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A STABILITY ANALYSIS FOR HAMILTON-JACOBI EQUATIONS

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ABSTRACT. We prove that vicosity solutions are stable under changes in the flux functions as well as boundary functions. This result can be used in the study of numerical approximation of Hamilton-Jacobi equations.

1. Introduction

We consider the Dirichlet problem for Hamilton-Jacobi equations with boundary condition

$$\text{(H-J)} \qquad \begin{aligned} u(x) + f(\nabla u(x)) &= m(x) \quad \text{in} \quad \Omega, \\ u(x) &= z \quad \text{on} \quad \partial\Omega, \end{aligned}$$

where Ω is any bounded open ball centered at the origin in $\mathbb{R}^N$. This problem is a nonlinear first-order problem and it is well known that this problem does not have a classical solution even though the Hamiltonian f is smooth. Therefore we have to deal with non-smooth solutions if we want a solution of (H-J) which satisfy the equations almost everywhere. The theory of first-order partial differential equations of Hamilton-Jacobi type has substantially developed with the introduction by Crandall and Lions [1] of the class of viscosity solutions, which turns out to be the correct class of generalized solutions for such type of equations. They also showed the uniqueness of generalized solutions that satisfy a so-called "viscosity" condition. Papers by Jensen and Souganidis [4] and Souganidis [5] provided a view of the scope of the references to much of the recent literature. Cauchy problem of Hamilton-Jacobi equations

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was studied by Crandall and Lions [2]. Hong [3] showed some regularity results for Cauchy problem of Hamilton-Jacobi equations.

This paper is organized as follows. In chapter 2, we give the definition of viscosity solutions of (H-J) in several space dimensions. We also review both uniqueness and stability of the viscosity solutions.

In chapter 3, we prove the following theorem that is the main result of this paper.

THEOREM 1.1. *Let u and v be the viscosity solutions of*

$$\begin{aligned} u(x) + f(\nabla u(x)) = m(x) \quad &in \quad \Omega, \\ u(x) = z_1 \quad &on \quad \partial\Omega, \end{aligned}$$

and

$$\begin{aligned} v(x) + g(\nabla v(x)) = n(x) \quad &in \quad \Omega, \\ v(x) = z_2 \quad &on \quad \partial\Omega. \end{aligned}$$

repectively, where f and g are Lipschitz continuous and convex, and m and n are bounded and continuous on $\partial\Omega \cup \Omega$. Then

$$\|u - v\|_{L^\infty(\Omega)} \le \|f - g\|_{L^\infty(\Omega)} + \max\{\|m - n\|_{L^\infty(\Omega)}, \|u - v\|_{L^\infty(\partial\Omega)}\}.$$

This stability result gives us an error estimate if we approximate the viscosity solutions of (H-J).

2. Viscosity solutions of (H-J)

The general reference for this section is [1]. To repeat, one cannot in general find a classical solution of (H-J) on a bounded open interval in $\mathbb{R}$, while bounded Lipschitz continuous "generalized solutions" in the almost-everywhere sense exist but are not unique. For example,

$$\begin{aligned} u_x^2 = 1, \quad &\text{on} \quad (-1, 1), \\ u(-1) = u(1) = 0, \end{aligned}$$

has two solutions that satisfy the equation almost everywhere, namely, $u = 1 - |x|$ and $u = |x| - 1$. which satisfies the equation. Moreover, if u and v are generalized solutions of (H-J), then so are $\min(u, v)$ and $\max(u, v)$. In fact, if the problem is nonlinear, one can expect infinitely many generalized solutions. Crandall and Lions [1] resolved the uniqueness problem by introducing a notion of viscosity.

