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MMPP,M/G/1 RETRIAL QUEUE WITH TWO CLASSES OF CUSTOMERS

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ABSTRACT. We consider a retrial queue with two classes of customers where arrivals of class 1 (resp. class 2) customers are MMPP and Poisson process, respectively. In the case that arriving customers are blocked due to the channel being busy, the class 1 customers are queued in priority group and are served as soon as the channel is free, whereas the class 2 customers enter the retrial group in order to try service again after a random amount of time. We consider the following retrial rate control policy, which reduces their retrial rate as more customers join the retrial group; their retrial times are inversely proportional to the number of customers in the retrial group. We find the joint generating function of the numbers of customers in the two groups by the supplementary variable method.

1. Introduction

Retrial queueing systems are characterized by the feature that arrivals who find the channel busy join the retrial group to try again for their requests in the random order and at random intervals. Retrial queues have been widely used to model many problems in telephone switching systems, computer and communication systems. For comprehensive surveys of retrial queues, see Yang and Templeton[9] and Falin[5].

In this paper, we consider a retrial queue with two classes of customers where arrivals of class 1 (resp. class 2) customers are MMPP(Markov Modulated Poisson Process) and Poisson process, respectively. In the case that arriving customers are blocked due to the channel being busy, the class 1 customers are queued in priority group and are served as soon as the channel is free, whereas the class 2 customers enter the

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retrial group in order to try service again after a random amount of time. We assume the following retrial rate control policy, which reduces their retrial rate as more customers join the retrial group; their retrial times are inversely proportional to the number of customers in the retrial group. M/G/1 retrial queue with one class of customers and MMPP, M/G/1 queue without retrial are special case of our model. Choi et al.[3] calculated the Laplace-Stieltjes transform of the virtual waiting time distribution and the generating function for the number of retrials for ordinary M/G/1 retrial queue with one class of customers. Choi and Han[1] obtained the joint generating function of the number of customers in the two groups for ordinary MMPP, M/G/1 queue.

The main purpose of this paper is to find the joint generating function of the numbers of customers in the two groups by the supplementary variable method.

This paper is organized as follows. In section 2, we describe the mathematical queueing model in detail. In section 3, we derive the joint generating function of the numbers of customers in the two groups by the supplementary variable method. In section 4, special case is treated. As the rate of exponential retrial time tends to infinite, queue length distribution for queueing system with retrial will approach to that for queueing system without retrial. We show that the above facts hold for MMPP, M/G/1 queue without retrial (Choi and Han[1]).

2. The model

We consider a single channel retrial queueing system with two classes of customers, called class 1 (resp. class 2) customers. The arrival process of class 1 customers is 2-state MMPP as follows. Let $\{J(t),\, t \geq 0\}$ be an underlying Markov chain on the state space $E = \{1, 2\}$ with generator $\begin{bmatrix} -\gamma_1 & \gamma_1 \\ \gamma_2 & -\gamma_2 \end{bmatrix}$. If $J(t) = i$, then class 1 customers arrive according to a Poisson process with intensity α_i $(i = 1, 2)$. Class 2 customers arrive according to a Poisson process with intensity β.

If a class 2 customer upon arrival finds the channel free, he immediately occupies the channel and leaves the system after service. If he finds the channel busy on his arrival, he enters the retrial group in order to seek service again after a random amount of time. He persists this way until he succeeds the connection. The retrial time is exponentially distributed with rate $\frac{\nu}{n}$, where n is the number of customers in the retrial group and is independent of all previous retrial times and all other stochastic processes in the system. The class 1 customers are queued in a priority group after blocked. As soon as the channel is free, a class 1 customer occupies the channel immediately, so class 2 customers in the retrial group will be served only when there are no class 1 customers in the priority group. According to the above rule class 1 customers in the priority group have non-preemptive priority over class 2 customers.

The service times of both classes of customers are independent and identically distributed with *p.d.f.* $b(x)$ and mean b. Let

$$\tilde{b}(\theta) = \int_0^\infty e^{-\theta x} b(x)\, dx$$

be the Laplace transform of the *p.d.f.* of the service time. It is easy to show that the system is stable provided that $\rho = \frac{\gamma_2\lambda_1+\gamma_1\lambda_2}{\gamma_1+\gamma_2} \cdot b < 1$, where $\lambda_i = \alpha_i + \beta$. We consider only stable systems in this paper.

