Comm. Korean Math. Soc. **11** (1996), No. 4, pp. 895–902

A NOTE ON $\mathcal{I}$-IDEALS IN BCI-SEMIGROUPS

Sun Shin Ahn and Hee Sik Kim*

ABSTRACT. In this paper, we describe the ideal generated by non-empty stable set in a BCI-group as a simple form, and obtain an equivalent condition of prime $\mathcal{I}$-ideal.

1. Introduction

The notion of BCK-algebras was proposed by Y. Imai and K. Iséki in 1966. In the same year, K. Iséki ([3]) introduced the notion of BCI-algebra which is a generalization of a BCK-algebra. The ideal theory plays an important role in studying BCK-algebras and BCI-algebras, and some interesting results have been obtained by several authors ([1,2,9]). In particular, the study of prime ideals is also an important part of the theory of BCK-algebras ([1]). In 1993, Y.B. Jun and et. al. ([6]) introduced the notion of BCI-semigroups/monoid, and studied their properties. They also considered the concept of $\mathcal{I}$-ideals and of zero-divisors in BCI-semigroups. Some authors ([7, 8]) studied BCI-semigroups with the notion of fuzzy (commutative) $\mathcal{I}$-ideals. Every p-semisimple BCI-algebra gives naturally an abelian group by defining $x + y := x * (0 * y)$, and hence p-semisimple BCI-semigroup leads to the ring structure. On the while, every ring gives a BCI-algebra by defining $x * y := x - y$ and hence we can construct a BCI-semigroup. Hence the BCI-semigroup is a generalization of the ring. In this paper, we describe the ideal generated by a non-empty stable set in a BCI-group as a simple form, and obtain an equivalent condition of a prime $\mathcal{I}$-ideal. Let us recall definitions and some propertites.

Received March 18, 1996. Revised July 26, 1996.

1991 AMS Subject Classification: 06F35, 06A06.

Key words and phrases: BCI-semigroup(group), (prime) $\mathcal{I}$-ideal, stable .

* This paper was supported (in part) by BSRI program, MOE, 1995, Project No. BSRI-95-1423.

DEFINITION 1.1 ([6]). A *BCI-semigroup* is a non-empty set X with two binary operations "$*$" and "$\cdot$" and constant 0 satisfying the following axioms:

(1) $(X; *, 0)$ is a BCI-algebra,

(2) $(X, \cdot)$ is a semigroup,

(3) the operation "$\cdot$" is distributive (on both sides) over the operation "$*$", that is, $x \cdot (y * z) = (x \cdot y) * (x \cdot z)$ and $(x * y) \cdot z = (x \cdot z) * (y \cdot z)$ for all $x, y, z \in X$.

EXAMPLE 1.2. Define two binary operations "$*$" and "$\cdot$" on a set $X := \{0, 1, 2, 3\}$ as follows:

$*$	0	1	2	3
0	0	0	2	2
1	1	0	3	2
2	2	2	0	0
3	3	2	1	0

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

Then, by routine calculations, we can see that $(X; *, \cdot, 0)$ is a BCI-semigroup.

EXAMPLE 1.3. Define two binary operations "$*$" and "$\cdot$" on a set $X := \{0, a, b, c\}$ as follows:

$*$	0	a	b	b
0	0	0	c	d
a	a	0	c	d
b	b	b	0	c
c	c	c	b	o

$\cdot$	0	a	b	b
0	0	0	0	0
0	0	0	0	0
b	0	0	b	c
c	0	0	c	b

Then it is easy to see that $(X; *, \cdot, 0)$ is a BCI-semigroup.

If a BCI-semigroup X contains an element 1_X such that $1_X \cdot x = x \cdot 1_X = x$ for all $x \in X$, then X is called a *BCI-monoid*, and we call 1_X the *multiplicative identity*. If every non-zero element of a BCI-monoid X has a multiplicative inverse, then X is called a *BCI-group*. In what follows, for convenience, we shall write the multiplication $x \cdot y$ by xy. We give some examples of a BCI-semigroup which is a generalization of the ring.

EXAMPLE 1.4. Let Q be the set of all rational numbers. Then $(Q, -, 0)$ is a BCI-algebra which is not a BCK-algebra, since $0 - x \neq 0$ for any non-zero x in Q. It is easily verified that $Q = (Q, -, \cdot, 0, 1)$ is a BCI-group, where " $\cdot$ " is the ordinary multiplication on Q.