DEFINITION 2.1. A *viscosity subsolution* (respectively, *supersolution*) of (H-J) with $H \in C(\mathbb{R}^N)$ is a bounded function $u \in C(\Omega)$ such that for every $\phi \in C^1(\Omega)$:

$$\text{If } x_0 \text{ is a local maximum point of } u - \phi \text{ on } \Omega, \text{ then } u(x_0) + f(\nabla\phi(x_0)) \leq m(x_0). \tag{2.1.1}$$

(respectively,

$$\text{If } x_0 \text{ is a local minimum point of } u - \phi \text{ on } \Omega, \text{ then } u(x_0) + f(\nabla\phi(x_0)) \geq m(x_0).) \tag{2.1.2}$$

DEFINITION 2.2. A *viscosity solution* of (H-J) is a bounded function $u \in C(\Omega)$ for which both (2.1.1) and (2.1.2) hold (i.e. u is both a viscosity subsolution and a viscosity supersolution).

REMARK. If u is a bounded classical solution of (H-J), then it is a viscosity solution, and if u is a viscosity solution of (H-J), then $u(x_0) + f(\nabla u(x_0)) = m(x_0)$ at any point (x_0) where u is differentiable.

3. Stability of two viscosity solutions

We prove that viscosity solutions are stable under changes in the nonlinear Hamiltonians f as well as changes in the functions m and z. To prove that, we prepare a theorem and two lemmas.

THEOREM 3.1. *Suppose that bounded and cotinuous functions w and v are viscosity solutions of*

$$w(x) + g(\nabla w(x)) = m(x) \quad in \quad \Omega,$$
$$w(x) = z_1 \quad on \quad \partial\Omega,$$

and

$$v(x) + g(\nabla v(x)) = n(x) \quad in \quad \Omega,$$
$$v(x) = z_2 \quad on \quad \partial\Omega.$$

respectively. Then

$$\|w - v\|_{L^\infty(\Omega)} \leq \max\{\|m - n\|_{L^\infty(\Omega)}, \|u - v\|_{L^\infty(\partial\Omega)}\}.$$

PROOF. Suppose that $\bar{w}$ is an extension of w defined by $\bar{w} = w$ in $\bar{\Omega} = \Omega \cup \partial\Omega$. For any $\epsilon > 0$, let $w_\epsilon = \bar{w}\beta_\epsilon$ where β is smooth,

$$\beta_\epsilon(x) = \frac{1}{\epsilon^N}\beta(\frac{x}{\epsilon}) \quad \text{and} \quad \int_{\mathbb{R}^N} \beta(x)\,dx = 1.$$

Then $\lim_{\epsilon\to 0} w_\epsilon = w$ in $C(\bar{\Omega})$ and $\lim_{\epsilon\to 0} g(\nabla w_\epsilon) = g(\nabla w)$ a.e. since $\lim_{\epsilon\to 0} \nabla w_\epsilon = \nabla w$. Therefore, for any $\epsilon_0 < \epsilon$,

$$g(\nabla v) - g(\nabla w_\epsilon) + (v - w_\epsilon) \le (n - m) + h_\epsilon$$

in $\Omega_{\epsilon_0} = \{x \in \Omega \mid |x| < \frac{1}{\epsilon_0},\ \mathrm{dist}(x, \partial\Omega) > \epsilon_0\}$, where $h_\epsilon(x)$ is bounded in $L^\infty(\Omega)$ and $\lim_{\epsilon\to 0} h_\epsilon = 0$ a.e.. If we set $g(\nabla v) - g(\nabla w_\epsilon) = C_\epsilon(x)\cdot\nabla(v-w_\epsilon)$ where $C_\epsilon(x) = \int_0^1 g'(t\nabla v+(1-t)\nabla w_\epsilon)\,dt$ is bounded in $L^\infty(\Omega)$, then by maximum principle,

$$\|v - w\|_{L^\infty(\Omega_{\epsilon_0})} \le \max\{\sup_{\partial\Omega_{\epsilon_0}}(v - w)^+, \|(n - m)^+\|_{L^\infty(\Omega)}\},$$

where $(v - w)^+ = \max\{v - w, 0\}$. If we let $\epsilon \to 0$, then we have

$$\|v - w\|_{L^\infty(\Omega)} \le \max\{\|(v - w)^+\|_{L^\infty(\partial\Omega)}, \|(n - m)^+\|_{L^\infty(\Omega)}\}.$$

