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C^1-BICUBIC SPLINE INTERPOLANT ON AN IRREGULAR MESH

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ABSTRACT. In the course of working on the preconditioning of C^1-bicubic collocation method, one has to deal with the C^1-bicubic splines. In this paper we are concerned with C^1-bicubic spline interpolant for a given function. We construct a basis for the space of C^1-bicubic splines for a given partition and find the C^1-bicubic spline interpolant for a given function defined on a set.

1. Introduction

Let $I = [0, 1]$ be the unit interval and let $\Omega = I \times I$ be the unit square. Define Δ^t as a partition of I for t-directions($t = x$ or y) such as

$$\Delta^t \; : \; 0 = t_0 < t_1 < \cdots < t_N = 1, \quad h_i^t = t_i - t_{i-1} \tag{1.1}$$

where N is a positive integer. Let $\Delta := \Delta^t$ for the one dimensional case and let $\Omega_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ be a partition $\pi := \Delta^x \times \Delta^y$ of Ω.

Define the space S^t of C^1–cubic splines for a partition Δ^t as

(1.2)
$$S^t = \{f \in C^1(I) : f|_{[t_{i-1}, t_i]} \text{ is polynomial of degree} \le 3, \; i = 1, \cdots, N\}.$$

Define the space S^π of C^1-bicubic splines for the partition π as

(1.3)
$$S^\pi = \{\sum_j c_j \; f_j(x) \; g_j(y) \;\; : f_j \in S^x, \; g_j \in S^y, \; c_j's \text{ are real numbers } \}.$$

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Let $\{\xi_i\}_{i=0}^{2N+1}$ be a set of numbers satisfying

(1.4) $\quad \xi_0 = 0, \quad t_{i-1} < \xi_{2i-1} < \xi_{2i} < t_i, \quad \xi_{2N+1} = 1, \quad i = 1, \cdots, N.$

In this paper we will construct the interpolatory basis $\{\phi_i\}_{i=0}^{2N+1}$ for S^t satisfying

(1.5)
$$\phi_i(\xi_k) = \delta_{ik}, \quad i, k = 0, 1, \cdots, 2N+1.$$

The existence and uniqueness of $\{\phi_i\}_{i=0}^{2N+1}$ can be checked by using Schoenberg-Whitney conditions (see [5],[6],[7]). We extend the results to the space S^π of C^1-bicubic splines generated by $\phi_i \, \psi_j$, $i = 0, 1, \cdots, 2N+1$, $j = 0, 1, \cdots, 2M+1$ where ϕ_i and ψ_j are the interpolatory basis functions of S^x and S^y respectively.

For a given function g defined on a set $\{(\xi_i, \eta_j) \in \Omega : i = 0, 1, \cdots, 2N+1, \quad j = 0, 1, \cdots, 2M+1\}$ and a partition π such that $\xi_i{}'s$ and $\eta_j{}'s$ satisfy (1.4) for x and y-directions, respectively, g has the unique C^1-bicubic spline interpolant on S^π.

We will discuss these splines on a uniform mesh and give some computational experiments with figures in section 4.

2. Some estimates for C^1-cubic spline

LAMMA 2.1. *Let f be a cubic polynomial on $[0, h]$ vanishing at p and q where $0 < p < q < h$ and $p \neq q$. Then there is a matrix $M[p, q, h]$ such that*

(2.1)
$$\begin{bmatrix} f(h) \\ f'(h) \end{bmatrix} = M[p, q, h] \begin{bmatrix} f(0) \\ f'(0) \end{bmatrix}$$

where

(2.2a)
$$M[p, q, h](1, 1) = 1 + \frac{h^3(p+q) - h^2(p^2 + q^2 + pq)}{p^2 q^2},$$

(2.2b)
$$M[p, q, h](1, 2) = h + \frac{h^3 - h^2(p+q)}{pq},$$

(2.2c)
$$M[p, q, h](2, 1) = \frac{3h^2(p+q) - 2h(p^2 + q^2 + pq)}{p^2 q^2}$$

and

$$M[p,q,h](2,2) = 1 + \frac{3h^2 - 2h(p+q)}{pq}. \tag{2.2d}$$

PROOF. Let $f(t) = at^3 + bt^2 + ct + d, a \neq 0$. Then we have

$$f(0) = d \quad \text{and} \quad f'(0) = c. \tag{2.3}$$

Since $f(p) = f(q) = 0$, we have

$$ap^3 + bp^2 + f'(0)p + f(0) = 0, \tag{2.4a}$$

$$aq^3 + bq^2 + f'(0)q + f(0) = 0. \tag{2.4b}$$

Hence we have a matrix form

$$\begin{bmatrix} p^3 & p^2 \\ q^3 & q^2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = - \begin{bmatrix} 1 & p \\ 1 & q \end{bmatrix} \begin{bmatrix} f(0) \\ f'(0) \end{bmatrix}. \tag{2.5}$$

