norm estimates. Recall that the essential norm is the distance to compact operators. To be more precise, let H_1, H_2 be Hilbert spaces and $T : H_1 \rightarrow H_2$ be bounded. Then, the essential norm $\|T\|_e$ is defined by

$$\|T\|_e = \inf \|T - S\|$$

where infimum is taken over all compact operators $S : H_1 \rightarrow H_2$. In the following, the statement $a \rightarrow \partial\Omega$ simply means that the euclidean distance $d(a, \partial\Omega)$ between $a \in \Omega$ and the topological boundary $\partial\Omega$ of Ω has the property $d(a, \partial\Omega) \rightarrow 0$.

THEOREM 9. Let u satisfy condition (*). Then the following inequalities hold.

$$(a) \limsup_{a \rightarrow \partial\Omega} \int_{\Omega} |P(u \circ \varphi_a)|^2 dV \leq \|T_u\|_e^2.$$

(b) For each $r > 0$, there exists a constant $C = C(r)$ such that

$$\limsup_{a \rightarrow \partial\Omega} \frac{1}{|E_r(a)|} \int_{E_r(a)} |P(u \circ \varphi_a)(\varphi_a)|^2 dV \leq C \limsup_{a \rightarrow \partial\Omega} \int_{\Omega} |P(u \circ \varphi_a)|^2 dV.$$

(c) For each $r = r(u, \Omega)$ sufficiently large, there exists a constant $C = C(r)$ such that

$$\|T_u\|_e^2 \leq C \limsup_{a \rightarrow \partial\Omega} \frac{1}{|E_r(a)|} \int_{E_r(a)} |P(u \circ \varphi_a)(\varphi_a)|^2 dV.$$

PROOF. We first prove (a). Since the normalized kernel k_a converges uniformly to 0 on compact subsets of Ω as $a \rightarrow \partial\Omega$, one can easily see that k_a converges weakly to 0 in A^2 as $a \rightarrow \partial\Omega$. Hence, for any compact operator S ,

$$\int_{\Omega} |Sk_a|^2 dV \rightarrow 0 \quad \text{as} \quad a \rightarrow \partial\Omega$$

and thus, by (8), we have

$$\begin{aligned} \|T_u - S\| &\geq \limsup_{a \rightarrow \partial\Omega} \left(\int_{\Omega} |(T_u - S)k_a|^2 dV \right)^{1/2} \\ &\geq \limsup_{a \rightarrow \partial\Omega} \left\{ \left(\int_{\Omega} |T_u k_a|^2 dV \right)^{1/2} - \left(\int_{\Omega} |Sk_a|^2 dV \right)^{1/2} \right\} \\ &= \limsup_{a \rightarrow \partial\Omega} \left(\int_{\Omega} |T_u k_a|^2 dV \right)^{1/2} \\ &= \limsup_{a \rightarrow \partial\Omega} \left(\int_{\Omega} |P(u \circ \varphi_a)|^2 dV \right)^{1/2}. \end{aligned}$$

It follows that

$$\limsup_{a \rightarrow \partial\Omega} \int_{\Omega} |P(u \circ \varphi_a)|^2 dV \leq \|T_u\|_{\epsilon}^2.$$

Therefore we obtain (a).

Part (b) is a consequence of Lemma 5.

Finally we show (c). For each $\rho > 0$, put

$$\Omega_{\rho} = \{z \in \Omega : d(z, \partial\Omega) \geq \rho\}$$

and let M_{ρ} be the multiplication by the characteristic function of Ω_{ρ} , acting on L^2 . Since the symbol of M_{ρ} is supported on a compact subset of Ω , the operator M_{ρ} is compact when restricted to A^2 . Thus the operator $M_{\rho}T_u^* : A^2 \rightarrow L^2$ is compact. Accordingly,

$$\|JT_u^*\|_{\epsilon} \leq \|JT_u^* - M_{\rho}T_u^*\| \quad (\rho > 0)$$

where $J : A^2 \rightarrow L^2$ denotes the inclusion operator. Put $G_\rho = JT_u^* - M_\rho T_u^*$ for simplicity and pick any $h \in A^2$, $a \in \Omega$. Then, by Proposition 2, we have