PROPOSITION 1.5 ([6]). *Let X be a BCI-semigroup. Then*
(i) $0x = x0 = 0$,
(ii) $x \leq y$ implies that $xz \leq yz$ and $zx \leq zy$, for all $x, y, z \in X$.

DEFINITION 1.6 ([6]). A non-empty subset A of a BCI-semigroup X is called a *left*(*right*) *$\mathcal{I}$-ideal* of X if
(i) A is an ideal of a BCI-algebra X,
(ii) $x \in X$ and $a \in A$ imply that $xa \in A$ ($ax \in A$). Both left and right $\mathcal{I}$-ideal is called *two-sided $\mathcal{I}$-ideal* or simply *$\mathcal{I}$-ideal.*

2. Main Results

In this section, we describe the ideal generated by a non-empty stable set in a BCI-group as a simple form, and obtain an equivalent condition of a prime $\mathcal{I}$-ideal.

THEOREM 2.1 ([6]). *Let $\{A_i\}$ be a collection of $\mathcal{I}$-ideals of the BCI-semigroup X, where i ranges over some index set. Then $\cap A_i$ is also an $\mathcal{I}$-ideal of X.*

DEFINITION 2.2. Let $(X : *, \cdot, 0)$ be a BCI-semigroup and let A be a subset of X. Then the intersection of all $\mathcal{I}$-ideals of X containing A is said to be the *ideal generated by A.*

Notice that this definition is well-defined sinces there is always at least one ideal containing A, i.e., X itself. For convenience the ideal generated by A will be denoted by $< A >$. We follow the convention: $< \phi >= 0$, and $< \{a_1, \cdots, a_n\} >=< a_1, \cdots, a_n >$. The elements $a_1, \cdots, a_n$ are said to be the *generators* of $< a_1, \cdots, a_n >$. An ideal $< a >$ generated by a single element is called a *principal $\mathcal{I}$-ideal.* A *principal BCI-semigroup* is a BCI-semigroup in which every $\mathcal{I}$-ideal is principal.

DEFINITION 2.3. A non-empty subset A of a semigroup $(X, \cdot)$ is called *left* (*right*) *stable* if for any $x \in X$ and any $a \in A$, $x \cdot a \in A$ ($a \cdot x \in A$). Both left and right stable is *two-sided stable* or simply *stable*.

EXAMPLE 2.4. In the Example 1.2, the set $\{0, 1\}$ is stable, while $\{0, 3\}$ is not stable.

THEOREM 2.5. *Let X be a BCI-group and commutative with respect the operation "$\cdot$" and A be a non-empty stable subset of X. Then*

$$< A >= \{x \in X | \exists a_1, \cdots, a_n \in A \text{ and } \exists r_1, \cdots, r_n \in X - \{0\} \text{ such that } r_n(\cdots(r_2(r_1(x * a_1) * a_2) * \cdots) * a_n) = 0\} \qquad (*)$$

PROOF. Denote the right of $(*)$ by B. Clearly $0 \in B$. Let $x * y \in B$ and $y \in B$. Then there exist $a_1, \cdots, a_n, b_1, \cdots, b_m \in A$ and $r_1, \cdots, r_n, s_1, \cdots, s_m \in X - \{0\} (n \geqq m)$ such that

$$r_n(\cdots(r_2(r_1((x * y) * a_1) * a_2) * \cdots) * a_n) = 0,$$

$$s_m(\cdots(s_2(s_1(y * b_1) * b_2) * \cdots) * b_m) = 0.$$

By the Proposition 1.5-(i), we may assume that $n \geqq m$.
So $r_n(\cdots(r_2(r_1(x * a_1) * a_2) * \cdots) * a_n) * r_n \cdots r_1 y = 0$, and hence

$$r_n(\cdots(r_2(r_1(x * a_1) * a_2) * \cdots) * a_n) \leqq r_n \cdots r_1 y.$$

Leftly "$\cdot$"-multiplying both sides of the above inequality by s_1, we have

$$s_1(r_n(\cdots(r_2(r_1(x * a_1) * a_2) * \cdots) * a_n)) \leqq s_1 r_n \cdots r_1 y = r_n \cdots r_1 s_1 y.$$