Then we have

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{p^2q^2} \begin{bmatrix} p+q & pq \\ -(p^2+q^2+pq) & -pq(p+q) \end{bmatrix} \begin{bmatrix} f(0) \\ f'(0) \end{bmatrix}. \tag{2.6}$$

Note that

$$f(h) = ah^3 + bh^2 + f'(0)h + f(0), \tag{2.7a}$$

$$f'(h) = 3ah^2 + 2bh + f'(0). \tag{2.7b}$$

Therefore we have

$$\begin{bmatrix} f(h) \\ f'(h) \end{bmatrix} = \begin{bmatrix} h^3 & h^2 \\ 3h^2 & 2h \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \begin{bmatrix} f(0) \\ f'(0) \end{bmatrix}. \tag{2.8}$$

Substituting (2.6) to (2.8), we have the conclusion.

COROLLARY 2.1. *Under the assumption of Lemma 2.1, the function f is represented as*

$$f(t) = at^3 + bt^2 + f'(0)t + f(0), \quad t \in [0, h] \tag{2.9}$$

with

$$\begin{bmatrix} a \\ b \end{bmatrix} = D[p, q] \begin{bmatrix} f(0) \\ f'(0) \end{bmatrix} \tag{2.10}$$

where

$$D[p, q] := \frac{1}{p^2q^2} \begin{bmatrix} p+q & pq \\ -(p+q)^2 + pq & -pq(p+q) \end{bmatrix}. \tag{2.11}$$

By changing (2.4a) in Lemma 2.1 as

$$ap^3 + bp^2 + f'(0)p + f(0) = 1 \tag{2.12}$$

and simply repeating some modifications of the arguments in Lemma 2.1, we have the following lemma and corollary:

LEMMA 2.2. *Let f be a cubic polynomial on $[0, h]$ satisfying $f(p) = 1$ and $f(q) = 0$ where $0 < p < q < h$ and $p \neq q$. Then we have*

$$\begin{bmatrix} f(h) \\ f'(h) \end{bmatrix} = M[p, q, h] \begin{bmatrix} f(0) \\ f'(0) \end{bmatrix} + \frac{1}{p^2(p-q)} \begin{bmatrix} 1-q \\ 3-2q \end{bmatrix}. \tag{2.13}$$

COROLLARY 2.2. *Under the assumption of Lemma 2.2, the function f is represented as*

$$f(t) = at^3 + bt^2 + f'(0)t + f(0), \quad t \in [0, h] \tag{2.14}$$

with

$$\begin{bmatrix} a \\ b \end{bmatrix} = D[p, q] \begin{bmatrix} f(0) \\ f'(0) \end{bmatrix} + \frac{1}{p^2(p-q)} \begin{bmatrix} 1 \\ -q \end{bmatrix}. \tag{2.15}$$

REMARK. If $p+q=h$, then $M[p,q,h]$ in Lemma 2.1 is a positive matrix such that

$$M[p,q,h] = \begin{bmatrix} 1+\dfrac{h^2}{pq} & h \\ \dfrac{h^3+2hpq}{p^2q^2} & 1+\dfrac{h^2}{pq} \end{bmatrix} \tag{2.16}$$

and

$$M[p,q,h](1,1) = M[p,q,h](2,2) = 1+\frac{h^2}{pq} > 5. \tag{2.17}$$

The positivity of $M[p,q,h]$ and (2.17) play an important role in an exponential decay for the C^1-cubic interpolatory splines (see [6],[7]).

3. C^1-bicubic spline interpolant

Consider the interpolatory basis $\{\phi_i\}_{i=0}^{2N+1}$ for S^t satisfying

$$\phi_i(\xi_j) = \delta_{ij}, \quad i,j=0,1,\cdots,2N+1 \tag{3.1}$$

where a set $\{\xi_i\}_{i=0}^{2N+1}$ is given in section 1.