$$G_\rho h(a) = \chi_\rho(a) \int_{\Omega} h(w) \overline{P(u \circ \varphi_a)(\varphi_a(w))} K(a, w) dV(w)$$

where χ_ρ denotes the characteristic function of the set $\Omega \setminus \Omega_\rho$. Given $r > 0$, decompose $G_\rho = U_{\rho,r} + V_{\rho,r}$ where $U_{\rho,r}h = \chi_\rho U_r h$ and $V_{\rho,r}h = \chi_\rho V_r h$. Here we use notations U_r and V_r introduced in the proof of Theorem 6. Since PJ is the identity on A^2 , it is not hard to see that $\|T_u^*\|_e = \|JT_u^*\|_e$ and thus

$$(14) \quad \|T_u\|_e \leq \|G_\rho\| \leq \|U_{\rho,r}\| + \|V_r\| \quad (\rho, r > 0).$$

By the Cauchy-Schwarz inequality

$$|U_{\rho,r}h(a)|^2 \leq \chi_\rho(a) I(a, r) \int_{E_r(a)} |h(w)|^2 E_r(a) |K(a, w)|^2 dV(w)$$

where $I(a, r)$ is as in the proof of Theorem 6 and hence, by Fubini's theorem and (10),

$$\int_{\Omega} |U_{\rho,r}h|^2 dV \leq C \left(\sup_{a \in \Omega \setminus \Omega_\rho} I(a, r) \right) \int_{\Omega} |h|^2 dV$$

so that

$$(15) \quad \|U_{\rho,r}\|^2 \leq C \sup_{a \in \Omega \setminus \Omega_\rho} \frac{1}{|E_r(a)|} \int_{E_r(a)} |P(u \circ \varphi_a)(\varphi_a)|^2 dV \quad (\rho, r > 0)$$

for some constant $C = C(r, \Omega)$ independent of ρ . Also, by (13), we have $\|V_r\| \leq \|T_u\|_e/2$ for all ρ and for all $r = r(u, \Omega)$ sufficiently large. For such r , we see from (14) and (15) that

$$\|T_u\|_e^2 \leq C \sup_{a \in \Omega \setminus \Omega_\rho} \frac{1}{|E_r(a)|} \int_{E_r(a)} |P(u \circ \varphi_a)(\varphi_a)|^2 dV \quad (r > 0)$$

for some constant $C = C(r, \Omega)$ independent of ρ . Now, taking $\rho \rightarrow 0$, we have (c). The proof is complete. \square

As a corresponding result to Corollary 8, we have the following by a similar proof.

COROLLARY 10. *Assume u satisfies condition $(*)$ and let p be the exponent as in $(*)$. Let $s > 0$, $4 - p \leq s \leq 1 + p/2$. Then the following inequalities hold.*

- (a) *There exist constants $C = C(p, s)$, $\delta = \delta(s)$ and $\varepsilon = \varepsilon(p, s)$ such that*

$$\limsup_{a \rightarrow \partial\Omega} \int_{\Omega} |P(u \circ \varphi_a)|^s dV \leq CM^\varepsilon \|T_u\|_e^\delta.$$

- (b) *For each $r > 0$, there exists a constant $C = C(r)$ such that*

$$\begin{aligned} & \limsup_{a \rightarrow \partial\Omega} \frac{1}{|E_r(a)|} \int_{E_r(a)} |P(u \circ \varphi_a)(\varphi_a)|^s dV \\ & \leq C \limsup_{a \rightarrow \partial\Omega} \int_{\Omega} |P(u \circ \varphi_a)|^s dV. \end{aligned}$$

- (c) *For each $r = r(u, \Omega)$ sufficiently large, there exist constants $C = C(r, s)$, $\delta = \delta(s)$ and $\varepsilon = \varepsilon(p, s)$ such that*

$$\|T_u\|_e^\delta \leq CM^\varepsilon \limsup_{a \rightarrow \partial\Omega} \frac{1}{|E_r(a)|} \int_{E_r(a)} |P(u \circ \varphi_a)(\varphi_a)|^s dV. \quad \square$$

As a consequence of Corollary 10, we recover characterizations ([Z], [CL]) of compact Toeplitz operators under the weaker condition $(*)$.

