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ABSTRACT DIFFERENTIATION ON CERTAIN GROUPOIDS

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ABSTRACT. On certain groupoids called LIR-groupoids, one can define abstract definitions of continuity and differentiation of functions. Many properties of this abstract continuity and differentiation have analogy to the ordinary continuity and differentiation of real-valued functions.

1. Introduction

An *LIR-groupoid* $(G, \cdot)$ is a set G equipped with a binary operation "$\cdot$" satisfying the following three identical relations :

$$(1.1)\quad \begin{cases} (x\cdot y)\cdot z = (x\cdot z)\cdot y & \text{(Left-normal law)}\\ x\cdot x = x & \text{(Idempotent law)}\\ x\cdot(y\cdot z) = x\cdot y. & \text{(Reduction law)}\end{cases}$$

The acronymous name "LIR-groupoid" from the above identities and much works on those groupoids can be found in [5], [6] and [7]. But certain examples of LIR-groupoids appeared in [4] before the name was invented. It is easy to see that LIR-groupoids may be characterized as groupoids satisfying the idempotent law, the reduction law and the identity

$$(x\cdot y)\cdot(z\cdot t) = (x\cdot z)\cdot(y\cdot t) \qquad \text{(Medial law)}$$

instead of the left-normal law. In fact, assuming (1.1), we have $(x\cdot y)\cdot(z\cdot t) = (x\cdot y)\cdot z = (x\cdot z)\cdot y = (x\cdot z)\cdot(y\cdot t)$, for all element x, y, z, t of the

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groupoid. Conversely, if we assume the idempotent law, the reduction law and the medial law, then $(x\cdot y)\cdot z = (x\cdot y)\cdot(z\cdot t) = (x\cdot z)\cdot(y\cdot t) = (x\cdot z)\cdot y$ for all element x, y, z, t of the groupoid, yielding the left-normal law.

In the present note, we will study how an LIR groupoid is obtained from a differential group and how abstract differentiation and continuity can be defined on LIR-groupoids, mainly based on [7]. From now on, we will use the simpler notations G for $(G, \cdot)$ and xy for $x \cdot y$ if it causes no confusion doing so.

2. Two relations on LIR-groupoids

Let G be an LIR-groupoid. For each element y of G, we define the *right translation* $R_y : G \to G$ by $R_y(x) = xy$ for all $x \in G$. Because of the idempotent and medial laws, we have $R_y(ab) = (ab)y = (ab)(yy) = (ay)(by) = R_y(a)R_y(b)$ for all $a, b \in G$. That is, R_y is an endomorphism of G. The set $B = \{R_y \mid y \in G\}$ may not be closed under composition, but it generates a submonoid $R(G)$ of the endomorphism monoid $\mathrm{End}(G)$. By the left-normal law, $R(G)$ is commutative. The following lemma is easy (see [1] and [2]).

LEMMA 2.1. *Let $(G, \cdot)$ be an LIR-groupoid and $\mathrm{End}(G)$ be its endomorphism monoid under composition. Define a binary operation '$*$' on $\mathrm{End}(G)$ by*

$$(\phi * \psi)(x) = \phi(x) \cdot \psi(x)$$

for all $x \in G$. Then $(\mathrm{End}(G), *)$ is a medial groupoid.

The map $R : (G, \cdot) \to (\mathrm{End}(G), *)$ defined by $y \mapsto R_y$ is a groupoid homomorphism, because $R_{xy}(a) = a(xy) = (aa)(xy) = (ax)(ay) = R_x(a) \cdot R_y(a) = (R_x * R_y)(a)$ for all $a \in G$. The image $(B, *)$ of this homomorphism is a left-zero semigroup, that is, a semigroup satisfying the identity $xy = x$, because $(R_x * R_y)(a) = R_x(a) \cdot R_y(a) = (ax)(ay) = (aa)(xy) = a(xy) = ax = R_x(a)$ for all $a \in G$ by (1.1), yielding $R_x * R_y = R_x$. The kernel γ of the homomorphism R is a congruence relation on G, and elements x, y in G are said to be *cocyclic* if $x \equiv_\gamma y$. Thus x and y are cocyclic if they define the same right multiplication, and $xy = R_y(x) = R_x(x) = xx = x$ in this case. Thus each congruence class of γ is a left-zero semigroup as a subgroupoid of G. Being isomorphic to $(B, *)$, G/γ is a left-zero semigroup.