For convenience, we denote $h_k = h_k^t$ for $1D$ case. Assume that for $i=1,2,\cdots,N$

$$\xi_{2i-1} = t_{i-1}+p_i \quad \text{and} \quad \xi_{2i} = t_{i-1}+q_i \tag{3.2}$$

where $0 < p_i < q_i < h_i$.

LEMMA 3.1. *Let $\{\phi_i\}_{i=0}^{2N+1}$ be the interpolatory basis for S^t. For $j,k=1,2,\cdots,N$, we have the following recursive relations :*

$$\begin{bmatrix} \phi_0(t_k) \\ \phi_0'(t_k) \end{bmatrix} = M[p_k,q_k,h_k] \begin{bmatrix} \phi_0(t_{k-1}) \\ \phi_0'(t_{k-1}) \end{bmatrix}, \tag{3.3}$$

$$\begin{bmatrix} \phi_{2N+1}(t_k) \\ \phi_{2N+1}'(t_k) \end{bmatrix} = M[p_k,q_k,h_k] \begin{bmatrix} \phi_{2N+1}(t_{k-1}) \\ \phi_{2N+1}'(t_{k-1}) \end{bmatrix}, \tag{3.4}$$

(3.5)
$$\begin{bmatrix} \phi_{2j-1}(t_k) \\ \phi'_{2j-1}(t_k) \end{bmatrix} = M[p_k, q_k, h_k] \begin{bmatrix} \phi_{2j-1}(t_{k-1}) \\ \phi'_{2j-1}(t_{k-1}) \end{bmatrix} + \frac{\delta_{jk}}{p_j^2(p_j - q_j)} \begin{bmatrix} 1 - q_j \\ 3 - 2q_j \end{bmatrix}$$

and

(3.6)
$$\begin{bmatrix} \phi_{2j}(t_k) \\ \phi'_{2j}(t_k) \end{bmatrix} = M[p_k, q_k, h_k] \begin{bmatrix} \phi_{2j}(t_{k-1}) \\ \phi'_{2j}(t_{k-1}) \end{bmatrix} + \frac{\delta_{jk}}{q_j^2(q_j - p_j)} \begin{bmatrix} 1 - p_j \\ 3 - 2p_j \end{bmatrix}.$$

PROOF. For the case ϕ_l $(l = 2j - 1)$, using the restriction and translation, we define $\phi_{l,k}$ on $[0, h_k]$ as follow:

$$\phi_{l,k}(t) = \phi_l(t_{k-1} + t), \quad k = 1, 2, \cdots, N. \tag{3.7}$$

Then we have, for $k = 1, 2, \cdots, N - 1$,

$$\phi_{l,k}(h_k) = \phi_l(t_k) = \phi_{l,k+1}(0), \quad \phi'_{l,k}(h_k) = \phi'_l(t_k) = \phi'_{l,k+1}(0), \tag{3.8}$$

Moreover, from (3.1), we have for $j, k = 1, 2, \cdots, N$,

$$\phi_{l,k}(p_k) = \phi_l(\xi_{2k-1}) = \delta_{jk}, \quad \phi_{l,k}(q_k) = \phi_l(\xi_{2k}) = 0. \tag{3.9}$$

Applying (3.8) and (3.9) to Lemma 2.1, 2.2, (3.5) follows. Similarly we will prove the other cases.

By Lemma 3.1, we can derive the $2N + 2$ second order linear systems for two linear equations:

(3.10)
$$\begin{bmatrix} 0 \\ \phi'_0(1) \end{bmatrix} = \tilde{M}[N, 1] \begin{bmatrix} 1 \\ \phi'_0(0) \end{bmatrix}, \qquad \begin{bmatrix} 1 \\ \phi'_{2N+1}(1) \end{bmatrix} = \tilde{M}[N, 1] \begin{bmatrix} 0 \\ \phi'_{2N+1}(0) \end{bmatrix}$$

and for $j = 1, 2, \cdots, N$

(3.11)
$$\begin{bmatrix} 0 \\ \phi'_{2j-1}(1) \end{bmatrix} = \tilde{M}[N, 1] \begin{bmatrix} 0 \\ \phi'_{2j-1}(0) \end{bmatrix} + \frac{1}{p_j^2(p_j - q_j)} \tilde{M}[N, j+1] \begin{bmatrix} 1 - q_j \\ 3 - 2q_j \end{bmatrix},$$