By symmetry, we have

$$\|w - v\|_{L^\infty(\Omega)} \le \max\{\|m - n\|_{L^\infty(\Omega)}, \|w - v\|_{L^\infty(\partial\Omega)}\}. \quad \square$$

LEMMA 3.2. *Let u and w be the viscosity solutions of*

$$\begin{aligned} u(x) + f(\nabla u(x)) &= m(x) \quad \text{in} \quad \Omega, \\ u(x) &= z_1(x) \quad \text{on} \quad \partial\Omega, \end{aligned}$$

and

$$\begin{aligned} w(x) + g(\nabla w(x)) &= m(x) \quad \text{in} \quad \Omega, \\ w(x) &= z_1(x) \quad \text{on} \quad \partial\Omega, \end{aligned}$$

respectively, where f and g are Lipschitz continuous. Then

$$\|u - w\|_{L^\infty(\Omega)} \le \|f - g\|_{L^\infty(\Omega)}.$$

Theorem 1.1 follows from combining Theorem 3.1 and Lemma 3.2.

LEMMA 3.3. *Assume that u and w are in Lemma 3.2. Let $\eta(z)$ be a smooth nonnegative function on $\mathbb{R}$ such that $\eta(-z) = \eta(z)$, $0 \le \eta(z) \le 1$, $\eta(0) = 1$ and $\eta(z) = 0$ if $|z| > 1$, and let $M = \max\{\|u\|_{L^\infty(\Omega)}, \|w\|_{L^\infty(\Omega)}\}$. Suppose that*

$$\sigma := \sup_{\Omega}(u(x) - w(x)) > 0$$

For any $\epsilon > 0$, define

$$\psi(x,y) = u(x) - w(y) + (3M + \frac{\sigma}{2})\beta_\epsilon(x-y),$$

where $\beta_\epsilon(x)$ is defined on Ω by $\beta_\epsilon(x) = \prod_{i=1}^N \eta(\frac{x_i}{\epsilon})$. If

$$\sup_{|x|\ge R} |u(x)| \quad \text{and} \quad \sup_{|x|\ge R} |w(x)| \to 0 \quad \text{as } R \to \infty, \tag{3.3.1}$$

then there exists a point $(x_0, y_0) \in \Omega \times \Omega$ such that $\psi(x_0, y_0) \ge \psi(x,y)$ on $\Omega \times \Omega$.

PROOF. Fix $\epsilon > 0$. If there is a sequence $\{(x_i, y_i)\}_{i\ge 1}$ in $\Omega \times \Omega$ such that

$$\psi(x_i, y_i) \to \sup_{\Omega\times\Omega} \psi, \tag{3.3.2}$$

then (x_i, y_i) remains bounded by the following arguments.

First,

$$\begin{aligned}\sup_{\Omega\times\Omega} \psi &\ge u(x) - w(x) + (3M + \frac{\sigma}{2})\beta_\epsilon(x-x)\\ &= u(x) - w(x) + 3M + \frac{\sigma}{2} \quad \text{for all } x \in \Omega.\end{aligned}$$

Therefore,

$$\begin{aligned}\sup_{\Omega\times\Omega} \psi &\ge \sup_{\Omega}(u(x) - w(x)) + 3M + \frac{\sigma}{2}\\ &= \sigma + 3M + \frac{\sigma}{2}\\ &= 3M + \frac{3}{2}\sigma.\end{aligned} \tag{3.3.3}$$