COROLLARY 11. *Assume u satisfies condition $(*)$ and let p be the exponent as in $(*)$. If $4 - p \leq s \leq 1 + p/2$, $s > 0$, then the following statements are equivalent.*

- (a) T_u is compact on A^2 .
- (b) $\int_{\Omega} |P(u \circ \varphi_a)|^s dV \rightarrow 0$ as $a \rightarrow \partial\Omega$.
- (c) $\frac{1}{|E_r(a)|} \int_{E_r(a)} |P(u \circ \varphi_a)(\varphi_a)|^s dV \rightarrow 0$ as $a \rightarrow \partial\Omega$ for all $r > 0$. \square

4. Remarks

For a function $u \in L^2$, the Hankel operator H_u with symbol u is defined by

$$H_u f = (I - P)(uf)$$

for functions $f \in A^2$. As in the case of Toeplitz operators, the operator $H_u : A^2 \rightarrow (A^2)^\perp$ is densely defined and not necessarily bounded in general. Luecking [Lu] characterized L^2 -symbols for corresponding Hankel operators to be bounded and compact on the disk and later Li [L] gave some extension of Luecking's result to strongly pseudoconvex domains. In a more recent paper [LL] Li and Luecking characterized bounded and compact Hankel operators with L^2 -symbols on strongly pseudoconvex domains. Also, some results computing essential norms of Hankel operators were obtained on the disk in [LR] with L^2 -symbols and in [A] with antiholomorphic symbols.

It is not hard to see that the arguments of the present paper work with H_u and $I - P$ in place of T_u and P . Thus Theorems 6, Theorem 9, Corollary 8, and Corollary 10 remain valid with H_u and $I - P$ in place of T_u and P . Although those results are not contained in results mentioned above, they may be considered as results of similar BMO(VMO) type. For example, we obtain the following VMO type characterization of compact Hankel operators, which recover results of [S1], [S2], [Z], [CL].

COROLLARY 12. *Assume u satisfies condition (*) and let p be the exponent as in (*). If $s > 0$, $4 - p \leq s \leq 1 + p/2$, then the following statements are equivalent.*

(a) H_u is compact on A^2 .

(b) $\int_{\Omega} |u \circ \varphi_a - P(u \circ \varphi_a)|^s dV \rightarrow 0 \quad \text{as } a \rightarrow \partial\Omega.$

(c) $\frac{1}{|E_r(a)|} \int_{E_r(a)} |u - P(u \circ \varphi_a)(\varphi_a)|^s dV \rightarrow 0 \quad \text{as } a \rightarrow \partial\Omega \quad \text{for all } r > 0. \quad \square$

Having characterizations of compact Toeplitz operators (Corollary 11) and compact Hankel operators (Corollary 12), one can go a little bit further as in [CL] by using well-known characterizations of compact Toeplitz operators with positive symbols to obtain the following.

COROLLARY 13. *Let u satisfy condition (*). Then $T_{|u|^2}$ is compact if and only if T_u and H_u are both compact. \square*

Appendix.