(3.12)
$$\begin{bmatrix} 0 \\ \phi'_{2j}(1) \end{bmatrix} = \tilde{M}[N,1] \begin{bmatrix} 0 \\ \phi'_{2j}(0) \end{bmatrix} + \frac{1}{q_j^2(q_j - p_j)} \tilde{M}[N, j+1] \begin{bmatrix} 1 - p_j \\ 3 - 2p_j \end{bmatrix},$$

where $\tilde{M}[N,k] = 0, \ k > N$ and
(3.13)
$$\tilde{M}[N,k] = M[p_N, q_N, h_N] M[p_{N-1}, q_{N-1}, h_{N-1}] \cdots M[p_k, q_k, h_k], \ k \leq N.$$

First, from these systems we find $\phi'_i(0)$ and $\phi'_i(1)$ and then using Lemma 3.1, find $\phi_i(t_k)$ and $\phi'_i(t_k)$ for $i = 0, 1, \cdots, 2N+1, \ k = 0, 1, \cdots, N$. Now, by Corollary 2.1, 2.2, we have the following lemma:

LEMMA 3.2. *On each subinterval* $[t_{k-1}, t_k]$, $k = 1, 2, \cdots, N$, *we have the following representation for* ϕ_i, $i = 0, 1, \cdots, 2N+1$,

$$\phi_i(t) = a_i(t - t_{k-1})^3 + b_i(t - t_{k-1})^2 + \phi'_i(t_{k-1})t + \phi_i(t_{k-1}) \tag{3.14}$$

with

$$\begin{bmatrix} a_i \\ b_i \end{bmatrix} = D[p_k, q_k] \begin{bmatrix} \phi_i(t_{k-1}) \\ \phi'_i(t_{k-1}) \end{bmatrix} + R_i \tag{3.15}$$

where

$$R_i := \begin{cases} \dfrac{1}{p_k^2(p_k - q_k)} \begin{bmatrix} 1 \\ -q_k \end{bmatrix} & \text{for} \quad i = 2k-1, \\ \dfrac{1}{q_k^2(q_k - p_k)} \begin{bmatrix} 1 \\ -p_k \end{bmatrix} & \text{for} \quad i = 2k, \\ 0 & \text{otherwise.} \end{cases} \tag{3.16}$$

By applying Lemma 3.1 and 3.2 to two dimensional case, we have the following theorem :

THEOREM 3.1. *Given a function* g *defined on a set* $\{(\xi_i, \eta_j) \in \Omega : i = 0, 1, \cdots, 2N+1, \quad j = 0, 1, \cdots, 2M+1\}$ *and a partition* $\pi := \Delta^x \times \Delta^y$ *that satisfies*

(3.17)
$$0 \leq x_{i-1} < \xi_{2i-1} < \xi_{2i} < x_i \leq 1, \quad i = 1, 2, \cdots, N,$$

(3.18)
$$0 \leq y_{j-1} < \eta_{2j-1} < \eta_{2j} < y_j \leq 1, \quad j = 1, 2, \cdots, M$$

and

(3.19)

$$(x_0, y_0) = (\xi_0, \eta_0) = (0,0), \quad (x_N, y_M) = (\xi_{2N+1}, \eta_{2M+1}) = (1,1),$$

g has the unique C^1-bicubic spline interpolant $s \in S^\pi$ such that

$$(3.20) \qquad s(x,y) = \sum_{i=0}^{2N+1} \sum_{j=0}^{2M+1} g(\xi_i, \eta_j)\, \phi_i(x)\, \psi_j(y) \quad \text{for} \quad (x,y) \in \Omega.$$

where $\{\phi_i\}_{i=0}^{2N+1}$ and $\{\psi_j\}_{j=0}^{2M+1}$ are the base of S^x and S^y, respectively, satisfying

$$(3.21) \qquad \phi_i(\xi_k) = \delta_{ik} \quad \text{and} \quad \psi_j(\eta_l) = \delta_{jl}.$$

4. C^1-bicubic splines on uniform mesh

In this section we consider C^1-cubic splines for the partition $\Delta = \{t_i\}_{i=0}^N$ such that $t_k = k\ h\ , k = 0, 1, \cdots, N$ where $h = 1/N$. Assume that for $i = 1, 2, \cdot, N$,

$$(4.1) \qquad \xi_{2i-1} = t_{i-1} + p \quad \text{and} \quad \xi_{2i} = t_{i-1} + q$$

where $0 < p < q < h$ and $p + q = h$.

