Comm. Korean Math. Soc. **11** (1996), No. 4, pp. 881–885

PICARD GROUP OF A SURFACE IN $\mathbb{P}^3$

SUNG-OCK KIM

ABSTRACT. We give the optimal lower bound for the Picard number of certain surfaces in the Noether-Lefschetz locus.

1. Introduction

We work over the complex numbers $\mathbb{C}$. A surface or a curve is a projective variety of dimension two or one, respectively. Lefschetz (1,1) Theorem asserts that there is a one to one correspondence between the curves on a smooth surface and the integral (1,1) cohomology classes of the surface. So the Picard group $Pic(S)$ of a smooth surface S in $\mathbb{P}^n$ is identified with $H^{1,1}(S) \cap H^2(S, \mathbb{Z})$, where, of course, $H^2(S, \mathbb{Z})$ is its image under the natural inclusion in $H^2(S, \mathbb{R})$. Moreover, a curve on S which is not a complete intersection of S with another hypersurface in $\mathbb{P}^n$ corresponds to a primitive integral (1,1)-class of S. That is, such a curve provides a generator other than the hyperplane class of S in $Pic(S)$.

Noether-Lefschetz theorem([5]) says that $Pic(S) \cong \mathbb{Z}$ for a general surface of degree $d \geq 4$ in $\mathbb{P}^3$.

The word "general" is used in the sense that a property is said to hold at a general point of a projective variety V if the property holds at all the points of V but the points in a countable union of subvarieties of V.

Since Lefschetz gave the complete proof of the above theorem, many different ways of proving the same theorem and improvements of the theorem in various directions have been done by several mathematicians. One direction is to study and compute the Picard number of a surface,

Received May 11, 1995. Revised October 8, 1996.
1991 AMS Subject Classification: 14J10.
Key words and phrases: Picard number.

which is the rank of the Picard group. For example, A. Lopez([6]) characterized the Picard group of a general surface in $\mathbb{P}^3$ containing a fixed curve.

In this paper, we will provide a lower bound for the Picard number of certain surfaces(Theorem 3.1). The method is infinitesimal Hodge theoretic and we will use one of M. Green's theorem.

2. Preliminaries

Let Y_d be the set of smooth surfaces of degree d in $\mathbb{P}^3$.

DEFINITION 2.1. The *Noether-Lefschetz locus* Σ *in* Y_d is defined as

$$\Sigma = \{S \in Y_d \mid Pic(S) \not\cong \mathbb{Z}\}.$$

M. Green([2]) proved that the codimension of an irreducible component of Σ in Y_d is at least $d-3$ for $d \geq 4$. The following theorem is also proven by M. Green.

THEOREM 2.2 (M.GREEN). *Let $W \subseteq H^0(O_{\mathbb{P}^n}(d))$ be a base point free linear subspace of codimension c. Let μ_k denote the multiplication map*

$$W \otimes H^0(O_{\mathbb{P}^n}(k)) \overset{\mu_k}{\longrightarrow} H^0(O_{\mathbb{P}^n}(d+k))$$

and

$$c_k = \operatorname{codim}(\operatorname{image} \mu_k).$$

If $c \leq d$ and $c_{c-1} \neq 0$, then $c_k = c-k$ for $0 \leq k \leq c$.

3. Main Results

THEOREM 3.1. *Let $d \geq 5$. Let Z be an irreducible component of Σ, and $c = Codim_{Y_d} Z$. If $c \leq d$ and $c_{c-1} \neq 0$, then the Picard number of S is $\geq c-d+5$, for a smooth point S of Z.*

PROOF. Let T_1 be the Zariski tangent space of Z at S. Since S is in Σ, there exists a curve C which is not a complete intersection. Let $L = O_S(C)$ and $\gamma = c_1(L) \in H^{1,1}_{prim}(S)$. An extension M of the tangent sheaf Θ_S of S by O_S is defined by the exact sequence

$$0 \to O_S \to M_S \to \Theta_S \to 0$$

with the extension class $c_1(L)$. We can choose C so that the image T_γ of T_1 under the Kodaira-Spencer map is in the kernel of the map

$$H^1(S, \Theta_S) \to H^2(S, O_S)$$

which is the cup product map with $\gamma = c_1(L)$. Equivalently, the image of $T_\gamma \otimes H^0(S, K_S)$ is contained in $c_1(L)^\perp$. Using the standard identification (cf. [1] or [4]), this is the multiplication map

$$\frac{H^0(\mathbb{P}^3, O_{\mathbb{P}^3}(d))}{J_d} \otimes H^0(\mathbb{P}^3, O_{\mathbb{P}^3}(d-4)) \to \frac{H^0(\mathbb{P}^3, O_{\mathbb{P}^3}(2d-4))}{J_{2d-4}}$$

