Comm. Korean Math. Soc. **11** (1996), No. 2, pp. 503–514

ON THE $L_2(\Omega)$-ERROR FOR THE P-VERSION UNDER NUMERICAL QUADRATURE RULES

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Abstract. We consider non-constant coefficient elliptic equations of the type $-\mathrm{div}(a\nabla u) = f$,and employ the P-version of the finite element method as a numerical method for the approximate solutions. To compute the integrals in the variational form of the discrete problem we need the numerical quadrature rule scheme. In practice the integrations are seldom computed exactly. In this paper, we give an $L_2(\Omega)$-error estimate of $\|u - \widetilde{u}_p\|_{0,\Omega}$ in comparison with $\|u - \widetilde{u}_p\|_{1,\Omega}$,under numerical quadrature rules which are used for calculating the integrations in each of the stiffness matrix and the load vector.

1. Introduction

Let Ω be a closed line segment $I = [-1, 1]$ in R^1 or $I \times I$ in R^2 with boundary Γ , and Sobolev spaces
(1.1) $H^{m,p}(\Omega) \equiv$ The completion of $\{u \in C^m(\Omega) : \|u\|_{m,p,\Omega} < \infty\}$, equipped with norm

$$(1.2) \qquad \|u\|_{m,p,\Omega} = \left(\sum_{0 \le |i| \le m} \|\partial^i u\|^p_{0,p,\Omega} \right)^{1/p} \quad \text{if} \quad 1 \le p < \infty,$$

$$(1.3) \qquad \|u\|_{m,\infty,\Omega} = \max_{0 \le |i| \le m} \|\partial^i u\|_{0,\infty,\Omega},$$

where $\|\cdot\|_{0,p,\Omega}$ is the usual $L_p(\Omega)$-norm, and the subscript p may be dropped when $p = 2$.

We define a space $H^m_0(\Omega) = \{u \in H^m(\Omega) \; : \; u \text{ vanishes on } \Gamma\}$,

Received July 26, 1995. Revised December 28, 1995.
1991 AMS Subject Classification: 65G99.
Key words and phrases: numerical quadrature rules, non-constant coefficient elliptic equations, p-version, $L_2(\Omega)$ -error.

and consider the following problem: To find $u \in H_0^1(\Omega)$ such that

$$(1.4) \qquad -\frac{d}{dx}(a\frac{du}{dx}) = f \quad \text{in} \quad \Omega \subset R^1,$$

$$(1.5) \qquad -\operatorname{div}(a\nabla u) = f \quad \text{in} \quad \Omega \subset R^2.$$

Here, the above problem is supposed to be well-posed in the sense that the bilinear form $B(\cdot,\cdot)$ is elliptic and continuous over $(H_0^1(\Omega))^2$,

$$(1.6) \qquad B(u,v) = \int_\Omega a\nabla u \cdot \nabla v \, dx.$$

For sake of simplicity, we assume that

$$(1.7) \qquad 0 < A_1 \le a(x) \le A_2 \quad \text{for all} \quad x \in \Omega,$$

and

$$(1.8) \qquad f \in L_2(\Omega).$$

Using the p-version of the finite element method over a single element we have the following discrete variational problem of (1.4)-(1.5): To find $u_p \in S_{p,0}(\Omega)$ such that

$$(1.9) \qquad B(u_p, v_p) = (f, v_p) \quad \text{for all} \quad v_p \in S_{p,0}(\Omega),$$

where

$$(1.10) \qquad (f,v) = \int_\Omega fv \, dx,$$

$$(1.11) \qquad U_p(\Omega) = \{\mathrm{t} : \mathrm{t} \text{ is a polynomial of degree} \le p \text{ in each variable on } \Omega\},$$

and

$$(1.12) \qquad S_{p,0}(\Omega) = U_p(\Omega) \cap H_0^1(\Omega).$$

Here, we have a $L_2(\Omega)$-error estimate for the p-version:

$$(1.13) \qquad \|u - u_p\|_{0,\Omega} \le C\, p^{-k} \|u\|_{k,\Omega} \quad \text{for all} \quad u \in H_0^k(\Omega), k \ge 1.$$

It follows from the assumption of exact integrations in (1.9), and the following Lemma can be found in [2] and [6].