Define two matrices $M_t = M[p, q, h]$ and $D_t = D[p, q]$. Let $\{\phi_i\}_{i=0}^{2N+1}$ be the interpolatory basis for S^t satisfying

$$\phi_i(\xi_k) = \delta_{ik}, \quad i, k = 0, 1, \cdots, 2N+1.$$

THEOREM 4.1. *For $k = 0, 1, \cdots, N$, we have*

$$(4.2) \qquad \phi_{2k}(t) = \phi_{2(N-k)+1}(1-t) \quad \text{for} \quad t \in I.$$

PROOF. Define $f(t) = \phi_{2(N-k)+1}(1-t)$ for $t \in I$. Since $p+q=h$, we can easily check that

$$(4.3) \qquad \xi_i = 1 - \xi_{2N-i+1}, \quad i = 0, 1, \cdots, 2N+1.$$

So f is a C^1-cubic spline satisfying

$$(4.4) \qquad f(\xi_{2k}) = 1 \quad \text{and} \quad f(\xi_j) = 0 \quad \text{for all } j \neq 2k.$$

By the uniqueness of ϕ_{2k}, f coincides with ϕ_{2k}. Therefore we have the conclusion.

Now we need to determine $(N+1)$ basis functions $\phi_1, \phi_3, \cdots, \phi_{2N+1}$ for S^t.
From Lemma 3.1 and (3.10)-(3.12), we have the following recursive relations for $i = 1, 2, \cdots, N+1$, $k = 1, 2, \cdots, N$,

$$(4.5) \qquad \begin{bmatrix} \phi_{2i-1}(t_k) \\ \phi'_{2i-1}(t_k) \end{bmatrix} = M_t \begin{bmatrix} \phi_{2i-1}(t_{k-1}) \\ \phi'_{2i-1}(t_{k-1}) \end{bmatrix}, \quad k < i$$

and

$$(4.6) \qquad \begin{bmatrix} \phi_{2i-1}(t_k) \\ \phi'_{2i-1}(t_k) \end{bmatrix} = (M_t)^{-1} \begin{bmatrix} \phi_{2i-1}(t_{k+1}) \\ \phi'_{2i-1}(t_{k+1}) \end{bmatrix}, \quad k \geq i$$

with $\phi'_{2i-1}(0)$ and $\phi'_{2i-1}(1)$ satisfying
(4.7)

$$\begin{bmatrix} \phi_{2i-1}(1) \\ \phi'_{2i-1}(1) \end{bmatrix} = (M_t)^N \begin{bmatrix} 0 \\ \phi'_{2i-1}(0) \end{bmatrix} + \frac{1}{p^2(p-q)} (M_t)^{N-i} \begin{bmatrix} 1-q \\ 3-2q \end{bmatrix}, i \neq n+1$$

where $\phi_{2N+1}(1) = 1$ and $\phi_{2i-1}(1) = 0$ $(i \neq n+1)$. And on each subinterval $[t_{k-1}, t_k]$, ϕ_{2i-1} is represented as
(4.8)

$$\phi_{2i-1}(t) = a_{2i-1}(t-t_{k-1})^3 + b_{2i-1}(t-t_{k-1})^2 + \phi'_{2i-1}(t_{k-1})t + \phi_{2i-1}(t_{k-1})$$

with

$$(4.9) \qquad \begin{bmatrix} a_{2i-1} \\ b_{2i-1} \end{bmatrix} = D_t \begin{bmatrix} \phi_{2i-1}(t_{k-1}) \\ \phi'_{2i-1}(t_{k-1}) \end{bmatrix} + \frac{\delta_{ik}}{p^2(p-q)} \begin{bmatrix} 1 \\ -q \end{bmatrix}.$$

EXAMPLE. Using two Legendre-Gauss points

$$p = \frac{h}{2}(1 - \frac{1}{\sqrt{3}}) \quad \text{and} \quad q = \frac{h}{2}(1 + \frac{1}{\sqrt{3}}) \quad \text{on} \quad [0, h], \tag{4.10}$$

we compute

$$M_t = \begin{bmatrix} 7 & h \\ \frac{48}{7} & 7 \end{bmatrix} \quad \text{and} \quad D_t = \frac{6}{h^3} \begin{bmatrix} 6 & h \\ -5h & -h^2 \end{bmatrix} \tag{4.11}$$

