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ON THE JOHNSON'S QUESTION ABOUT THE KIM-KOSTRIKIN GROUP

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ABSTRACT. In this paper, we shall prove that the Kim-Kostrikin group is isomorphic to a hyperbolic orbifold group and so it is infinite.

1. Introduction

In the present note we study the balanced group $G_1(1)$ with the presentation

$$(*) \qquad G_1(1) = \langle a,b,c,d \mid ada = cb,\ bdb = ac,\ cdc = ba,\ abc = 1 \rangle,$$

that was called by D. Johnson the *Kim-Kostrikin group*.

The interest to the studying of the above group is motivated by the following. In [5] A. C. Kim and A. I. Kostrikin considered the infinite series $G_k(n)$, $n \geq 1$, $k = 1,2,3$, of groups with balanced presentation (the number of generators is equal to the number of relations). Considering the tessellations on the boundary of the three-dimensional sphere, they constructed the infinite series of closed orientable three-dimensional manifolds $M_k(n)$ where $n \geq 2$ for $k = 1,2$, and $n \geq 1$ for $k = 3$, whose fundamental groups are isomorphic to $G_k(n)$. The first exceptional group $G_1(1)$ is exactly the Kim-Kostrikin group. The second exceptional group $G_2(1)$ is isomorphic to the free product of two groups of order three.

Moreover, they studied the algebraic structure of the groups $G_k(n)$, and proved that *all group $G_k(n)$ are infinite except, possibly, the group $G_1(1)$*.

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In [4] D. Johnson asked *if the exceptional group $G_1(1)$ is infinite?* The aim of the present note is to give the affirmative answer on the Johnson's question.

2. Theorem and Proof

THEOREM. *The Kim-Kostrikin group $G_1(1)$ is infinite.*

PROOF. We first eliminate c and d from the presentation (*). From the relations

$$c = b^{-1}a^{-1}, \quad d = ab^2a^2b,$$

we get the 2-generator presentation

$$G_1(1) = \langle a, b \mid a^2b^2a^2 = b^{-1}a^{-1}ba^{-1}b^{-1}, b^2a^2b^2 = a^{-1}b^{-1}ab^{-1}a^{-1}\rangle.$$

It is obvious from the presentation (*), that the group $G_1(1)$ has an automorphism ρ of order three such that

$$\rho(a) = b, \quad \rho(b) = c, \quad \rho(c) = a, \quad \rho(d) = d.$$

For the 2-generator presentation the automorphism ρ can be defined as

$$\rho(a) = b, \quad \rho(b) = b^{-1}a^{-1}.$$

Let us consider the split extension $\widehat{G}_1(1)$ of $G_1(1)$ by $\langle\rho\rangle$, that is a semidirect product of G and $\langle\rho\rangle \cong \mathbb{Z}_3$, and then, we get

$$(**) \quad \begin{aligned} \widehat{G}_1(1) = \langle a, b, \rho \mid & a^2b^2a^2 = b^{-1}a^{-1}ba^{-1}b^{-1}, \\ & b^2a^2b^2 = a^{-1}b^{-1}ab^{-1}a^{-1}, \\ & \rho a\rho^{-1} = b, \rho b\rho^{-1} = b^{-1}a^{-1}, \rho^3 = 1\rangle. \end{aligned}$$

Here we see that $a^{-1} = b\rho b\rho^{-1}$, whence $a = \rho b^{-1}\rho^{-1}b^{-1}$. And so,

$$\rho a = \rho^{-1}b^{-1}\rho^{-1}b^{-1} = (b\rho)^{-2}.$$

But, since $\rho a = b\rho$, we get $(\rho a)^3 = (b\rho)^3 = 1$. Denoting $\tau = \rho a$, we get $a = \rho^{-1}\tau$ and $b = \tau\rho^{-1}$. Then from the presentation (**), we get the relation

$$(\rho^{-1}\tau)^2(\tau\rho^{-1})^2(\rho^{-1}\tau)^2 = (\tau\rho^{-1})^{-1}(\rho^{-1}\tau)^{-1}(\tau\rho^{-1})(\rho^{-1}\tau)^{-1}(\tau\rho^{-1})^{-1},$$

that is equivalent to

$$\rho\tau\rho^{-1}\tau^{-1}\rho^{-1}\tau\rho\tau = \tau\rho\tau\rho^{-1}\tau^{-1}\rho^{-1}\tau\rho,$$

that can be written in the form

$$w\tau = \tau w,$$

where

$$w = \rho\tau\rho^{-1}\tau^{-1}\rho^{-1}\tau\rho.$$

Therefore, the group $\widehat{G}_1(1)$ has the 2-generator presentation

$$\widehat{G}_1(1) = \langle \rho, \tau \mid w\tau = \tau w, \quad \rho^3 = \tau^3 = 1 \rangle.$$

Because $w = \rho^{i_1}\tau^{i_2}\rho^{i_3}\tau^{i_4}\rho^{i_5}\tau^{i_6}\rho^{i_7}$, where i_j is the sign of the number $3j$ by mod 16 on the segment $[-8, 8]$, the word w corresponds to the $(8/3)$-rational link, that is the well-known 2-component Whitehead link W [3]. In this case, the group

$$\pi(W) = \langle \rho, \tau \mid w\tau = \tau w \rangle,$$

where w is as above, is the group of the Whitehead link W. Comparing above presentations of $\widehat{G}_1(1)$ and $\pi(W)$, by [1], we see that the group $\widehat{G}_1(1)$ is the group of the orbifold $W(3,3)$ whose underlying space is the three-dimensional sphere and singular set is the Whitehead link W with both branched indices equal three.

We recall [3], that the orbifold $W(3,3)$ is a compact orientable hyperbolic orbifold. Whence its orbifold group $\widehat{G}_1(1)$ is hyperbolic, and so, infinite. The fact that the group $G_1(1)$ is the normal subgroup of $\widehat{G}_1(1)$ of index three, completes the proof.

REMARK. As we see from the proof, the Kim-Kostrikin group $G_1(1)$ is the fundamental group of the three-fold cyclic covering $M_1(1)$ of the three-dimensional sphere, branched over the Whitehead link W. The n-fold cyclic coverings, branched over the link W were studied in [2], and $M_1(1)$ is the manifold $M_{3,1}$ from [2]. Moreover, the Kim-Kostrikin group coincides with the group $\Gamma_{3,1}$ from Theorem 3.3 [2], that leads to another proof for the affirmative answer on the Johnson's question.

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