LEMMA 1.1. *For each integer* $l \geq 0$, *there exists a sequence of projections*

(1.14) $\qquad \Pi_p^l : H^l(\Omega) \to U_p(\Omega), \quad p = 1, 2, 3, \cdots, \quad$ *such that*

(1.15) $\qquad \Pi_p^l v_p = v_p \quad$ *for all* $\quad v_p \in U_p(\Omega)$,

(1.16) $\qquad \|u - \Pi_p^l u\|_{s,\Omega} \leq C\, p^{-(r-s)} \|u\|_{r,\Omega} \quad$ *for all* $\quad u \in H^r(\Omega)$
$\qquad\qquad 0 \leq s \leq l \leq r,$

where C *is a constant independent of* p *and* u *but depend upon* r *and* l.

In this paper, under a family $G_p = \{I_k\}$ of numerical quadrature rules with respect to $U_p(\Omega)$ we shall give an actual problem of (1.9), and derive an estimate of $\|u - \widetilde{u}_p\|_{0,\Omega}$ in comparison with $\|u - \widetilde{u}_p\|_{1,\Omega}$, where $\widetilde{u}_p$ is an approximation of u_p which satisfies (2.2).

2. Preliminaries

Let I_k be numerical quadrature rules defined on Ω by

$$I_k(f) = \sum_{i=1}^{n(k)} w_i^k f(x_i^k) \sim \int_\Omega f(x)\, dx. \tag{2.1}$$

We consider a family $G_p = \{I_k\}$ of quadrature rules with respect to $U_p(\Omega)$, $p = 1, 2, 3, \cdots$, satisfying the following properties:
For each $I_k \in G_p$,

(K1) $w_i^k > 0$ and $x_i^k \in \Omega$ for $i = 1, \cdots, n(k)$.

(K2) $I_k(f^2) \leq C \|f\|_{0,\Omega}^2$ for all $f \in U_p(\Omega)$.

(K3) $C \|f\|_{0,\Omega}^2 \leq I_k(f^2)$ for all $f \in \widetilde{U}_p(\Omega)$,
where $\widetilde{U}_p(\Omega) = \{ \dfrac{\partial f}{\partial x_i} : f \in U_p(\Omega) \} \subset U_p(\Omega)$.

(K4) $I_k(f) = \int_\Omega f(x)\, dx$ for all $f \in U_{d(k)}(\Omega)$,
where $d(k) \geq \widetilde{d}(p) > 0$.

In particular, let L_q be the q-point Gauss-Legendre (G-L) rule in R^1, or the cross-product of q-point G-L rules along the x and y axes in R^2 respectively. Then, $\{L_q\}_{q \geq l(p)}$ is a family of G-L rules with respect to

$U_p(\Omega)$, which satisfies the properties (K1)-(K4) with $d(q) = 2q-1$. Here, $l(p)$ denotes p in R^1, and $p+1$ in R^2 respectively.

Hence,using numerical quadrature rules I_m and I_l in G_p we obtain the following actual problem of (1.9): To find $\widetilde{u}_p \in S_{p,0}(\Omega)$ such that

$$(2.2) \qquad B_m(\widetilde{u}_p, v_p) = (f, v_p)_l \quad \text{for all} \quad v_p \in S_{p,0}(\Omega),$$

where

$$(2.3) \qquad (u,v)_q = I_q(uv),$$

$$(2.4) \qquad B_m(u,v) = \sum_{i=1}^{n} \left(a\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \right)_m \quad \text{on} \quad \Omega \subset R^n, n = 1 \text{ or } 2.$$

In addition, due to (1.7) and (K3) we easily see that $B_m(\cdot,\cdot)$ is elliptic. This allows us to solve the above approximate problem (2.2).