and for $i = 1, 2, \cdots, N$,

$$\phi'_{2i-1}(0) = \frac{6[(10 + 5\sqrt{3})\lambda^i + (2 - \sqrt{3})\lambda^{2N-i}]}{h(1 - \lambda^{2N})}, \tag{4.12}$$

$$\phi'_{2i-1}(1) = \frac{6[(10 + 5\sqrt{3})\lambda^{N+i} + (2 - \sqrt{3})\lambda^{N-i}]}{h(1 - \lambda^{2N})}, \tag{4.13}$$

$$\phi'_{2N+1}(0) = \frac{8\sqrt{3}\lambda^N}{h(1 - \lambda^{2N})} \tag{4.14}$$

and

$$\phi'_{2N+1}(1) = \frac{4\sqrt{3}(1 + \lambda^{2N})}{h(1 - \lambda^{2N})} \tag{4.15}$$

where $\lambda = 7 - 4\sqrt{3}$. Then from (4.5),(4.6) and (4.8), we can find the basis functions $\{\phi_i\}_{i=0}^{2N+1}$ for S^t. Let S^π be the space of C^1-bicubic splines generated by $\phi_i(x)\phi_j(y)$, $i, j = 0, 1, \cdots, 2N + 1$.

For one dimensional case, we show the figures of two basis functions ϕ_5 ($N = 4$) and ϕ_{10} ($N = 8$) for S^t in Figure 1 and the figures of two C^1-cubic spline interpolants $s_f(t)$ and $s_g(t)$ for $f(t) = \sin(3\pi t)$ and $g(t) = \cos(2\pi t)$, respectively, in Figure 2,3 when $N = 1, 2, 4, 8$.

For two dimensional case, we show the figures of two basis function $\phi_5(x)\phi_5(y)$ ($N = 4$) and $\phi_{10}(x)\phi_{10}(y)$ ($N = 8$) for S^π in Figure 4 and the figures of C^1-bicubic spline interpolant $s(x, y)$ for $G(x, y) = \sin(3\pi x)\cos(2\pi y)$ in Figure 5,6 when $N = 1, 2, 4, 8$.

Figure 1. The basis functions ϕ_5 ($N = 4$) and ϕ_{10} ($N = 8$).