We let define a measure dV_α ($\alpha > -1$ is fixed throughout this section) on Ω by $dV_\alpha(z) = \prod_{j=1}^m (1 - |z^j|^2)^\alpha dV_j(z^j)$ where each dV_j is the normalized volume measure on B_{n_j} . Corresponding to a complex number $s = \sigma + it$ ($\sigma > -1$, $-\infty < t < \infty$), we define a kernel

$$K_s(z, w) = \prod_{j=1}^m \frac{(1 - |w^j|^2)^\sigma}{(1 - z^j \cdot \overline{w^j})^{n_j+1+s}} \quad (z \in \Omega, w \in \Omega)$$

and an integral operator

$$P_s f(z) = \lambda_s \int_{\Omega} K_s(z, w) f(w) dV(w) \quad (z \in \Omega)$$

whenever the integral makes sense. Here $\lambda_s = \prod_{j=1}^m \Gamma(n_j + 1 + s) / \Gamma(n_j + 1) \Gamma(s + 1)^m$ and the complex powers that occur in the kernel K_s are understood to be the usual principal branches. In particular, $dV_0 = dV$ and $P_0 = P$, the Bergman projection. Then we have

THEOREM 14. (a) *For $1 \leq p < \infty$, P_s is a bounded operator on $L^p(\Omega, dV_\alpha)$ if and only if $p(1 + \sigma) > 1 + \alpha$.*

(b) *If $p(1 + \sigma) > 1 + \alpha$, then $P_s f = f$ and $P_s \bar{f} = \bar{f}(0)$ for every holomorphic f in $L^p(\Omega, dV_\alpha)$.*

The above theorem can be found in several cases, such as [C], [FR], [Ko] and [Z2]. To obtain the above we first define

$$Q_\sigma f(z) = \int_{\Omega} \prod_{j=1}^m \frac{(1 - |w^j|^2)^\sigma}{|1 - z^j \cdot \overline{w^j}|^{n_j+1+\sigma}} f(w) dV(w) \quad (z \in \Omega).$$

We first prove a preliminary version of Theorem 14.

PROPOSITION 15. For $1 \leq p < \infty$, Q_σ is a bounded operator on $L^p(\Omega, dV_\alpha)$ if and only if $p(1 + \sigma) > 1 + \alpha$.

PROOF. We first consider the case $p = 1$. Assume that Q_σ is a bounded operator on $L^1(\Omega, dV_\alpha)$. Note that the adjoint operator Q_σ^* of the operator Q_σ is given by

$$Q_\sigma^* f(z) = \int_{\Omega} \prod_{j=1}^m \frac{(1 - |z^j|^2)^{\sigma-\alpha}}{|1 - z^j \cdot \overline{w^j}|^{n_j+1+\sigma}} f(w) dV_\alpha(w) \quad (z \in \Omega).$$

The boundedness of Q_σ on $L^1(\Omega, dV_\alpha)$ implies the boundedness of Q_σ^* on $L^\infty(\Omega)$. It follows from Fubini's theorem that

$$\sup \prod_{j=1}^m (1 - |z^j|^2)^{\sigma-\alpha} \int_{B_{n_j}} \frac{(1 - |w^j|^2)^\alpha}{|1 - z^j \cdot \overline{w^j}|^{n_j+1+\sigma}} dV_j(w^j) < \infty$$

where the supremum is taken over all $z \in \Omega$. By Proposition 1.4.10 of [Ru], this happens if and only if $\sigma > \alpha$. Now assume that $\sigma > \alpha$ and show that Q_σ is bounded on $L^1(\Omega, dV_\alpha)$. To do so, pick any $f \in L^1(\Omega, dV_\alpha)$. Then, by Fubini's theorem and Proposition 1.4.10 of [Ru] again, we have

$$\begin{aligned} & \int_{\Omega} |Q_\sigma f| dV_\alpha \\ & \leq \int_{\Omega} |f(w)| \prod_{j=1}^m (1 - |w^j|^2)^\sigma \int_{B_{n_j}} \frac{(1 - |z^j|^2)^\alpha}{|1 - z^j \cdot \overline{w^j}|^{n_j+1+\sigma}} dV_j(z^j) dV(w) \\ & \leq C \int_{\Omega} |f(w)| \prod_{j=1}^m (1 - |w^j|^2)^\alpha dV(w) \\ & = C \int_{\Omega} |f| dV_\alpha \end{aligned}$$

for some constant C independent of f . This completes the proof for the case $p = 1$.