The following Lemma can be seen in [5].

LEMMA 2.1. *For* $\Omega \subset R^n, n = 1$ *or* 2, *let* $u \in H^\gamma(\Omega)$ *with* $\gamma \geq n$. *Then the projection* Π_p^n *from Lemma 1.1 satisfies*

$$(2.5) \qquad \|u - \Pi_p^n u\|_{0,\infty,\Omega} \leq C p^{-(\gamma - \frac{n}{2})} \|u\|_{\gamma,\Omega}.$$

3. Main results

To estimate the error $\|u - \widetilde{u}_p\|_{0,\Omega}$ we start with the following Lemma.

LEMMA 3.1. *Let* u *be the exact solution of (1.4)-(1.5) and* u_p *the p-version solution of (1.9). Then, for an approximate solution* $\widetilde{u}_p$ *of* u_p *which satisfies (2.2) we have*

$$(3.1) \qquad \begin{aligned} \|u - \widetilde{u}_p\|_{0,\Omega} &\leq \|u - u_p\|_{0,\Omega} \\ &+ \sup_{w_p \in S_{p,0}(\Omega)} \frac{1}{\|w_p\|_{0,\Omega}} (|B(\widetilde{u}_p, w) - B_m(\widetilde{u}_p, w)| \\ &+ |(f,w) - (f,w)_l|), \end{aligned}$$

where for each $w_p \in S_{p,0}(\Omega)$, $w \in S_{p,0}(\Omega)$ *denotes the solution of discrete variational problem:*

$$(3.2) \qquad B(w, v_p) = (w_p, v_p) \quad \textit{for all} \quad v_p \in S_{p,0}(\Omega).$$

PROOF. By the triangle inequality we have

$$\|u-\widetilde{u}_p\|_{0,\Omega} \le \|u-u_p\|_{0,\Omega} + \|u_p-\widetilde{u}_p\|_{0,\Omega}. \tag{3.3}$$

Since $u_p - \widetilde{u}_p \in S_{p,0}(\Omega)$ the last term of the right side in (3.3) can be characterized as

$$\|u_p-\widetilde{u}_p\|_{0,\Omega} = \sup_{w_p\in S_{p,0}(\Omega)} \frac{|(w_p, u_p-\widetilde{u}_p)|}{\|w_p\|_{0,\Omega}}. \tag{3.4}$$

Hence we obtain from (3.2) that

$$\begin{aligned} |(w_p, u_p-\widetilde{u}_p)| &= |B(w, u_p-\widetilde{u}_p)| \\ &\le |B(w,u_p) - B_m(w,\widetilde{u}_p)| + |B_m(w,\widetilde{u}_p) - B(w,\widetilde{u}_p)|. \end{aligned} \tag{3.5}$$

Due to the fact that $B(\cdot,\cdot)$ is symmetric and $w \in S_{p,0}(\Omega)$, it follows from (1.9) and (2.2) that

$$|(w, u_p-\widetilde{u}_p)| \le |B(\widetilde{u}_p, w) - B_m(\widetilde{u}_p, w)| + |(f,w)-(f,w)_l|. \tag{3.6}$$

This completes the proof.

The above Lemma indicates that the error $\|u-\widetilde{u}_p\|_{0,\Omega}$ will depend on several terms. The first term $\|u-u_p\|_{0,\Omega}$ in (3.1) was already discussed in (1.13), which depends on the smoothness of the exact solution $u(x)$. The other terms will depend upon the smoothness of $a(x)$ and $f(x)$. In this connection, first we give the following Lemma.