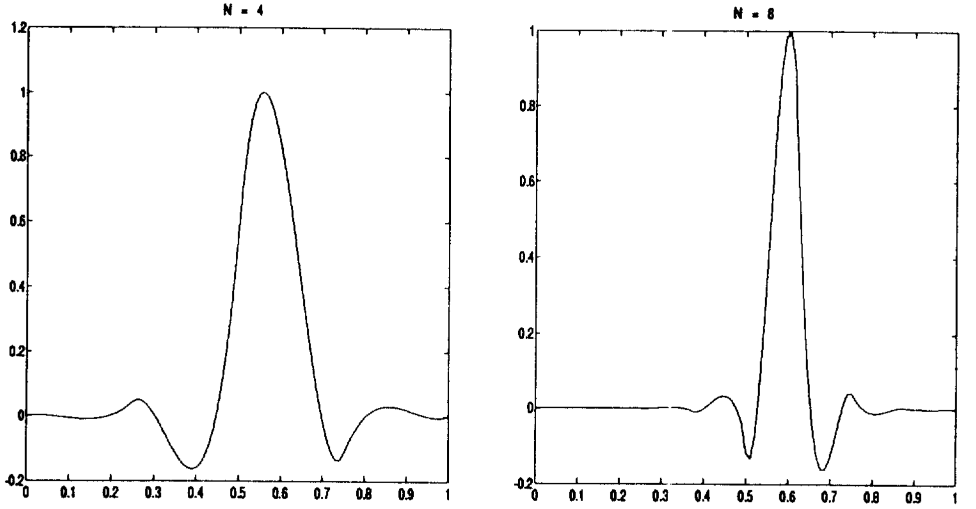
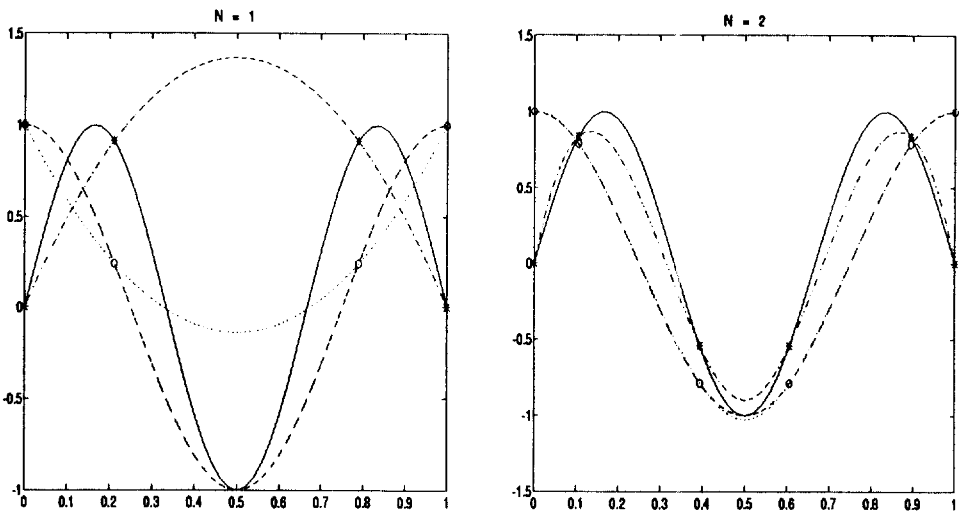
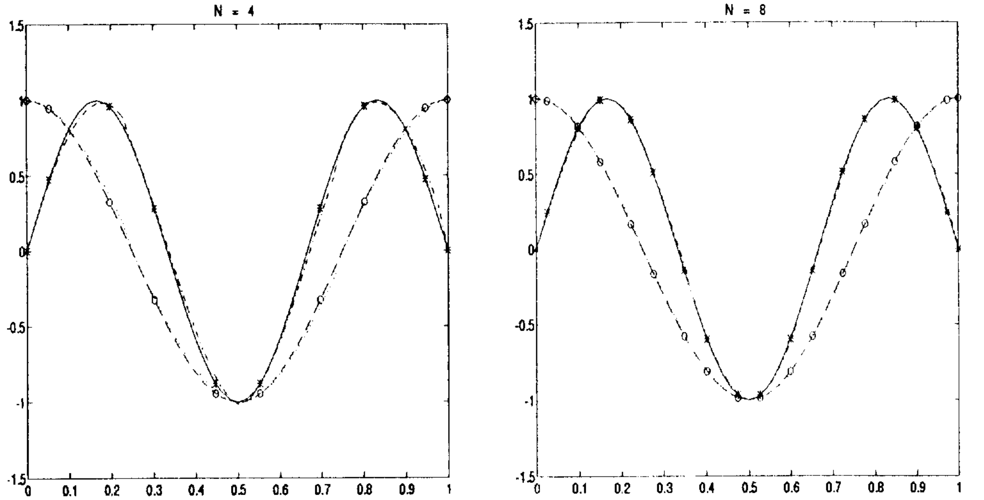


Figure 2. Cubic spline interpolants when $N = 1, 2$.



$-$: $z = f(t)$, $-\cdot$: $z = s_f(t)$ and $- -$: $z = g(t)$, $\cdots$: $z = s_g(t)$.

Figure 3. Cubic spline interpolants when $N = 4, 8$.



$-$: $z = f(t)$, $-\cdot$: $z = s_f(t)$ and - - : $z = g(t)$, $\cdots$: $z = s_g(t)$.

Figure 4. The basis functions $\phi_5(x)\phi_5(y)$ $(N = 4)$ and $\phi_{10}(x)\phi_{10}(y)$ $(N = 8)$.