For an element x of $(G, \cdot)$, the *orbit* of x is defined to be the set $R(G)x = \{\theta(x) \mid \theta \in R(G)\}$. We define a relation β on G by $x \equiv_\beta y$ if and only if there is an element z in the intersection of the orbit of x and the orbit of y, i.e., there are ϕ, ψ in $R(G)$ such that $\phi(x) = \psi(y)$. Two elements related by β are said to be *cobordic*, and β is called the *cobordism relation*. For every x, y in G, $xy \equiv_\beta x$ because $R_1(xy) = (xy)1 = xy = R_y(x)$, where 1 is the identity element of the monoid $R(G)$. Thus, $[x]_\beta[y]_\beta = [x]_\beta$ for all x, y in G. That is, G/β is a left-zero semigroup.

PROPOSITION 2.2. *The cobordism relation β is the smallest congruence relation on $(G, \cdot)$ such that G/β is a left-zero semigroup.*

PROOF. By definition, β is easily seen to be transitive and symmetric. Suppose, $x \equiv_\beta y$ and $y \equiv_\beta z$. Then there are θ, η, ϕ, ψ in $R(G)$ such that $\theta(x) = \eta(y)$ and $\phi(y) = \psi(z)$. Since $R(G)$ is commutative, we have $\phi\theta(x) = \phi\eta(y) = \eta\phi(y) = \eta\psi(z)$, and so $x \equiv_\beta z$. Thus β is transitive, hence β is an equivalence relation. Suppose that $x \equiv_\beta y$ and $t \equiv_\beta u$. Then $xt \equiv_\beta x \equiv_\beta y \equiv_\beta yu$, implying $xt \equiv_\beta yu$, which proves that β is a congruence relation. Finally, suppose α is a congruence relation such that $(G/\alpha, \cdot)$ is a left-zero semigroup. Suppose $x \equiv_\beta y$ with $x\theta = y\eta$ for some θ, η in $R(G)$, and assume $\theta = R_{a_1} \cdots R_{a_m}$ and $\eta = R_{b_1} \cdots R_{b_n}$. Then

$$\begin{aligned}
[x]_\alpha &= (\cdots([x]_\alpha[a_m]_\alpha)\cdots)[a_1]_\alpha = [(\cdots(xa_m)\cdots)a_1]_\alpha \\
&= [R_{a_1} \cdots R_{a_m}(x)]_\alpha = [\theta(x)]_\alpha = [\eta(y)]_\alpha = [R_{b_1} \cdots R_{b_n}(y)]_\alpha \\
&= [(\cdots(yb_n)\cdots)b_m]_\alpha = (\cdots([y]_\alpha[b_n]_\alpha)\cdots)[b_1]_\alpha = [y]_\alpha
\end{aligned}$$

because of the reduction law. So, $x \equiv_\alpha y$. Thus, β is contained in α.

Since G/γ is a left-zero semigroup, β is smaller than γ as a corollary to the above theorem. Thus, cobordic elements are cocyclic.

By the terminology of Mac Lane [3], a *differential group* $(K, +, d)$ is an abelian group $(K, +)$ together with a group endomorphism d satisfying

$d^2 = 0$. If $(K, +, d)$ is a differential group, then elements of $\operatorname{Ker} d$ are called *cycles* and elements of $\operatorname{Im} d$ are called the *boundaries*. There is an easy way of associating an LIR-groupoid with a differential group, as the following proposition shows.

PROPOSITION 2.3. ([7]) *Let $(K, +, d)$ be a differential group. Define a binary operation "$\cdot$" on K by $x \cdot y = x - dx + dy$ for all x, y in K. Then $(K, \cdot)$ is an LIR-groupoid.*

As a result of the above proposition, LIR-groupoids are occasionally called *differential groupoids* [7], and the groupoid defined in the above proposition is called the differential groupoid associated with the differential group $(K, +, d)$. As in the homology theory of groups or complexes, the following hold.