Rightly "$*$"-multiplying both sides of the above inequality by $s_1 r_n \cdots r_1 b_1$, by Proposition 1.5-(ii), we have

$$s_1(r_n(\cdots(r_2(r_1(x * a_1) * a_2) * \cdots) * a_n)) * s_1 r_n \cdots r_1 b_1$$
$$\leqq r_n \cdots r_1(s_1(y * b_1))$$

and hence

$$s_1(r_n(\cdots(r_2(r_1(x * a_1) * a_2) * \cdots) * a_n) * r_n \cdots r_1 b_1) \leqq r_n \cdots r_1(s_1(y * b_1)).$$

Leftly "$\cdot$"-multiplying both sides of the above inequality by s_2, we have

$$s_2(s_1(r_n(\cdots(r_2(r_1(x * a_1) * a_2) * \cdots) * a_n) * r_n \cdots r_1 b_1)) \leqq s_2(r_n \cdots r_1(s_1(y * b_1))).$$

Rightly "$*$"-multiplying both sides of the above inequality by $s_2 r_n \cdots r_1 b_2$, we have

$$s_2(s_1(r_n(\cdots(r_2(r_1(x * a_1) * a_2) * \cdots) * a_n) * r_n \cdots r_1 b_1) * r_n \cdots r_1 b_2) \leqq r_n \cdots r_1(s_2(s_1(y * b_1) * b_2)).$$

Repeating the above argument m-times we obtain

$$s_m(\cdots(s_1(r_n(\cdots(r_2(r_1(x * a_1) * a_2) * \cdots) * a_n) * r_n \cdots r_1 b_1) * \cdots) * r_n \cdots r_1 b_m) \leqq r_n \cdots r_1(s_m(\cdots(s_2(s_1(y * b_1) * b_2) * \cdots) * b_m)) = 0.$$

Consequently,

$$s_m(\cdots(s_1(r_n(\cdots(r_2(r_1(x * a_1) * a_2) * \cdots) * a_n) * r_n \cdots r_1 b_1) * \cdots) * r_n \cdots r_1 b_m) = 0.$$

This implies $x \in B$.

For any $k \in X$ and $x \in B$, there exist $a_1, \cdots, a_n \in A$ and $r_1, \cdots, r_n \in X$ such that

$$r_n(\cdots(r_2(r_1(x * a_1) * a_2) * \cdots) * a_n) = 0.$$

Since A is stable, for any $k \in X$, $ka_i \in A$ (and $a_i k \in A$). So

$$\begin{aligned} r_n(\cdots(r_2(r_1(kx * ka_1) * ka_2) * \cdots) * ka_n) &= k(r_n(\cdots(r_2(r_1(x * a_1) * a_2) * \cdots) * a_n)) \\ &= k \cdot 0 \\ &= 0. \end{aligned}$$

Hence $kx \in B$ (and $xk \in B$). Summarizing the above facts B is an $\mathcal{I}$-ideal of a BCI-semigroup X. Obviously, $A \subseteq B$.

Let I be any $\mathcal{I}$-ideal containing A. In order to prove $B \subseteq I$, we assume that $x \in B$. Then there are $a_1, \cdots, a_n \in A$ and $r_1, \cdots, r_n \in X$ such that

$$r_n(\cdots(r_2(r_1(x * a_1) * a_2) * \cdots) * a_n) = 0.$$

Since $0 \in I$, we have

$$r_n(\cdots(r_2(r_1(x * a_1) * a_2) * \cdots) * a_n) \in I,$$

so

$$r_n r_{n-1}(\cdots(r_2(r_1(x * a_1) * a_2) * \cdots) * a_{n-1}) * r_n a_n \in I.$$

Since I is an $\mathcal{I}$-ideal and $r_n a_n \in I$, it follows that

$$r_n r_{n-1}(\cdots(r_2(r_1(x * a_1) * a_2) * \cdots) * a_{n-1}) \in I.$$

Repeating this argument n times we obtain

$$r_n \cdots r_1 x \in I.$$

Since X is a BCI-group, we obtain $x \in I$. Hence $B \subseteq I$ and $B = < A >$, proving the theorem. □

DEFINITION 2.6. An $\mathcal{I}$-ideal $P \neq X$ in a BCI-semigroup X is said to be *prime* if it has the following property: If A and B are $\mathcal{I}$-ideals in X such that $AB \subseteq P$, then $A \subseteq P$ or $B \subseteq P$.

EXAMPLE 2.7. In Example 1.2, the set $\{0, 1\}$ is prime $\mathcal{I}$-ideal of the BCI-semigroup X.