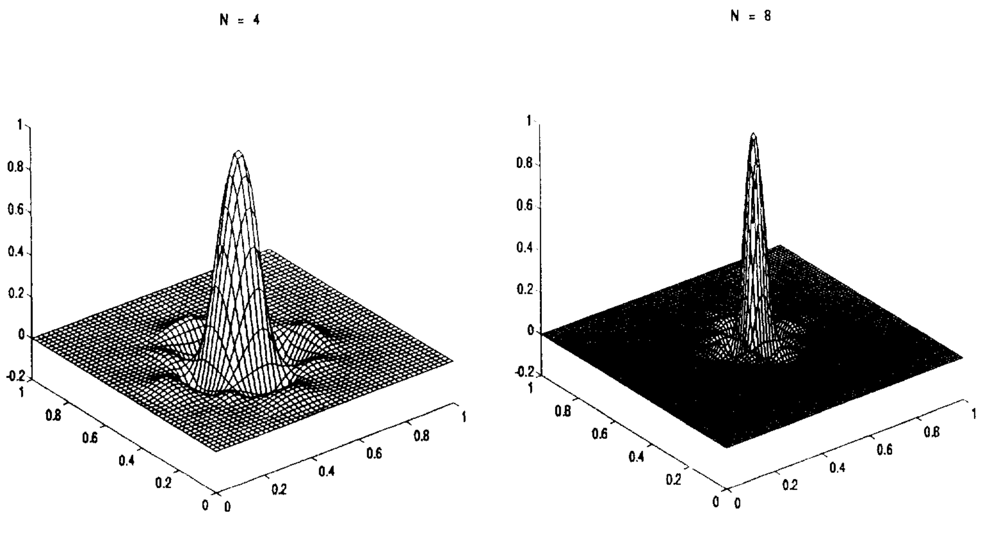
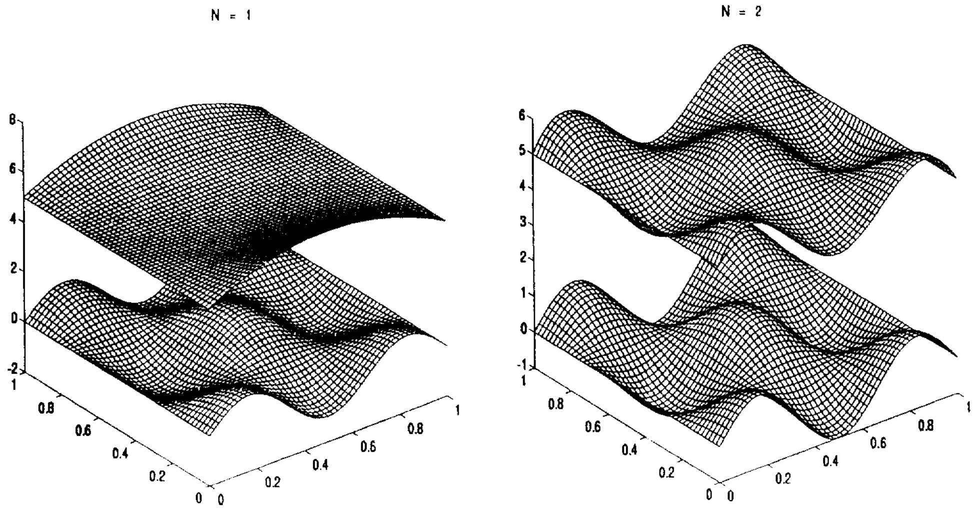
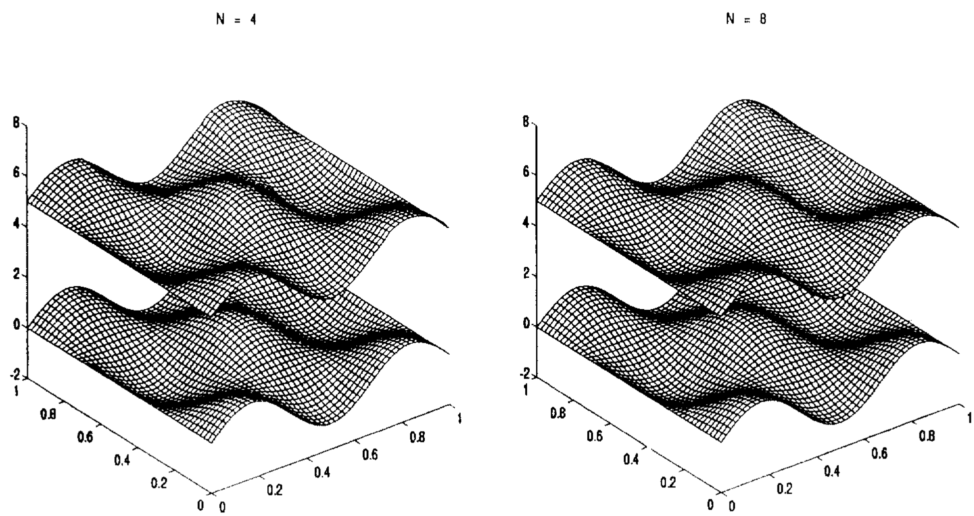


Figure 5. Bicubic spline interpolant for $G(x, y)$ when $N = 1, 2$.



above : $z = s(x, y) + 5$ and below : $z = G(x, y)$.

Figure 6. Bicubic spline interpolant for $G(x, y)$ when $N = 4, 8$.



above : $z = s(x, y) + 5$ and below : $z = G(x, y)$.

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