3. Joint probabilities of queue lengths

We will investigate the joint distribution of the queue lengths of both classes of customers at departure points and at arbitrary time points simultaneously by the supplementary variable method. Here we take supplementary variable as the remaining service time.

At an arbitrary time, the steady state of the system can be characterized by the following random variables;

J = the phase of underlying Markov chain,

N_1 = the number of class 1 customers in priority group (excluding the customer in service),

N_2 = the number of class 2 customers in retrial group,

$\tilde{S}$ = the remaining service time of the customer in service,

$$\xi = \begin{cases} 0, & \text{when channel is idle,} \\ 1, & \text{when channel is busy.} \end{cases}$$

Define the probabilities;

$$q_j^i = P[J = i,\, N_2 = j,\, \xi = 0],$$
$$p_{jk}^i(x)dx = P[J = i,\, N_1 = j,\, N_2 = k,\, \tilde{S} \in (x, x + dx], \xi = 1],$$

and their Laplace transforms

$$\tilde{p}_{jk}^i(\theta) = \int_0^\infty e^{-\theta x} p_{jk}^i(x)\, dx.$$

Note that $\tilde{p}_{jk}^i(0) = P[J = i,\, N_1 = j,\, N_2 = k,\, \xi = 1]$ is the steady state probability that $J = i$ and there are j customers in the priority group, k customers in the retrial group and the channel is busy.

For simplicity of expression, we denote $i' = 3 - i$ throughout this paper. Using a typical argument of the supplementary variable method, we have the following system of differential difference equations; for $i = 1, 2$,

(3.1a)

$$-\frac{dp_{0k}^i(x)}{dx} = -(\lambda_i + \gamma_i)p_{0k}^i(x) + \beta p_{0,k-1}^i(x) + \gamma_{i'} p_{0k}^{i'}(x) + b(x)[p_{1k}^i(0) + \nu q_{k+1}^i + \lambda_i q_k^i],\ k \geq 0,$$

(3.1b)

$$-\frac{dp_{jk}^i(x)}{dx} = -(\lambda_i + \gamma_i)p_{jk}^i(x) + \alpha_i p_{j-1,k}^i(x) + \beta p_{j,k-1}^i(x) + \gamma_{i'} p_{jk}^{i'}(x) + b(x)p_{j+1,k}^i(0),\ j \geq 1,\ k \geq 0,$$

(3.2)

$$(\lambda_i + \gamma_i + \nu)q_k^i = \gamma_{i'} q_k^{i'} + p_{0k}^i(0),\ k \geq 0,$$

where $p_{j,-1}^i(x) = 0$. By taking Laplace transforms of (3.1), it follows that for $i = 1, 2$,

(3.3a)
$$(\theta - \lambda_i - \gamma_i)\tilde{p}^i_{0k}(\theta) + \beta\tilde{p}^i_{0,k-1}(\theta) + \gamma_{i'}\tilde{p}^{i'}_{0k}(\theta) = p^i_{0k}(0) - \tilde{b}(\theta)[p^i_{1k}(0) + \nu q^i_{k+1} + \lambda_i q^i_k],\ k \geq 0,$$

(3.3b)
$$(\theta - \lambda_i - \gamma_i)\tilde{p}^i_{jk}(\theta) + \alpha_i\tilde{p}^i_{j-1,k}(\theta) + \beta\tilde{p}^i_{j,k-1}(\theta) + \gamma_{i'}\tilde{p}^{i'}_{jk}(\theta) = p^i_{jk}(0) - \tilde{b}(\theta)p^i_{j+1,k}(0),\ j \geq 1,\ k \geq 0.$$

For complex z_2 with $|z_2| \leq 1$, define the generating function;

$$\tilde{P}^i_j(\theta, z_2) = \sum_{k=0}^{\infty} \tilde{p}^i_{jk}(\theta)\, z_2^k,\ i = 1,2,\ j \geq 0,$$

$$P^i_j(z_2) = \sum_{k=0}^{\infty} p^i_{jk}(0)\, z_2^k,\ i = 1,2,\ j \geq 0,$$

$$Q_i(z_2) = \sum_{k=0}^{\infty} q^i_k\, z_2^k,\ i = 1,2.$$

From equations (3.2), (3.3a) and (3.3b) we obtain, for $i = 1,2$,

(3.4a)
$$(\theta - \lambda_i + \beta z_2)\tilde{P}^i_0(\theta, z_2) + \gamma_{i'}\tilde{P}^{i'}_0(\theta, z_2) = P^i_0(z_2) - \tilde{b}(\theta)\left[P^i_1(z_2) + \frac{\nu}{z_2}(Q_i(z_2) - q^i_0) + \lambda_i Q_i(z_2)\right],$$