THEOREM 2.4. ([7]) *Let $(K, +, d)$ be a differential group and $(K, \cdot)$ be its associated differential groupoid. Then, for any x, y in K,*

(1) *x and y are cocyclic if and only if $x - y$ is a cycle, and*
(2) *x and y are cobordic if and only if $x - y$ is a boundary.*

3. Differentiation

Let $\mathbf{R}$ be the set of all real numbers and d be any symbol which does not belong to $\mathbf{R}$. Put $\mathbf{R}[d] = \{a + dx \mid a,\ x \in \mathbf{R}\}$ and define an addition and a multiplication on $\mathbf{R}[d]$ by $(a + dx) + (b + dy) = (a + b) + d(x + y)$ and $(a + dx)(b + dy) = (ab) + d(ay + bx)$, respectively, for all a, b, x, y in $\mathbf{R}$. Then $\mathbf{R}[d]$ becomes a commutative ring with unity, called the ring of the *dual numbers* over $\mathbf{R}$. Note that $d\mathbf{R}$ is an ideal of $\mathbf{R}$ and $\mathbf{R}[d] = \mathbf{R} \oplus d\mathbf{R}$. The element d has the property $d^2 = 0$ and d is called the *differential.* The elements of $d\mathbf{R}$ are called *infinitesimal* elements of $\mathbf{R}[d]$.

Note that d acts as an endomorphism of the abelian group $(\mathbf{R}[d], +)$, and $(\mathbf{R}[d], +, d)$ is a differential group. Let $(\mathbf{R}[d], \cdot)$ be the differential groupoid associated with it. By the elementary calculus, if $f : \mathbf{R} \to \mathbf{R}$ is a differentiable function, then, at each $a \in \mathbf{R}$, we can approximate f with a linear function f_a such that

$$f_a(a + x) = f(a) + f'(a)x \tag{3.1}$$

for all $x \in \mathbf{R}$. The graph of f_a is the tangential line of the graph of f at a. The following is obvious.

LEMMA 3.1. *Let $f : \mathbf{R} \to \mathbf{R}$ be a differentiable function and f_a be the linear function defined in (3.1), then $f_a(a) = f(a)$ and $f_a'(x) = f'(a)$ for all $x \in \mathbf{R}$.*

Using this linear approximation, we can extend any function $f : \mathbf{R} \to \mathbf{R}$ to a function $f : \mathbf{R}[d] \to \mathbf{R}[d]$, using the same notation, by the rule

$$f(a + dx) = f(a) + f'(a)dx \tag{3.2}$$

for all a, $x \in \mathbf{R}$. Because $f_a(a+dx) = f_a(a)+f_a'(a)dx = f(a)+f'(a)dx = f(a + dx)$ by the previous lemma and (3.2), we can say that if a is in $\mathbf{R}$ and $u \in \mathbf{R}[d]$ is infinitesimally close to a then $f(u) = f_a(u)$. That is, the approximation f_a is exact for the infinitesimally close neighborhood of a.

Let ϕ and ψ be the projections of $\mathbf{R}[d]$ onto $\mathbf{R}$ and $d\mathbf{R}$, respectively, that is, for every dual number $u = a + dx$, $u\phi = a$ and $u\psi = dx$. Thus, $u = u\phi + u\psi$.

THEOREM 3.2. *Let $f : \mathbf{R} \to \mathbf{R}$ be a differentiable function, and extend it to $f : \mathbf{R}[d] \to \mathbf{R}[d]$ by the rule in (3.2). Then, for any u, v in $\mathbf{R}[d]$, $f(u \cdot v) = f(u) \cdot f_a(v)$. Furthermore, for each $a \in \mathbf{R}$, if $g : \mathbf{R}[d] \to \mathbf{R}[d]$ satisfies the equation $f(a \cdot v) = f(a) \cdot g(v)$ for every v, then g is infinitesimally close to f_a.*

PROOF. Let $u = a + dx$ and $v = b + dy$. Due to Lemma 3.1, we have

$$\begin{aligned} f_a(v) &= f_a(b + dy) = f_a(b) + f_a'(b)\,dy = f_a(a + (b - a)) + f_a'(b)\,dy \\ &= f_a(a) + (b - a)f_a'(a) + f_a'(b)\,dy = f(a) + (b - a)f'(a) + f'(a)\,dy \\ &= f(a) + f'(a)\,(b + dy - a) = f(a) + f'(a)\,(v - a). \end{aligned}$$

Thus,

$$\begin{aligned} f(u)\cdot f_a(v) \equiv& f(a+dx)\cdot f_a(v)\\ =&(f(a)+f'(a)\,dx)\cdot(f(a)+f'(a)\,(v-a))\\ =&f(a)+f'(a)\,dx-d(f(a)+f'(a)\,dx)+d(f(a)+f'(a)\,(v-a))\\ =&f(a)+f'(a)\,dx-df(a)+f'(a)\,d^2x+df(a)+f'(a)\,d(b+dy-a)\\ =&f(a)+f'(a)\,d(x-a+b)\\ =&f(a+d(x-a+b))\\ =&f((a+dx)-d(a+dx)+d(b+dy))\\ =&f((a+dx)\cdot d(b+dy))\\ =&f(u\cdot v). \end{aligned}$$