THEOREM 2.8. *If P is an $\mathcal{I}$-ideal of a BCI-semigroup X such that $P \neq X$ and for all $a, b \in X$*

$$ab \in P \Rightarrow a \in P \quad \text{or} \quad b \in P \qquad (**)$$

then P is prime. Conversely, if $(X; *, 0)$ is an associative BCI-algebra, $(X, \cdot)$ is a commutative semigroup and the operation "$\cdot$" is distributive on both side over the operation "$*$", then any prime $\mathcal{I}$-ideal P satisfies the condition $(**)$.

PROOF. If A and B are $\mathcal{I}$-ideals such that $AB \subseteq P$ and $A \not\subseteq P$, then there exists an element $a \in A - P$. Since $ab \in AB \subseteq P$ for any $b \in B$, we have $b \in P$ by applying the condition (**). Hence $B \subseteq P$. This means P is a prime $\mathcal{I}$-ideal of X.

Conversely, let P be a $\mathcal{I}$-ideal of X and $ab \in P$. Then $< ab > \subseteq P$. We claim that $< a >< b > \subseteq < ab >$. Let $x \in < a >$ and $y \in < b >$. Then by Theorem 2.5 there are $r, s \in X - \{0\}$ such that $r(x * a) = 0$ and $s(y * b) = 0$. Hence

$$\begin{aligned}
rs(xy * ab) =& rs(xy * ab) * (sb0 * ra0) \\
=& rs(xy * ab) * (sb \cdot r(x * a) * ra \cdot s(y * b)) \\
=& rs(xy * ab) * rs(b(x * a) * a(y * b)) \\
=& rs((xy * ab) * (b(x * a) * a(y * b))) \\
=& rs((xy * ab) * ((bx * (ay * ab)) * ba)) \\
=& rs((xy * ab) * (((bx * ay) * ab) * ba)) \\
=& rs((xy * ab) * ((bx * ay) * (ab * ba))) \\
=& rs((xy * ab) * (bx * ay)) \\
=& rs(xy * ab) * rs(xb * ay) \\
=& ((rxsy * rasb) * rsxb) * rasy \\
=& ((rxsy * rasb) * rasy) * rsxb \\
=& ((rxsy * rasy) * rasb) * rxsb \\
=& (rxsy * (rasy * rasb)) * rxsb \\
=& (rx \cdot sy * rx \cdot sb) * (ra \cdot sy * ra \cdot sb) \\
=& rx(sy * sb) * ra(sy * sb) \\
=& (rx * ra) \cdot (sy * sb) \\
=& 0.
\end{aligned}$$

This means that $xy \in < ab >$. Hence $< a >< b > \subseteq < ab > \subseteq P$. Since P is prime, $< a > \subseteq P$ or $< b > \subseteq P$, whence $a \in P$ or $b \in P$. □

ACKNOWLEDGEMENTS. The authors express their thanks to the referee for his/her valuable suggestions.

References

1. J. Ahsan, E. Y. Deeba and A. B. Thaheem, *On prime ideals of BCK-algebrs*, Math. Japonica **36** (1991), 875-882.
2. Z. Chen and H. Wang, *On simple BCI-algebras*, Math. Japonica **36** (1991), 627-632.
3. K. Iséki, *An algebra related with a propositional calculus*, Proc. Japan. Acad. **42** (1966), 351-366.
4. K. Iséki and S. Tanaka, *An introduction to the theory of BCK-algebras*, Math. Japonica **23** (1978), 1-26.
5. S. S. Ahn and H. S. Kim, *On isomorphism theorems in BCI-semigroups*, J. Chung cheong Math. Soc. **9** (1996), 1-9.
6. Y. B. Jun, S. M. Hong and E. H. Roh, *BCI-semigroups*, Honam. Math. J. **15** (1993), 59-64.
7. Y. B. Jun, Y. H. Kim, J. Y. Kim and H. S. Kim, *Fuzzy commutative $\mathcal{I}$-ideals in BCI-semigroups*, submitted.
8. Y. B. Jun, S. S. Ahn, J. Y. Kim and H. S. Kim, *Fuzzy $\mathcal{I}$-ideals in BCI-semigroups*, submitted.
9. C .Z. Mu and W. H. Xiong, *On ideals in BCK-algebras*, Math. Japonica **36** (1991), 497-501.

Sun Shin Ahn
Dept. of Mathematics Education
Dongguk University
Seoul 100-715, Korea.

Hee Sik Kim
Dept. of Mathematics Education
Chungbuk National University
Chongju 361-763, Korea
heekim@cbucc.chungbuk.ac.kr