(3.4b)
$$(\theta - \lambda_i + \beta z_2)\tilde{P}^i_j(\theta, z_2) + \alpha_i P^i_{j-1}(\theta, z_2) + \gamma_{i'}\tilde{P}^{i'}_j(\theta, z_2) = P^i_j(z_2) - \tilde{b}(\theta)P^i_{j+1}(z_2),\ i = 1,2,\ j \geq 1,$$

(3.5)
$$(\lambda_i + \gamma_i)Q_i(z_2) + \nu(Q_i(z_2) - q^i_0) = \gamma_{i'}Q_{i'}(z_2) + P^i_0(z_2).$$

Next we introduce the generating function of $\tilde{P}^i_j(\theta, z_2)$ and $P^i_j(z_2)$;

$$\tilde{P}_i(\theta, z_1, z_2) = \sum_{j=0}^{\infty} \tilde{P}_j^i(\theta, z_2) z_1^j, \ i = 1, 2,$$

$$P_i(z_1, z_2) = \sum_{j=0}^{\infty} P_j^i(z_2) z_1^j, \ i = 1, 2.$$

Note that $\tilde{P}_i(0, z_1, z_2) = E[z_1^{N_1} z_2^{N_2}; J = i, \xi = 1]$ is the joint generating function of (N_1, N_2) when the channel is busy and the phase of underlying Markov chain is i. From equation (3.4), we obtain, for $i = 1, 2$,

(3.6)

$$(\theta - \alpha_i(1 - z_1) - \beta(1 - z_2) - \gamma_i)\tilde{P}_i(\theta, z_1, z_2) + \gamma_{i'} \tilde{P}_{i'}(\theta, z_1, z_2)$$
$$= \left(1 - \frac{\tilde{b}(\theta)}{z_1}\right) P_i(z_1, z_2) + \tilde{b}(\theta) \left(\frac{P_i(0, z_2)}{z_1} - (\lambda_i + \frac{\nu}{z_2})Q_i(z_2) + \frac{\nu q_0^i}{z_2}\right).$$

Let

$$\begin{aligned} A(\theta, z_1, z_2) = &(\theta - \alpha_1(1 - z_1) - \beta(1 - z_2) - \gamma_1) \\ &\cdot (\theta - \alpha_2(1 - z_1) - \beta(1 - z_2) - \gamma_2) - \gamma_1\gamma_2. \end{aligned} \tag{3.7}$$

By solving simultaneous equation (3.6), we obtain our main result for $\tilde{P}_i(\theta, z_1, z_2)$.

THEOREM 3.1. *The solutions $\tilde{P}_i(\theta, z_1, z_2)$ of equations (3.6) are given by*

(3.8)

$$\begin{bmatrix} \tilde{P}_1(\theta, z_1, z_2) \\ \tilde{P}_2(\theta, z_1, z_2) \end{bmatrix} = \begin{bmatrix} a_{11}(\theta, z_1, z_2) & a_{12}(\theta, z_1, z_2) \\ a_{21}(\theta, z_1, z_2) & a_{22}(\theta, z_1, z_2) \end{bmatrix}$$
$$\cdot \begin{bmatrix} P_1(z_1, z_2) & P_1(0, z_2) & (\lambda_1 + \frac{\nu}{z_2})Q_1(z_2) - \frac{\nu q_0^1}{z_2} \\ P_2(z_1, z_2) & P_2(0, z_2) & (\lambda_2 + \frac{\nu}{z_2})Q_2(z_2) - \frac{\nu q_0^2}{z_2} \end{bmatrix}$$
$$\cdot \begin{bmatrix} 1 - \frac{\tilde{b}(\theta)}{z_1} \\ \frac{\tilde{b}(\theta)}{z_1} \\ -\tilde{b}(\theta) \end{bmatrix} \cdot \frac{1}{A(\theta, z_1, z_2)},$$

where

$$\begin{aligned} a_{11}(\theta, z_1, z_2) &= \theta - \alpha_2(1 - z_1) - \beta(1 - z_2) - \gamma_2, \\ a_{12}(\theta, z_1, z_2) &= -\gamma_2, \\ a_{21}(\theta, z_1, z_2) &= -\gamma_1, \\ a_{22}(\theta, z_1, z_2) &= \theta - \alpha_1(1 - z_1) - \beta(1 - z_2) - \gamma_1. \end{aligned}$$