LEMMA 3.2. *Let* $u_p, w_p \in U_p(\Omega)$ *and* $f \in L_\infty(\Omega)$. *Then, for all* $v_q \in U_q(\Omega)$, $f_r \in U_r(\Omega)$ *with* $0 < q \le p$ *and* $r = d(m) - p - q > 0$ *we have*

$$\begin{aligned} &|(f\,u_p, w_p) - (f\,u_p, w_p)_m| \\ &\le C\{\|f_r\|_{0,\infty,\Omega}\|u_p - v_q\|_{0,\Omega} + \|f-f_r\|_{0,\infty,\Omega}\|u_p\|_{0,\Omega}\}\|w_p\|_{0,\Omega} \end{aligned} \tag{3.7}$$

where C *is independent of* p, q *and* m.

PROOF. It is similar to that in [5, Lemma 3.3].

Now, for each $t \in U_p(\Omega)$ we denote

$$\Xi_q(t) = \max_i \left\| \left(\frac{\partial t}{\partial x_i} + t\right) - \Pi_q^1\left(\frac{\partial t}{\partial x_i} + t\right)\right\|_{0,\Omega}, \quad 0 < q \le p. \tag{3.8}$$

Then, we obtain

(3.9) $\Xi_q(t) \le C\, q^{-(\lambda-1)} \|t\|_{\lambda,\Omega}$ for all $t \in U_p(\Omega)$,

where λ is a sufficiently large number. Moreover, it follows from (1.15) that

(3.10) $\Xi_p(t) = 0$ for all $t \in U_p(\Omega)$.

Here,we have the following proposition.

PROPOSITION 3.3. *Let $u \in H_0^\sigma(\Omega)$ and $a \in H^\rho(\Omega)$ with $\rho \ge n$. Then, for any $w \in S_{p,0}(\Omega)$ we have*

$$(3.11)\quad \begin{aligned} &|B(\widetilde{u}_p, w) - B_m(\widetilde{u}_p, w)| \\ &\le C\,\{\Xi_q(\widetilde{u}_p) + q^{-\sigma}\|u\|_{\sigma,\Omega} + q^{-1}\|u-\widetilde{u}_p\|_{1,\Omega} \\ &\quad + r^{-(\rho-\frac{n}{2})}(\|u-\widetilde{u}_p\|_{1,\Omega} + \|u\|_{1,\Omega})\|a\|_{\rho,\Omega}\}\,\|w\|_{1,\Omega}, \end{aligned}$$

where q is a positive integer such that $0 < q \le p$ and $r = d(m)-p-q > 0$.

PROOF. For $w \in S_{p,0}(\Omega)$ we have

$$(3.12)\quad \begin{aligned} &|\,B(\widetilde{u}_p, w) - B_m(\widetilde{u}_p, w)\,| \\ &\le C\,\{\max_i |\left(a\,\frac{\partial \widetilde{u}_p}{\partial x_i}, \frac{\partial w}{\partial x_i}\right) - \left(a\,\frac{\partial \widetilde{u}_p}{\partial x_i}, \frac{\partial w}{\partial x_i}\right)_m |\}. \end{aligned}$$

Let q be any integer such that $0 < q \le p$ and $r = d(m)-p-q > 0$. Then, for any $i = 1, \cdots, n$, due to Lemma 3.2 with $f_r = \Pi_r^n a$ and $v_q \in U_q(\Omega)$, we have

$$(3.13)\quad \begin{aligned} &|\left(a\,\frac{\partial \widetilde{u}_p}{\partial x_i}, \frac{\partial w}{\partial x_i}\right) - \left(a\,\frac{\partial \widetilde{u}_p}{\partial x_i}, \frac{\partial w}{\partial x_i}\right)_m | \\ &\le C\{\,\|\Pi_r^n a\|_{0,\infty,\Omega}\,\|\frac{\partial \widetilde{u}_p}{\partial x_i} - v_q\|_{0,\Omega} \\ &\quad + \|a - \Pi_r^n\, a\|_{0,\infty,\Omega}\|\frac{\partial \widetilde{u}_p}{\partial x_i}\|_{0,\Omega}\,\}\,\|\frac{\partial w_p}{\partial x_i}\|_{0,\Omega}. \end{aligned}$$