where J_k denotes the Jacobian ideal of S in degree k. Let W be the preimage of T_γ in $H^0(\mathbb{P}^3, O_{\mathbb{P}^3}(d))$. Then the multiplication map

$$W \otimes H^0(\mathbb{P}^3, O_{\mathbb{P}^3}(d-4)) \to H^0(\mathbb{P}^3, O_{\mathbb{P}^3}(2d-4))$$

is not surjective. W contains J_d and J_d is base point free since S is smooth. Hence W is a base point free linear subspace of codimension c. For c such that $c \leq d$ and $c_{c-1} \neq 0$, the codimension of the image of $W \otimes H^0(\mathbb{P}^3, O_{\mathbb{P}^3}(d-4))$ is $c-d+4$ by Theorem (2.3). Hence the codimension of $T_\gamma \otimes H^0(S, K_S)$ in $H^{1,1}_{prim}(S)$ is $c-d+4$. So the $\dim H^{1,1}_{prim}(S) \geq c-d+4$ and rank $Pic(S) \geq c-d+5$. □

REMARK. It is well known that the component of Σ whose generic member S contains a line has codimension $d-3$([3]). In this case, the inequality in Theorem (3.1) says that rank $\mathrm{Pic}(S) \geq 2$, which is trivial. A. Lopez proved that the equality holds in this case.

THEOREM 3.2 (A.LOPEZ). *Let $d \geq 5$. Let Z_1 denote an irreducible component of Σ whose generic member contains a line. Then for a general S in Z_1, the Picard number is 2.*

His proof can be divided into two steps. The first step is to show that the codimension of $T_\gamma \otimes H^0(S, K_S)$ in $H^{1,1}_{prim}(S)$ is 1. The second step shows that this fact implies the Picard number of S to be 2. By using Theorem (3.1), we can prove the first part as follows.

LEMMA 3.3. *The codimension of $T_\gamma \otimes H^0(S, K_S)$ in $H^{1,1}_{prim}(S)$ is 1.*

PROOF. Let S be a smooth point of Z_1. Then the codimension of Z_1 is $d-3$. Let C denote a line contained is S. Then we can choose C so that the argument in the proof of Theorem (3.1) can hold. Hence for $\gamma = c_1(O_S(C)) \in H^{1,1}_{prim}(S)$, the codimension of $T_\gamma \otimes H^0(S, K_S)$ in $H^{1,1}_{prim}(S)$ is 1. □

For the second step showing how the above lemma implies this theorem, we will restate Lopez's proof.

PROOF. ([7]) By the semicontinuity theorem, it is enough to prove that for each class $\gamma' \in H^{1,1}_{prim}(S, \mathbb{C}) \backslash \mathbb{C}\gamma$, there exists a deformation $\eta \in T_\gamma$ such that, when we deform S in the direction of η to a surface S', the class γ' is not of type $(1,1)$ in $H^2(S', \mathbb{Z}) \cong H^2(S, \mathbb{Z})$.

But, by the Lemma (3.3),

$$H^{1,1}_{prim}(S, C) \cong \mathbb{C}(\gamma) \oplus \text{image}(T_\gamma \otimes H^0(S, K_S)).$$

Let γ'' be the nonzero component of γ' in $im(T_\gamma \otimes H^0(S, K_S))$. If $T_\gamma \subset T_{\gamma''}$, then

$$\gamma'' \in im(T_\gamma \otimes H^0(S, K_S)) \subset im(T_{\gamma''} \otimes H^0(S, K_S)) \subset (\gamma'')^\perp$$

which is a contradiction. □

References

1. J. Carlson and P. Griffiths, *Infinitesimal variation of Hodge structure and the global Torelli problem*, J. Geometrie Algebrique d'Angers, Sijthoff and Noordhoff (1980), 51-76.

2. M. Green, *A new proof of the explicit Noether-Lefschetz theorem*, J. Differential Geom. **27** (1988), 155-159.
3. ———, *Components of maximal dimension in the Noether-Lefschetz locus*, J. Differential Geom. **29** (1989), 295-302.
4. S. Kim, *Noether-Lefschetz locus for surfaces*, Trans. Amer. Math. Soc. **324** (1991), 369-384.
5. S. Lefschetz, *On certain numerical invariants of algebraic varieties*, Trans. Amer. Math. Soc. **22** (1921), 326-363.
6. A. Lopez, *Noether-Lefschetz theory and the Picard group of projective surfaces*, Memoirs of the Amer. Math. Soc. **89** (1991).
7. ———, *Hodge theory on the Fermat surface and the Picard number of a general surface in $\mathbb{P}^3$ containing a plane curve*, Sezione Scientifica, Bollettino U.M.I. **7-B** (1993), 1-22.

Hang-Dong University
Pohang 791-940, Korea