In the remainder of this paper, we will express the functions $P_i(z_1, z_2)$, $P_i(0, z_2)$ and $Q_i(z_2)$ explicitly in terms of known parameters ($i = 1, 2$). To do this, we need the following lemma;

LEMMA 3.2. *Let $A(\theta, z_1, z_2)$ be the function defined as (3.7). Then for a fixed $|z_1| \leq 1$ and $|z_2| \leq 1$ except that $z_1 = z_2 = 1$, the following equation in θ*

$$A(\theta, z_1, z_2) = 0 \tag{3.9}$$

has exactly two solutions in the region $\{\theta \,|\, Re\,\theta > 0\}$.

PROOF. Define

$$\begin{aligned} f(\theta, z_1, z_2) &= (\theta - \alpha_1(1 - z_1) - \beta(1 - z_2) - \gamma_1) \\ &\quad \cdot (\theta - \alpha_2(1 - z_1) - \beta(1 - z_2) - \gamma_2) \\ g(\theta, z_1, z_2) &= \gamma_1 \gamma_2 \end{aligned}$$

and let C_ϵ be the contour which consists of the imaginary axis from $-\epsilon i$ to ϵi and the semicircle of radius ϵ in the right half plane. Then for $\epsilon > \max\{\alpha_1, \alpha_2\} + \beta + 2\max\{\gamma_1, \gamma_2\}$, we can easily deduce that

$$|f(\theta, z_1, z_2)| > |g(\theta, z_1, z_2)|,$$

on the contour C_ϵ. Therefore by Rouche's theorem, $A(\theta, z_1, z_2) = 0$ has exactly two solutions in the region $\{\theta \,|\, \mathrm{Re}\,\theta > 0\}$. □

REMARK. When $z_1 = z_2 = 1$, the solutions of $A(\theta, z_1, z_2) = 0$ are $\theta = 0$ and $\theta = \gamma_1 + \gamma_2$.

Let $\theta_i(z_1, z_2)$, $i = 1, 2$, denote the two solutions of (3.9). Here we assume that $\theta_1(z_1, z_2) \neq \theta_2(z_1, z_2)$. Since $\tilde{P}_1(\theta, z_1, z_2)$ is analytic in the region $\mathrm{Re}\,\theta > 0$, $|z_i| < 1$, whenever the denominator of $\tilde{P}_1(\theta, z_1, z_2)$

vanishes, the numerator must also vanish. Since the denominator of $\tilde{P}_1(\theta, z_1, z_2)$ vanishes at $\theta_i(z_1, z_2)$, $i = 1, 2$, we have from equation (3.8) the following two equations;

$$\begin{aligned} a_i(z_1,z_2)\Big\{&(1-\frac{\tilde{b}(\theta_i(z_1,z_2))}{z_1})P_1(z_1,z_2)+\frac{\tilde{b}(\theta_i(z_1,z_2))}{z_1}P_1(0,z_2)\\ &-\tilde{b}(\theta_i(z_1,z_2))\big((\lambda_1+\frac{\nu}{z_2})Q_1(z_2)-\frac{\nu q_0^1}{z_2}\big)\Big\}\\ =\gamma_2\Big\{&(1-\frac{\tilde{b}(\theta_i(z_1,z_2))}{z_1})P_2(z_1,z_2)+\frac{\tilde{b}(\theta_i(z_1,z_2))}{z_1}P_2(0,z_2)\\ &-\tilde{b}(\theta_i(z_1,z_2))\big((\lambda_2+\frac{\nu}{z_2})Q_2(z_2)-\frac{\nu q_0^2}{z_2}\big)\Big\},\\ i=1,2,& \end{aligned} \tag{3.10}$$

where $a_i(z_1, z_2) = \theta_i(z_1, z_2) - \alpha_2(1 - z_1) - \beta(1 - z_2) - \gamma_2$.