Since $\|\Pi_r^n\, a\|_{0,\infty,\Omega} \le \|a - \Pi_r^n\, a\|_{0,\infty,\Omega} + \|a\|_{0,\infty,\Omega}$ we easily see from Lemma 2.1 and (1.7) that $\|\Pi_r^n\, a\|_{0,\infty,\Omega}$ is bounded by a fixed constant for any $r = d(m) - p - q > 0$. Moreover, taking $v_q = \Pi_q^1(\frac{\partial \widetilde{u}_p}{\partial x_i} + \widetilde{u}_p) + \Pi_q^1(u - \widetilde{u}_p) - \Pi_q^1 u$ in $U_q(\Omega)$ we have from Lemma 1.1 that

$$(3.14)\quad \|\Pi_r^n a\|_{0,\infty,\Omega} \|\frac{\partial \widetilde{u}_p}{\partial x_i} - v_q\|_{0,\Omega}$$
$$\leq C \left\{ \|(\frac{\partial \widetilde{u}_p}{\partial x_i} + \widetilde{u}_p) - \Pi_q^1(\frac{\partial \widetilde{u}_p}{\partial x_i} + \widetilde{u}_p)\|_{0,\Omega} \right.$$
$$\left. + \|(u - \widetilde{u}_p) - \Pi_q^1(u - \widetilde{u}_p)\|_{0,\Omega} + \|u - \Pi_q^1 u\|_{0,\Omega} \right\}$$
$$\leq C \left\{ \Xi_q(\widetilde{u}_p) + q^{-1}\|u - \widetilde{u}_p\|_{1,\Omega} + q^{-\sigma}\|u\|_{\sigma,\Omega} \right\},$$

where C is independent of p and q.

In addition, we obtain from Lemma 2.1 that

$$(3.15)\quad \|a - \Pi_r^n a\|_{0,\infty,\Omega} \|\frac{\partial \widetilde{u}_p}{\partial x_i}\|_{0,\Omega}$$
$$\leq C r^{-(\rho - \frac{n}{2})} \|a\|_{\rho,\Omega} \|\widetilde{u}_p\|_{1,\Omega}$$
$$\leq C r^{-(\rho - \frac{n}{2})} \|a\|_{\rho,\Omega} (\|u - \widetilde{u}_p\|_{1,\Omega} + \|u\|_{1,\Omega}).$$

Thus, substituting (3.14) and (3.15) in (3.13) we complete the proof, since $\|\frac{\partial w}{\partial x_i}\|_{0,\Omega} \leq C \|w\|_{1,\Omega}$.

To estimate the third factor that $\|u - \widetilde{u}_p\|_{0,\Omega}$ depends upon we will use the following proposition.

PROPOSITION 3.4. *Let $I_l \in G_p$ be a quadrature rule on $\Omega \subset R^n$, which satisfies $d(l) - p - 1 > 0$. Let $f \in H^\mu(\Omega)$ with $\mu \geq n$. Then, for any $w \in S_{p,0}(\Omega)$ we have*

$$(3.16)\quad |(f, w) - (f, w)_l| \leq C r^{-(\mu - \frac{n}{2})} \|f\|_{\mu,\Omega} \|w\|_{1,\Omega},$$

where $r = d(l) - p$.

