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TWISTED PRODUCT REPRESENTATION OF REFLECTED BROWNIAN MOTION IN A CONE

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ABSTRACT. Consider a strong Markov process X^0 that has continuous sample paths in the closed cone $\overline{G}$ in $R^d (d \geq 3)$ such that the process behaves like a ordinary Brownian motion in the interior of the cone, reflects instantaneously from the boundary of the cone and is absorbed at the vertex of the cone.

It is shown that $X^0(t)$ has a representation $R(t)\Theta(t)$ where $R(t) \in [0, \infty)$ and $\Theta(t) \in S^{d-1}$, the surface of the unit ball.

1. Introduction

Let Ω be a subdomain of the unit sphere S^{d-1} in $R^d (d \geq 3)$ such that $S^{d-1} \setminus \overline{\Omega}$ is nonempty and the boundary $\partial\Omega$ of Ω in S^{d-1} is C^3. We may assume $(0, ..., 0, -1) \notin \Omega$. Define the open cone $G = \{r\omega : r > 0, \omega \in \Omega\}$. The closure and the boundary of G will be denoted by $\overline{G}$ and ∂G, respectively. The origin $\{0\}$ is the vertex of the cone. Let $\mathbf{v}$ be a C^2 d-dimensional vector field on $\partial G \setminus \{0\}$, such that $\mathbf{v}$ is constant on rays of the cone, i.e., for each $\omega \in \Omega$, $\mathbf{v}(r\omega) = \mathbf{v}(\omega)$ for all $r > 0$. We assume that the component of $\mathbf{v}$ in the inward normal direction to $\partial G \setminus \{0\}$ is positive. Indeed, without any loss of generality, by the scaling and continuity of $\mathbf{v}$, we may and do assume that $\mathbf{v} \cdot \mathbf{n} = 1$ on $\partial G \setminus \{0\}$, where $\mathbf{n}$ is the inward unit normal on $\partial G \setminus \{0\}$. Let v_r denote the component of $\mathbf{v}$ in the direction of the radial unit vector $\mathbf{e}_r$ in R^d, and define the vector $\mathbf{q} = \mathbf{v} - v_r\mathbf{e}_r - \mathbf{n}$. We assume $v_r \in C^3(\partial G \setminus \{0\})$ and $\mathbf{q} \in C^4(\partial G \setminus \{0\})$ as in [KW].

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We consider a strong Markov process X^0 that has continuous sample paths in $\overline{G}$ and the following three properties. We refer this process as reflected Brownian motion(abbreviated RBM) with reflection $\mathbf{v}$ absorbed at 0

(1.1) The state space is the (closed cone) $\overline{G}$ and the process behaves in the interior of the cone like ordinary Brownian motion.
(1.2) The process reflects instantaneously from the smooth part $\partial G \setminus \{0\}$ of the boundary of the cone, the direction of reflection being given by the vector field $\mathbf{v}$.
(1.3) X^0 is absorbed at the origin.

The question of existence and uniqueness (in law) of the process X^0 is established with the submartingale formulation by Kwon and Williams [KW]. In this paper, we give an alternative representation for X^0 as a "twisted product" such that $X^0(t) = R(t)\Theta(t)$ where $R(t) \in [0, \infty)$ and $\Theta(t) \in \overline{\Omega} \subset S^{d-1}$. This is similar to the skew product construction of Ito and Mckean ([IM Sec.7.15-7.17]) and twisted product representation of Brownian motion with polar drift by R.J.Williams ([W]). In [W], the twisted product representation provides intuition of the results on the behaviors of a diffusion process with generator $L = \frac{1}{2}\triangle + (2r)^{-1}\mu(\theta)\frac{\partial}{\partial r}$ near the origin. In our case, by [KW], we know a lot about behaviors of X^0 near the origin already. But we expect it gives useful bounds for the local time of the boundary for some special case of reflection, for example, where X^0 never hits 0 once it starts from $x \neq 0$. Moreover with bounds for local time, we may expect we can get some results for harmonic functions u in a cone with certain boundary conditions ($\frac{\partial u}{\partial n} = f$) since the key for representation of harmonic function via diffusions is bounds for the local time for the boundary.