If $\beta_\epsilon(x-y)=0$, then

$$\begin{aligned}\psi(x,y) &= u(x)-w(y)\\ &\le 2M.\end{aligned}$$

Hence, (3.3.2) implies that $\beta_\epsilon(x_i-y_i)>0$ for large i, whence $|x_i-y_i|<\epsilon$. If $|x_i|\to\infty$ and $|y_i|\to\infty$, then

$$\lim_{i\to\infty}\sup_{\Omega\times\Omega}\psi(x_i,y_i)\le 3M+\frac{\sigma}{2}\quad\text{by 3.3.1.}$$

This contradicts (3.3.2) and (3.3.3). Therefore, $\{(x_i,y_i)\}_{i\ge 1}$ is a bounded sequence and there is a convergent subsequence of $\{(x_i,y_i)\}_{i\ge 1}$. Let (x_0,y_0) be the limit of the above subsequence. This completes the proof. □

Proof of Lemma 3.2. We will prove that σ, defined in Lemma 3.3, satisfies

$$\sigma\le\|f-g\|_{L^\infty(\Omega)}.$$

By symmetry in u and w, we see that this implies

$$\|u-w\|_{L^\infty(\Omega)}\le\|f-g\|_{L^\infty(\Omega)}.$$

We first assume that

$$\text{(A)}\qquad \sup_{|x|\ge R}|u(x)|\quad\text{and}\quad\sup_{|x|\ge R}|w(x)|\to 0\quad\text{as } R\to\infty,$$

where R is the radius of the ball Ω. If $\sigma=0$, then we are done. Otherwise, by Lemma 3.3, for any $\epsilon>0$ we can find a point $(x_0,y_0)\in\Omega\times\Omega$ such that

$$u(x)-\Big(w(y_0)-(3M+\frac{\sigma}{2})\beta_\epsilon(x-y_0)\Big)$$

attains a local maximum at (x_0), whence

$$u(x_0)+f\Big(-(3M+\frac{\sigma}{2})\nabla_x\beta_\epsilon(x_0-y_0)\Big)\le f(x_0). \tag{3.2.1}$$

Similarly

$$-w(y) - \Big(-u(x_0) - (3M + \frac{\sigma}{2})\beta_\epsilon(x_0 - y)\Big)$$

attains local maximum at $y = y_0$ and therefore

$$w(y) - \Big(u(x_0) + (3M + \frac{\sigma}{2})\beta_\epsilon(x_0 - y)\Big)$$

attains local minimum at $y = y_0$. By the definition of the viscosity solution,

$$w(y_0) + g\Big((3M + \frac{\sigma}{2})\nabla_y \beta_\epsilon(x_0 - y_0)\Big) \geq f(y_0).$$

Since $\nabla_x \beta_\epsilon(x - y_0)|_{x=x_0} = -\nabla_y \beta_\epsilon(x_0 - y)|_{y=y_0}$,

$$w(y_0) + g\Big(-(3M + \frac{\sigma}{2})\nabla_x \beta_\epsilon(x_0 - y_0)\Big) \geq f(y_0). \tag{3.2.2}$$

Combining (3.2.1) and (3.2.2) gives

(3.2.3)

$$\begin{aligned} u(x_0) - w(y_0) \leq g\Big(-(3M + \frac{\sigma}{2})\nabla_x \beta_\epsilon(x_0 - y_0)\Big) \\ -f\Big((3M + \frac{\sigma}{2})\nabla_y \beta_\epsilon(x_0 - y_0)\Big) + f(x_0) - f(y_0). \end{aligned}$$

For all $x \in \Omega$,

$$\begin{aligned} u(x) - w(x) + 3M + \frac{\sigma}{2} &= \psi(x, x) \\ &\leq \psi(x_0, y_0) \\ &\leq u(x_0) - w(y_0) + 3M + \frac{\sigma}{2}. \end{aligned} \tag{3.2.4}$$