PROOF. Since $(\Pi_r^n f, w) = (\Pi_r^n f, w)_l$ for $w \in S_{p,0}(\Omega)$ and $r = d(l) - p$, we have

$$(3.17)\quad |(f, w) - (f, w)_l| \leq |(f - \Pi_r^n f, w)| + |(f - \Pi_r^n f, w)_l|.$$

Using the Schwarz inequality and Lemma 1.1 we have

$$(3.18)\quad |(f - \Pi_r^n f, w)| \leq C \|f - \Pi_r^n f\|_{0,\Omega} \|w\|_{0,\Omega}$$
$$\leq C r^{-\mu} \|f\|_{\mu,\Omega} \|w\|_{0,\Omega}.$$

Also, it follows from Lemma 2.1 and (K2) that

$$(3.19)\qquad \begin{aligned}|(f-\Pi_r^n f, w)_l| &\le C\,\|f-\Pi_r^n f\|_{0,\infty,\Omega}(w,w)_l^{\frac{1}{2}}\\ &\le C\,r^{-(\mu-\frac{n}{2})}\|f\|_{\mu,\Omega}\|w\|_{0,\Omega}.\end{aligned}$$

Since (3.18) is dominated by (3.19) it completes the proof from $\|w\|_{0,\Omega} \le C\,\|w\|_{1,\Omega}$.

Now, we shall state the main theorem.

THEOREM 3.5. *For any I_m, $I_l \in G_p$, let $u \in H_0^\sigma(\Omega)$ be the exact solution of (1.1)-(1.2) and $\widetilde{u}_p \in S_{p,0}(\Omega)$ an approximate solution of u_p which satisfies (2.2). We assume that $a \in H^\rho(\Omega)$ and $f \in H^\mu(\Omega)$ with $min(\rho,\mu) \ge n$. Then, for any integer q such that $0 < q \le p$ and $r = d(m) - p - q > 0$ we have*

$$(3.20)\qquad \begin{aligned}\|u-\widetilde{u}_p\|_{0,\Omega} &\le C\,\{q^{-\sigma}\|u\|_{\sigma,\Omega}\\ &+(q^{-1}+r^{-(\rho-\frac{n}{2})}\|a\|_{\rho,\Omega})\|u-\widetilde{u}_p\|_{1,\Omega}\\ &+r^{-(\rho-\frac{n}{2})}\|a\|_{\rho,\Omega}\|u\|_{1,\Omega}\\ &+(d(l)-p)^{-(\mu-\frac{n}{2})}\|f\|_{\mu,\Omega}+\Xi_q(\widetilde{u}_p)\},\end{aligned}$$

where C is independent of p and q.

PROOF. For each $w_p \in S_{p,0}(\Omega)$ let $w \in S_{p,0}(\Omega)$ be the solution of (3.2). Then, since $w \in S_{p,0}(\Omega)$ we have $B(w,w) = |(w_p,w)| \le \|w_p\|_{0,\Omega}\|w\|_{0,\Omega}$. In addition, due to Poincaré's inequality and (1.7), we easily see that there exists a fixed constant M such that

$$(3.21)\qquad \frac{\|w\|_{1,\Omega}}{\|w_p\|_{0,\Omega}} \le M.$$

Thus, by a direct application of proposition 3.3 and 3.4 to Lemma 3.1

we have

$$(3.22)\qquad \sup_{w_p\in S_{p,0}(\Omega)} \frac{1}{\|w_p\|_{0,\Omega}}(|B(\widetilde{u}_p,w)-B_m(\widetilde{u}_p,w)| + |(f,w)-(f,w)_l|) \le C\{q^{-\sigma}\|u\|_{\sigma,\Omega} + (q^{-1}+r^{-(\rho-\frac{n}{2})}\|a\|_{\rho,\Omega})\|u-\widetilde{u}_p\|_{1,\Omega} + r^{-(\rho-\frac{n}{2})}\|a\|_{\rho,\Omega}\|u\|_{1,\Omega} + (d(l)-p)^{-(\mu-\frac{n}{2})}\|f\|_{\mu,\Omega} + \Xi_q(\widetilde{u}_p)\}.$$

Moreover, it follows from (1.13) that the first term of the right side in (3.1) is dominated by the first term in (3.22). This completes the proof.