Suppose the additional condition holds, then, for all v in $\mathbf{R}[d]$,

$$\begin{aligned} f'(a)dv &= f(a+dv)) - f(a) = f(a-da+d(a+v)) - f(a)\\ &= f(a\cdot(a+v)) - f(a) = f(a)\cdot g(a+v) - f(a)\\ &= f(a) - df(a) + dg(a+v) - f(a) = dg(a+v) - df(a). \end{aligned}$$

So $d(g(a+v)-f'(a)v-f(a))=0$, and $g(a+v)$ is cocyclic with $f'(a)v+f(a)$, which is equal to $f_a(a+v)$. That is, g is infinitesimally close to f_a.

Let $(G,\cdot)$ be a differential groupoid and x is an element of G. A function $f:G\to G$ is called *differentiable* at x if there is an endomorphism f_x of $(G,\cdot)$ such that $f(x\cdot y)=f(x)\cdot f_x(y)$ for all in y in G. Such an endomorphism f_x is called a *derivative* of f at x. If f is differentiable at every point of G, then f is said to be differentiable on G. It should be noted that for elementary calculus every differentiable real valued function has a unique derivative, but, for an abstract differentiable function here, there may be many distinct derivatives.

EXAMPLE. (1) If $(G,\cdot)$ is a left-zero semigroup, then every function f is differentiable at every point of G, and every endomorphism is a derivative of f. In fact, for any x, y in G and any endomorphism g, we have $f(x\cdot y)=f(x)=f(x)\cdot g(y)$.

(2) Every endomorphism of $(G,\cdot)$ is differentiable and one of its own derivatives.

THEOREM 3.3. ([7]) *If $f : G \to G$ is differentiable at x and $g : G \to G$ is differentiable at $f(x)$, then the composition gf is differentiable at x and $g_{f(x)}f_x$ can be taken for $(gf)_x$.*

PROOF. For all $x, y \in G$, $(gf)(x \cdot y) = g(f(x) \cdot f_x(y)) = (gf)(x) \cdot (g_{f(x)}f_x)(y)$. Since $g_{f(x)}f_x$ is an endomorphism, $g_{f(x)}f_x$ can be taken for $(gf)_x$ by the definition.

Let $(G, \cdot)$ be a differential groupoid and x is an element of G. A function $f : G \to G$ is called *continuous* at x if $(x, y) \in \beta$ implies $(f(x), f(y)) \in \beta$ for every $y \in G$, where β is the cobordism relation defined in section 2. If f is continuous at every point of G, then f is said to be continuous on G.

THEOREM 3.4. *Let $(G, \cdot)$ be a differential groupoid and $f : G \to G$ be any function.*

(1) *If f is differentiable at each $y \in [x]_\beta$, then it is continuous at x.*
(2) *If f is differentiable, then it is continuous.*

PROOF. Suppose f is differentiable at each $y \in [x]_\beta$. First, we show that, for each $y \in [x]_\beta$ and $\theta \in R(G)$, there is θ_y in $R(G)$ such that $f(\theta(y)) = \theta_y(f(y))$, by the induction on the complexity of θ. For empty word 1, it is trivial. Suppose this is true for θ and prove it for $R_z\theta$ for each z. Since $\theta(y) \equiv_\beta y \equiv_\beta x$, f is differentiable at $\theta(y)$. Thus, for any z in G, we have $f(R_z(\theta(y))) = f(\theta(y) \cdot z) = f(\theta(y)) \cdot f_{\theta(y)}(z) = \theta_y(f(y)) \cdot f_{\theta(y)}(z) = R_{f_{\theta(y)}(z)}(\theta_y(f(y)))$. So, we can take the endomorphism $R_{f_{\theta(y)}(z)}\theta_y$ for $(R_z\theta(y))_y$.

Now suppose $y \equiv_\beta x$, and assume $\phi(x) = \psi(y)$ for some ϕ, ψ in $R(G)$. Then, by what was shown above, $\phi_x(f(x)) = f(\phi(x)) = f(\psi(y)) = \psi_y(f(y))$, and so $f(x) \equiv_\beta f(y)$. Thus, f is continuous at x. Thus, (1) is proved, and (2) is an obvious consequence of (1).

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