The precise mathematical formulation for (1.1)-(1.3) is given in precise mathematical terms as the question of existence and uniqueness of a solution of the following submartingale problem associated with $(G, \mathbf{v})$with absorption at the vertex in (2.1)-(2.3) (see the section 2 of [KW]). For this, let $C_{\overline{G}}$ denote the set of continuous function $w : [0, \infty) \to \overline{G}$ endowed with the topology of uniform convergence on compact subsets in $[0, \infty)$. For each $t \geq 0$, let $\mathcal{M}_t = \sigma\{w(s) : 0 \leq s \leq t\}$ denote the σ-algebra of subsets of $C_{\overline{G}}$ generated by the coordinate maps $w \to w(s) \in R^d$ for $0 \leq s \leq t$ and let $\mathcal{M} = \bigvee_{t \geq 0} \mathcal{M}_t = \sigma\{w(t) : 0 \leq t < \infty\}$. Equivalently, $\mathcal{M}_t$(resp. $\mathcal{M}$) is the Borel σ-algebra associated with the topology of uniform convergence on $[0,t]$ (resp. compact subsets of $[0,\infty)$). For each domain $D \subset R^d$ and $n \geq 0$, $C^n(D)$ denotes the set of functions $f : D \to R$ that are n-times continuously differentiable in D. The set of functions in $C^n(D)$ whose partial derivatives up to and including those of order n are bounded in D is denoted by $C_b^n(D)$. The symbol $C_c^n(D)$ denotes the set of functions in $C^n(D)$ that have compact supports in D. The same notations will be used with $R^1 \times S^{d-1}$ in place of R^d.

The Laplace operator on $R^d \setminus \{0\}$ is given in polar coordinates by

$$(1) \qquad \triangle = \frac{\partial^2}{\partial r^2} + (d-1)r^{-1}\frac{\partial}{\partial r} + r^{-2}\triangle_{S^{d-1}}$$

where $\triangle_{S^{d-1}}$ is the Laplace-Beltrami operator on S^{d-1}. The set S^{d-1} is endowed with the topology induced from R^d. The gradient operator is given by

$$(2) \qquad \nabla = \frac{\partial}{\partial r}\mathbf{e}_r + r^{-1}\nabla_{S^{d-1}}$$

where $\mathbf{e}_r$ is the unit vector in the radial direction and $\nabla_{S^{d-1}}$ is the tangential gradient operator on S^{d-1}.

2. twisted product representation

By [KW], X^0 satisfying (1.1)-(1.3) is characterized (in law) as the unique process that has continuous paths in $\overline{G}$ and associated probability measure on $(C_{\overline{G}}, \mathcal{M})$ $\{P_x^0\}$ (one each starting point $x \in \overline{G}$) satisfying the following three properties.

$$(2.1) \qquad P_x^0(X^0(0) = x) = 1$$

(2.2) Define $\tau_0 = \inf\{t \geq 0 : X^0(t) = 0\}$. For each$f \in C_b^2(\overline{G})$ that satisfies $\mathbf{v} \cdot \nabla f \geq 0$ on $\partial G \setminus \{0\}$, we have

$$f(X^0(t \wedge \tau_0)) - \frac{1}{2}\int_0^{t\wedge\tau_0} \triangle f(X^0(s))ds$$

is a P_x^0-submartingale with respect $\mathcal{F}_t^0 = \sigma\{X^0(s) : 0 \le s \le t\}$.

(2.3) $$P_x^0(X^0(t) = 0 \quad forall \quad t \ge \tau_0) = 1.$$

In this section, we give a twisted product representation for X^0 similar to [W]. We will have two processes $R(t)$, $\Theta(t)$ for X^0 such that $R(t)$ is some time change of one-dimensional log-like process and $\Theta(t)$ is a time change of reflected Brownian motion $\tilde{X}$ on $\overline{\Omega}$ with the reflection $\mathbf{n} + \mathbf{q}$. The way we get the representation is that we project $\tilde{X}$ by the stereographic projection with the pole (0,...,0,-1). Then we get a reflected diffusion Z in R^{d-1} which is equivalent to $\tilde{X}$. By looking Z downstairs, we get the right semimartingale for $\tilde{X}$.