From equation (3.10), we have the functions $P_i(z_1, z_2)$, $i = 1, 2$, in terms of $P_i(0, z_2)$ and $Q_i(z_2)$.

LEMMA 3.3. *The solutions $P_i(z_1, z_2)$ of equations (3.10) are given by*

$$\begin{aligned} \begin{bmatrix} P_1(z_1,z_2)\\ P_2(z_1,z_2)\end{bmatrix} &= \begin{bmatrix} b_{11}(z_1,z_2) & b_{12}(z_1,z_2)\\ b_{21}(z_1,z_2) & b_{22}(z_1,z_2)\end{bmatrix}\\ &\cdot\begin{bmatrix} P_1(0,z_2) & (\lambda_1+\frac{\nu}{z_2})Q_1(z_2)-\frac{\nu q_0^1}{z_2}\\ P_2(0,z_2) & (\lambda_2+\frac{\nu}{z_2})Q_2(z_2)-\frac{\nu q_0^2}{z_2}\end{bmatrix}\\ &\cdot\begin{bmatrix}1\\ -z_1\end{bmatrix}\cdot\frac{1}{B(z_1,z_2)}, \end{aligned} \tag{3.11}$$

where

$B(z_1,z_2)=\gamma_2(z_1-\tilde{b}(\theta_1(z_1,z_2)))(z_1-\tilde{b}(\theta_2(z_1,z_2)))(\theta_1(z_1,z_2)-\theta_2(z_1,z_2))$

$b_{11}(z_1,z_2)=\gamma_2\{a_2(z_1,z_2)[z_1-\tilde{b}(\theta_1(z_1,z_2))]\tilde{b}(\theta_2(z_1,z_2))$
$\quad -a_1(z_1,z_2)[z_1-\tilde{b}(\theta_2(z_1,z_2))]\tilde{b}(\theta_1(z_1,z_2))\}$,

$b_{12}(z_1,z_2)=\gamma_2^2\{[z_1-\tilde{b}(\theta_2(z_1,z_2))]\tilde{b}(\theta_1(z_1,z_2))$
$\quad -[z_1-\tilde{b}(\theta_1(z_1,z_2))]\tilde{b}(\theta_2(z_1,z_2))\}$,

$b_{21}(z_1,z_2)=a_1(z_1,z_2)a_2(z_1,z_2)\{[z_1-\tilde{b}(\theta_1(z_1,z_2))]\tilde{b}(\theta_2(z_1,z_2))$

$$-[z_1 - \tilde{b}(\theta_2(z_1,z_2))]\tilde{b}(\theta_1(z_1,z_2))\},$$

$$b_{22}(z_1,z_2) = \gamma_2\{a_2(z_1,z_2)[z_1 - \tilde{b}(\theta_2(z_1,z_2))]\tilde{b}(\theta_1(z_1,z_2)) - a_1(z_1,z_2)[z_1 - \tilde{b}(\theta_1(z_1,z_2))]\tilde{b}(\theta_2(z_1,z_2))\}.$$

Now we express the boundary functions $P_i(0,z_2)$, $i=1,2$, in terms of $Q_i(z_2)$. To determine the functions $P_i(0,z_2)$, we shall need the following lemma;

LEMMA 3.4. *For a given $|z_2| < 1$, the following equation in z_1*

$$(z_1 - \tilde{b}(\theta_1(z_1,z_2)))(z_1 - \tilde{b}(\theta_2(z_1,z_2))) = 0 \tag{3.12}$$

has exactly two solutions in the unit circle.

PROOF. From Lemma 3.2, we have, for $|z_1| = 1$

$$\operatorname{Re}\theta_i(z_1,z_2) > 0, \quad i = 1,2.$$

Therefore by using Rouche's theorem,

$$(z_1 - \tilde{b}(\theta_1(z_1,z_2)))(z_1 - \tilde{b}(\theta_2(z_1,z_2))) = 0$$

has exactly two solutions in the unit circle. □

Let $\phi_i(z_2)$, $i=1,2$, denote the two solutions of (3.12). For the simplicity of expression, we assume that $\phi_1(z_2) \neq \phi_2(z_2)$. Since $P_1(z_1,z_2)$ is analytic in the region $|z_i| < 1$, $i=1,2$, whenever the denominator of $P_1(z_1,z_2)$ vanishes, the numerator must also vanish. Since the denominator of $P_1(z_1,z_2)$ vanishes at $\phi_i(z_2)$, $i=1,2$, we have from equation (3.11) the following two equations;