In [5] we easily obtain the following estimate

$$(3.23)\qquad \|u-\widetilde{u}_p\|_{1,\Omega} \le C\{q^{-(\sigma-1)}\|u\|_{\sigma,\Omega} + r^{-(\rho-\frac{n}{2})}\|a\|_{\rho,\Omega}\|u\|_{1,\Omega} + (d(l)-p)^{-(\mu-\frac{n}{2})}\|f\|_{\mu,\Omega}\}.$$

Hence, when $d(m)$ and $d(l)$ are large enough with $q=p$, the rate of convergence for $\|u-\widetilde{u}_p\|_{1,\Omega}$ is asymptotically $O(p^{-(\sigma-1)})$, which coincides with that of $\|u-u_p\|_{1,\Omega}$. Also, it follows from (3.10) that the $L_2(\Omega)$ error $\|u-\widetilde{u}_p\|_{0,\Omega}$ in (3.20) is asymptotically $O(p^{-\sigma})$ under nearly exact integrations, which is the same with that of $\|u-u_p\|_{0,\Omega}$ in (1.13). Moreover, we see that under certain conditions the $L_2(\Omega)$ error $\|u-\widetilde{u}_p\|_{0,\Omega}$ has nearly $O(p^{-1})$ improvement over the H^1 error $\|u-\widetilde{u}_p\|_{1,\Omega}$.

Here, we see the following facts.

(1) In the case where a and f are sufficiently smooth, i.e., ρ and μ are large enough, even when $d(m)\approx 2p+1$ with $q=p$ and $d(l)\approx p+1$ the first term of the right side in (3.20) may dominate the other terms, so that the rate of convergence for $\|u-\widetilde{u}_p\|_{0,\Omega}$ is asymptotically $O(p^{-\sigma})$.

(2) When a or f is not smooth enough we may reduce the error $\|u-\widetilde{u}_p\|_{0,\Omega}$ by increasing the value of $d(m)$ or $d(l)$ respectively. In fact,

using overintegrations $I_m(m > p)$ or $I_l(l > p)$ we recover the optimal rate of convergence for $\|u - \widetilde{u}_p\|_{0,\Omega}$.

(3) We define the ratio

$$R(p,m,l) = \frac{\|u - \widetilde{u}_p\|_{0,\Omega}}{\|u - \widetilde{u}_p\|_{1,\Omega}} \quad \text{for} \quad m > p \quad \text{and} \quad l > p. \tag{3.24}$$

Then, for two sufficiently large m and l the ratio $R(p,m,l)$ is nearly $O(p^{-1})$. Moreover, in the case where $a(x)$ or $f(x)$ is not smooth enough overintegrations may be reguired to further reduce the ratio $R(p,m,l)$ until the term p^{-1} dominates again. But, when ρ and μ are large enough rather than σ we have no need of overintegrations, i.e., taking $d(m) = d(l) \approx 2p+1$ with $q \approx p$ we obtain the optimal $O(p^{-1})$.

Now, to confirm the main results we shall consider the following one-dimensional problem:

$$-\frac{d}{dx}(a\frac{du}{dx}) = f \quad \text{on} \quad I = [-1,1]$$

with $u(-1) = u(1) = 0$. Let $a(x) = \dfrac{1}{(x+1+w)}, w > 0$. We choose $f(x)$ in such a way that the exact solution is $u(x) = (x+1)^{\frac{3}{2}} - 2^{\frac{1}{2}}(x+1)$. Especially, taking $w = 0.01$ we see that the exact solution $u(x)$ is not smooth enough in comparison with that of $a(x)$. Hence, we expect that $R(p,m,l = 1000)$ is nearly $O(p^{-1})$, even when $m \approx p$. Figure 3 shows this phenomenon in the case where overintegrations $L_m(m \geq p)$ are used and $L_l(l = 1000)$. Moreover, since $a(x)$ has a pole at $x = -1.01$ which is very close to $x = -1$ the overintegrations $I_m(m > p)$ may be required to recover the optimal results, $O(p^{-\sigma})$ for $\|u - \widetilde{u}_p\|_{0,\Omega}$ and $O(p^{-(\sigma-1)})$ for $\|u - \widetilde{u}_p\|_{1,\Omega}$. When overintegrations $L_m(m > p)$ are used for $\|u - \widetilde{u}_p\|_{0,\Omega}$ and $\|u - \widetilde{u}_p\|_{1,\Omega}$, Figure 1 and Figure 2 represent those results respectively.