Assume that $1 < p < \infty$ and put

$$Q_\sigma(z, w) = \prod_{j=1}^m \frac{(1 - |w^j|^2)^{\sigma-\alpha}}{|1 - z^j \cdot \overline{w^j}|^{n_j+1+\sigma}},$$

so that

$$Q_\sigma f(z) = \int_{\Omega} Q_\sigma(z, w) f(w) dV_\alpha(w).$$

Let q be the conjugate exponent of p . If $p(1 + \sigma) \leq 1 + \alpha$, then $q(\sigma - \alpha) + \alpha \leq -1$ and thus

$$\int_{\Omega} |Q_\sigma(z, w)|^q dV_\alpha(w) = \infty$$

for every $z \in \Omega$, and thus $Q_\sigma f$ fails to exist for some $f \in L^p(\Omega, dV_\alpha)$. If $p(1 + \sigma) > 1 + \alpha$, then $\epsilon = \sigma - (1 + \alpha)/p > -1$, $\delta = (1 + \alpha)/p - 1 > -1$ and $\sigma - \alpha > 0$. Define $\varphi(z) = \prod_{j=1}^m (1 - |z^j|^2)^{-\frac{1+\alpha}{pq}}$. Then, by Fubini's Theorem and Proposition 1.4.10 of [Ru], we have

$$\begin{aligned} & \int_{\Omega} Q_\sigma(z, w) \varphi(w)^q dV_\alpha(w) \\ &= \prod_{j=1}^m \int_{B_{n_j}} \frac{(1 - |w^j|^2)^\epsilon}{|1 - z^j \cdot \overline{w^j}|^{n_j+1+\sigma}} dV_j(w^j) \leq [a\varphi(z)]^q \end{aligned}$$

and

$$\begin{aligned} & \int_{\Omega} Q_\sigma(z, w) \varphi(z)^p dV_\alpha(z) \\ &= \prod_{j=1}^m \int_{B_{n_j}} \frac{(1 - |w^j|^2)^{\sigma-\alpha} (1 - |z^j|^2)^\delta}{|1 - z^j \cdot \overline{w^j}|^{n_j+1+\sigma}} dV_j(z^j) \leq [b\varphi(w)]^p \end{aligned}$$

for some constants a and b depending only on m, α, σ and p . It follows from Schur's theorem (see, for example [Z2, Theorem 3.3.2]) that

$$\int_{\Omega} |Q_\sigma f|^p dV_\alpha \leq (ab)^p \int_{\Omega} |f|^p dV_\alpha$$

for $f \in L^p(\Omega, dV_\alpha)$. This completes the proof. \square

PROOF OF THEOREM 14. Part (a) of the theorem follows from Proposition 15 because

$$e^{-m\pi|t|/2} \leq |K_s(z, w)| |K_\sigma(z, w)|^{-1} \leq e^{m\pi|t|/2}$$

on $\Omega \times \Omega$. The second part can be easily seen from the special case [C, Theorem 1] when Ω is the unit ball. \square

As mentioned before $dV_0 = dV$ and $P_0 = P$. Hence Theorem 14 yields

COROLLARY 16. *The Bergman projection P is bounded on L^p if and only if $1 < p < \infty$.* \square

One immediate consequence of Theorem 14 is that the L^p -norm of any holomorphic function on Ω is controlled by the corresponding norm of its real part. In the setting of the unit ball the following can be found in Theorem 7.1.5 of [Ru] and Lemma 10 of [Z1] for the case $\alpha = 0$ and $\alpha > -1$, respectively. The proof of the following corollary is similar to that of the special case [Z1, Lemma 10] when Ω is the unit ball. We omit the details.

COROLLARY 17. *For each $p \geq 1$ and $\alpha > -1$ there exists a constant C , depending only on p , α and Ω , such that*

$$\int_{\Omega} |f|^p dV_{\alpha} \leq C \int_{\Omega} |\operatorname{Re} f|^p dV_{\alpha}$$

for all holomorphic functions f on Ω such that $f(0) = 0$. \square

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