$$\begin{aligned} &a_i(\phi_i(z_2),z_2)P_1(0,z_2) - \gamma_2 P_2(0,z_2) \\ &= \phi_i(z_2)\Big\{a_i(\phi_i(z_2),z_2)\big[(\lambda_1 + \frac{\nu}{z_2})Q_1(z_2) - \frac{\nu q_0^1}{z_2}\big] \\ &\qquad -\gamma_2\big[(\lambda_2 + \frac{\nu}{z_2})Q_2(z_2) - \frac{\nu q_0^2}{z_2}\big]\Big\}. \end{aligned} \tag{3.13}$$

From equation (3.13), we express the functions $P_i(0,z_2)$, $i=1,2$, in terms of $Q_i(z_2)$ as follows.

LEMMA 3.5. *The solutions $P_i(0,z_2)$ of equations (3.13) are given by*
(3.14)
$$\begin{bmatrix} P_1(0,z_2) \\ P_2(0,z_2) \end{bmatrix} = \begin{bmatrix} c_{11}(z) & c_{12}(z) \\ c_{21}(z) & c_{22}(z) \end{bmatrix} \cdot \begin{bmatrix} (\lambda_1 + \frac{\nu}{z_2})Q_1(z_2) - \frac{\nu q_0^1}{z_2} \\ (\lambda_2 + \frac{\nu}{z_2})Q_2(z_2) - \frac{\nu q_0^2}{z_2} \end{bmatrix} \cdot \frac{1}{C(z)},$$

where

$$\begin{aligned}
C(z_2) &= \gamma_2[\theta_2(\phi_2(z_2),z_2) - \theta_1(\phi_1(z_2),z_2) + \alpha_2(\phi_2(z_2) - \phi_1(z_2))], \\
c_{11}(z_2) &= \gamma_2[a_2(\phi_2(z_2),z_2)\phi_2(z_2) - a_1(\phi_1(z_2),z_2)\phi_1(z_2)], \\
c_{12}(z_2) &= \gamma_2^2[\phi_1(z_2) - \phi_2(z_2)], \\
c_{21}(z_2) &= a_1(\phi_1(z_2),z_2)a_2(\phi_2(z_2),z_2)[\phi_2(z_2) - \phi_1(z_2)], \\
c_{22}(z_2) &= \gamma_2\left[a_2(\phi_2(z_2),z_2)\phi_1(z_2) - a_1(\phi_1(z_2),z_2)\phi_2(z_2)\right].
\end{aligned}$$

Next we find the functions $Q_i(z_2)$, $i=1,2$. Substituting (3.14) into (3.5) yields the following equations for $Q_i(z_2)$, $i=1,2$.

LEMMA 3.6. *The solutions $Q_i(z)$ are given by*

$$\begin{bmatrix} Q_1(z) \\ Q_2(z) \end{bmatrix} = \begin{bmatrix} d_{11}(z) & d_{12}(z) \\ d_{21}(z) & d_{22}(z) \end{bmatrix} \cdot \begin{bmatrix} q_0^1 \\ q_0^2 \end{bmatrix} \cdot \frac{\nu}{D(z)}, \tag{3.15}$$

where

$$\begin{aligned}
D(z) = {} & [(\lambda_1+\gamma_1+\nu)zC(z) - (\lambda_1 z+\nu)c_{11}(z)] \\
& \cdot[(\lambda_2+\gamma_2+\nu)zC(z) - (\lambda_2 z+\nu)c_{22}(z)] \\
& -[\gamma_1 zC(z) + (\lambda_1 z+\nu)c_{21}(z)][\gamma_2 zC(z) + (\lambda_2 z+\nu)c_{12}(z)], \\
d_{11}(z) = {} & [zC(z) - c_{11}(z)][(\lambda_2+\gamma_2+\nu)zC(z) - (\lambda_2 z+\nu)c_{22}(z)] \\
& -c_{21}(z)[\gamma_2 zC(z) + (\lambda_2 z+\nu)c_{12}(z)], \\
d_{12}(z) = {} & [zC(z) - c_{22}(z)][\gamma_2 zC(z) + (\lambda_2 z+\nu)c_{12}(z)] \\
& -c_{12}(z)[(\lambda_2+\gamma_2+\nu)zC(z) - (\lambda_2 z+\nu)c_{22}(z)], \\
d_{21}(z) = {} & [zC(z) - c_{11}(z)][\gamma_1 zC(z) + (\lambda_1 z+\nu)c_{21}(z)] \\
& -c_{21}(z)[(\lambda_1+\gamma_1+\nu)zC(z) - (\lambda_1 z+\nu)c_{11}(z)], \\
d_{22}(z) = {} & [zC(z) - c_{22}(z)][(\lambda_1+\gamma_1+\nu)zC(z) - (\lambda_1 z+\nu)c_{11}(z)] \\
& -c_{12}(z)[\gamma_1 zC(z) + (\lambda_1 z+\nu)c_{21}(z)],
\end{aligned}$$