Therefore, by 3.2.3 and 3.2.4,

$$\begin{aligned} \sigma &\leq f\Big(- (3M + \frac{\sigma}{2})\nabla_x \beta_\epsilon(x_0 - y_0)\Big) - g\Big((3M + \frac{\sigma}{2})\nabla_y \beta_\epsilon(x_0 - y_0)\Big) \\ &\quad + f(x_0) - f(y_0) \\ &\leq \|f\Big(- (3M + \frac{\sigma}{2})\nabla_x \beta_\epsilon(x_0 - y_0)\Big) \\ &\quad - g\Big(-(3M + \frac{\sigma}{2})\nabla_x \beta_\epsilon(x_0 - y_0)\Big)\|_{L^\infty(\Omega)} + \omega_f(\epsilon) \end{aligned}$$

where $\omega_f(\epsilon)$ is the modulus of continuity of f. Since ϵ is arbitrary,

$$\sigma \leq \|f - g\|_{L^\infty(\Omega)}.$$

We now drop the assumption (A). For $R > 0$, let $\rho(x)$ be a smooth function having support in the ball $B(0, R+1) = \{x \in \Omega \mid |x| \leq R+1\}$ such that $\rho(x) = 1$ on $|x| \leq R$. Suppose that $u^\rho(x) = \rho(x)u(x)$ and $w^\rho(x) = \rho(x)w(x)$. Then the corresponding viscosity solutions $u^\rho(x)$ and $w^\rho(x)$ have the following properties:

$$\begin{aligned} u^\rho(x) &= u(x) \quad \text{on} \quad |x| < R - |f|_{\text{Lip}} \quad \text{and} \\ w^\rho(x) &= w(x) \quad \text{on} \quad |x| < R - |g|_{\text{Lip}}; \end{aligned}$$

see [1]. Let $L = \max\{|f|_{\text{Lip}}, |g|_{\text{Lip}}\}$. Then

$$\begin{aligned} \max_{|x|<R-L} |u(x) - w(x)| &= \max_{|x|<R-L} |u^\rho(x) - w^\rho(x)| \\ &\leq \|f - g\|_{L^\infty(\Omega)} \quad \text{by the previous argument.} \end{aligned}$$

Hence, letting $R \to \infty$, we have

$$\|u - w\|_{L^\infty(\Omega)} \leq \|f - g\|_{L^\infty(\Omega)}.$$

This completes the proof. □

We now prove Theorem 1.1.

PROOF OF THEOREM 1.1. In addition to the equation in the statement of Theorem 1.1, consider

$$\begin{aligned} u(x) + g(\nabla u(x)) &= m(x) \quad \text{in} \quad \Omega, \\ u(x) &= z_1(x) \quad \text{on} \quad \partial\Omega. \end{aligned}$$

Then, by Theorem 3.1 and Lemma 3.2,

$$\begin{aligned} \|u - v\|_{L^\infty(\Omega)} &\leq \|u - w\|_{L^\infty(\Omega)} + \|w - v\|_{L^\infty(\Omega)} \\ &\leq \|f - g\|_{L^\infty(\Omega)} + \max\{\|m - n\|_{L^\infty(\Omega)}, \|u - v\|_{L^\infty(\partial\Omega)}\}. \end{aligned}$$

This completes the proof. □

References

1. M. G. Crandall and P. L. Lions, *Viscosity solutions of Hamilton-Jacobi equations*, Tran. Amer. Math. Soc. **277** (1983), 1–52.
2. ———, *Two approximations of solutions of Hamilton-Jacobi equations*, Math. Comp. **43** (1984), 1–19.
3. B. I. Hong, *On Regularity and approximation for Hamilton-Jacobi equations*, J. Korean Math. Soc. **301** (1993), 93–104.
4. R. Jensen and P. E. Souganidis, *A regularity for Viscosity solutions of Hamilton-Jacobi equations in one space demension*, Tran. Amer. Math. Soc. **301** (1987), 137–147.
5. P. E. Souganidis, *Existence of Viscosity solutions of Hamilton-Jacobi equations in one space demension*, J. differential Equations **56** (1985), 345–390.

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