Suppose $(S, \mathcal{G})$ is a fixed measurable space on which is defined a standard one dimensional Brownian motion B and an independent reflected Brownian motion $\tilde{X}$ on $\overline{\Omega}$ with the reflection $\mathbf{n} + \mathbf{q}$. That is, $\tilde{X}$ is the diffusion such that the density $p(t, x, y)$ of $\tilde{X}$ satisfies $\frac{\partial p}{\partial t}(t, x, y) = \triangle_{S^{d-1}} p(t, x, y)$ for $x, y \in \Omega$ and $\nabla_{S^{d-1}} p(t, x, y) \cdot (\mathbf{n} + \mathbf{q}) = 0$ for $x \in \partial\Omega$. The existence of $p(t, x, y)$ is well known ([GT Sec.6.7]).

Let $\mathbf{p}$ be the streographic projection from $\overline{\Omega}$ to R^{d-1} with the pole $(0, ..., 0, -1)$. Then $\mathbf{p}$ is smooth and also $\mathbf{p}^{-1}$ by (0,...,0,-1)$\notin \overline{\Omega}$. Let $Z(t) = \mathbf{p}(\tilde{X}(t))$. Then $Z(t)$ is a reflected diffusion on a C^2 compact set $D = \mathbf{p}(\overline{\Omega}) \subset R^{d-1}$ with the reflection $\mathbf{v}' = \mathbf{p}_*(\mathbf{n} + \mathbf{q})$. That is, for $x \in \partial D$, $\mathbf{v}'(x)$ is the C^2 vector field projected from $\mathbf{n} + \mathbf{q}(\mathbf{p}^{-1}(x))$ in the sense that for any C^1 function f, $\nabla_{R^{d-1}} f \cdot \mathbf{v}'(x) = \nabla_{S^{d-1}}(f \circ \mathbf{p}) \cdot (\mathbf{n} + \mathbf{q})(\mathbf{p}^{-1}(x))$. Moreover $\mathbf{v}' \cdot \mathbf{n}' = 1$ where $\mathbf{n}'$ is the unit inward normal of D by $\mathbf{v} \cdot \mathbf{n} = 1$. Hence by Lion and Snitzman ([LS]) we get the following.

(3) $$Z(t) = \int_0^t \sigma(Z(s))dB(s) + \int_0^t b(Z(s))ds + \int_0^t \mathbf{v}'(Z(s))dL_Z(s)$$

where B is a (d-1)-dimensional Brownian motion and $L_Z(t) = \int_0^t 1_{(Z(s)\partial D)}$ $dL_Z(s)$ which is finite a.s.. Now we get σ and b explicitly. For $x = (x^1, ..., x^d) \in R^d$, let $\rho(x) = |x|$. Then for $x \in D$, $\triangle_{S^{d-1}}$ at $\mathbf{p}^{-1}(x)$, that is the Laplacian streographic projection is

$$\triangle_{S^{d-1}} = (\frac{1+\rho^2}{2})^2 \triangle_{R^{d-1}} - \frac{(d-3)(1+\rho^2)\rho}{2} \frac{\partial}{\partial \rho}.$$

The generator of $\tilde{X}$ is $\frac{1}{2}\triangle_{S^{d-1}}$ same as Z in the sense that for $x \in D$, $\mathbf{p}^{-1}(x) = \tilde{x}$ and $g \in C^2$, $\lim_{t\to 0} \frac{E^x g(Z(t)) - g(x)}{t} = \lim_{t\to 0} \frac{E^x g(\mathbf{p}(\tilde{X}(t)) - g(\tilde{x})}{t}$ $= \frac{1}{2}\triangle_{S^{d-1}} g(\mathbf{p}(\tilde{x}))$. Hence by (3) $L = \frac{1}{2}\sum_{i,j}(\sigma\sigma^T)_{ij}D_{ij} + \sum b_i D_i$ must be $\frac{1}{2}$ of the Laplacian projection from S^{d-1} to R^{d-1}. Therefore for $x \in D$ $\rho = |x|$,

$$\delta_{ij}(\frac{1+|x|^2}{2})^2 = \frac{1}{2}\sigma_{ij}(x) \tag{4}$$

and

$$-\frac{(d-3)(1+|x|^2)|x|}{4}\frac{\partial}{\partial\rho} = \sum_{i=1}^{d-1} b_i(x)D_i.$$