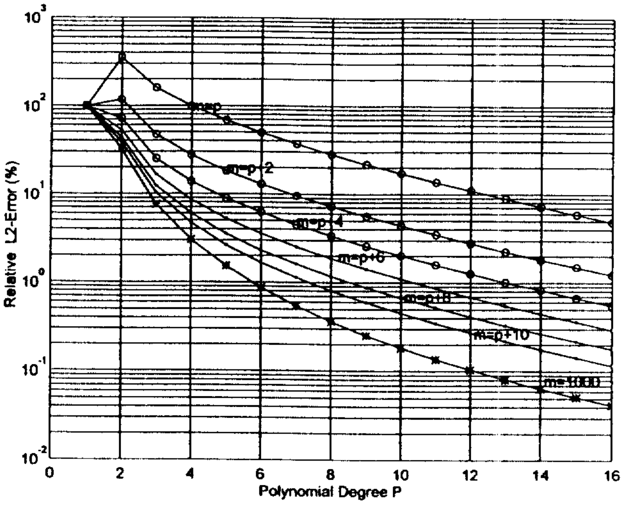

Figure 1

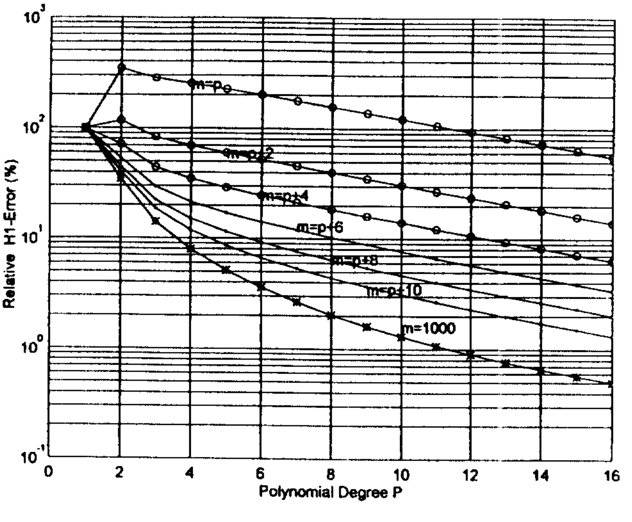

Figure 2

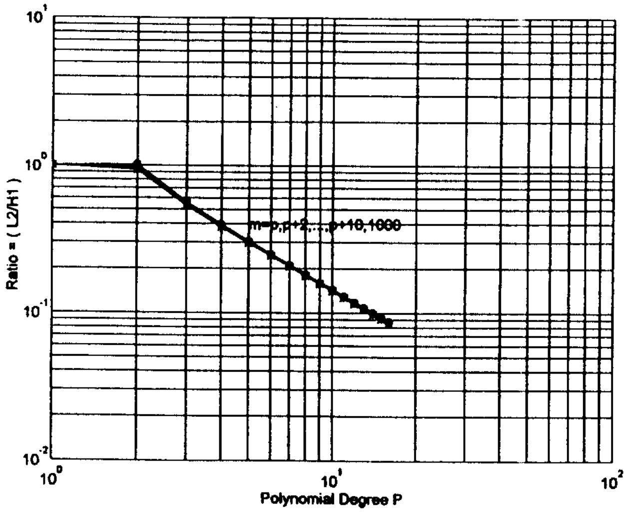

Figure 3

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