By the following lemma, we can determine the constants q_0^i, $i=1,2$.

LEMMA 3.7. *The following equation in z, $|z| \le 1$,*

$$(z - \tilde{b}(\theta_1(z,z)))(z - \tilde{b}(\theta_2(z,z))) = 0$$

has the solutions ϕ *and* 1.

From equation (3.11) and Lemma 3.7, we get

$$\begin{aligned}(3.16)\qquad &-a_1(\phi,\phi)\left[(\lambda_1(1-\phi)+\gamma_1)Q_1(\phi)-\gamma_2 Q_2(\phi)\right]\\ &\quad+\gamma_2\left[(\lambda_2(1-\phi)+\gamma_2)Q_2(\phi)-\gamma_1 Q_1(\phi)\right]=0,\end{aligned}$$

which yields

$$(3.17)\qquad Q_2(\phi)=\frac{a_1(\phi,\phi)}{\gamma_2}Q_1(\phi).$$

Next we find q_0^1 and q_0^2. By inserting $z_i=1$ and $\theta=0$ into (3.8) and using L'Hospital's theorem, we obtain, for $i=1,2$,

$$(3.18)\qquad \tilde{P}_i(0,1,1)=\frac{\gamma_{i'}[P_1(1,1)+P_2(1,1)]\cdot b+(\gamma_{i'}Q_{i'}(1)-\gamma_i Q_i(1))}{\gamma_1+\gamma_2}.$$

Note that $P_1(1,1)+P_2(1,1)$ is departure rate and departure rate must be the same as arrival rate at steady-state. Hence we have

$$(3.19)\qquad P_1(1,1)+P_2(1,1)=\frac{\gamma_2\lambda_1+\gamma_1\lambda_2}{\gamma_1+\gamma_2}.$$

From the total probability $1=\sum_{i=1}^{2}\left[Q_i(1)+\tilde{P}_i(0,1,1)\right]$, we obtain $Q_1(1)+Q_2(1)=1-\rho$ as the probability that the channel is idle, and $\tilde{P}_1(0,1,1)+\tilde{P}_2(0,1,1)=\rho$ as the probability that the channel is busy. Thus we obtain explicit expression for the boundary constants q_0^1 and q_0^2. By inserting (3.11), (3.14) and (3.15) into (3.8), we have expressed $\tilde{P}_i(\theta,z_1,z_2)$ explicitly in terms of known parameters α_i, β and γ_i.

4. Special case

As the rate of exponential retrial time tends to infinite, queue length distribution for queueing system with retrial approaches to that for queueing system without retrial. We show that the above facts hold for

MMPP, M/G/1 queue without retrial (Choi and Han[1]). Let $Q_i^{(o)}(z) = \lim_{\nu\to\infty} Q_i(z)$ and $q_0^{i(o)} = \lim_{\nu\to\infty} q_0^i$, $i = 1, 2$. From (3.15), we have

$$Q_i^{(o)}(z) = q_0^{i(o)}, \quad i = 1, 2. \tag{4.1}$$

From (4.1) and (3.17), we obtain explicit expression for the boundary constants $q_0^{i(o)}$, $i = 1, 2$,

$$\begin{bmatrix} q_0^{1(o)} \\ q_0^{2(o)} \end{bmatrix} = \begin{bmatrix} \gamma_2 \\ a_1(\phi,\phi) \end{bmatrix} \cdot \frac{1-\rho}{a_1(\phi,\phi)+\gamma_2}. \tag{4.2}$$

The above result agrees with the one for ordinary MMPP, M/G/1 queue (Choi and Han[1]).

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