Let $\tilde{L}$ be the boundary process related to $\tilde{X}$, equivalent to L_Z in the sense that $\tilde{L}(t) = \int_0^t 1_{(\tilde{X}(s)\in\partial\Omega)}d\tilde{L}(s) = \int_0^t 1_{(Z_s\in\partial D)}dL_Z(s) = L_Z(t)$.

Consider 1-dimensional log-like process $Y(t)$ such that

$$Y(t) = B^1(t) + \frac{(d-2)}{2}t + \int_0^t v_r(\tilde{X}(s))d\tilde{L}(s) \tag{5}$$

where $B^1(t)$ is a 1-dimensional Brownian motion independent of (d-1)-dimensional Brownian motion in (3). Now we show that the process X^0 in (2.1)-(2.3) in $\overline{G}$ before τ_0 is a skew product of $(Y(t), Z(t))$, that is, $X^0(t) = (Y(\gamma(t)), Z(\gamma(t)))$ where $\gamma(t)$ is a time change.

By the definition, $(Y, \tilde{X})$ is a diffusion with continuous paths in $R^1 \times \overline{\Omega}$ and $(Y, \tilde{X}) = (Y, \mathbf{p}^{-1}(Z))$. Now we compute the generator of $(Y, \tilde{X})$. First, for $g_1 \in C_c^2(R^1)$, we apply $Y(t)$ in (5), then by the Ito's formular, we have

(6)

$$\begin{aligned} g_1(Y(t)) = &\int_0^t g_1'(Y(s))dB^1(s) + \int_0^t \frac{(d-2)}{2}g_1'(Y(s))ds \\ &+ \int_0^t g_1'(Y(s))v_r(\tilde{X}(s))d\tilde{L}(s) + \frac{1}{2}\int_0^t g_1''(Y(s))ds \\ = &\text{ martingale} + \frac{1}{2}\int_0^t [\frac{\partial^2}{\partial y^2}g_1(Y(s)) + (d-2)\frac{\partial}{\partial y}g_1(Y(s))]ds \\ &+ \int_0^t \frac{\partial}{\dagger}g_1(Y(s))v_r(\tilde{X}(s))d\tilde{L}(s). \end{aligned}$$

Now for $g_2 \in C^2(\overline{\Omega})$, let $g = g_2 \circ \mathbf{p}^{-1}$, then $g \in C^2(D)$ and $g_2(\tilde{X}(t)) = g_2(\mathbf{p}^{-1}(Z(t)) = g(Z(t))$. Recall $Z(t)$ in (4). Then, by the Ito's formular,
(7)

$$g_2(\tilde{X}(t)) = g(Z(t)) = \int_0^t \nabla_{R^{d-1}} g \cdot \sigma(Z(s))dB(s) + \int_0^t \nabla_{R^{d-1}} g \cdot \mathbf{v}'(Z(s))dL_Z(s) + \int_0^t Lg(Z(s))ds$$

where $\nabla_{R^{d-1}} g \cdot \mathbf{v}'(Z(s)) = \nabla_{S^{d-1}} g_2 \cdot (\mathbf{n}+\mathbf{q})(\tilde{X}(s))$ and L is in (4) such that

$$Lg(x) = \frac{1}{2}\triangle_{S^{d-1}} g(\mathbf{p}\circ\mathbf{p}^{-1}(x)) = \frac{1}{2}\triangle_{S^{d-1}} g_2(\mathbf{p}^{-1}(x)).$$

Hence

$$g_2(\tilde{X}(t)) = \text{martingale} + \frac{1}{2}\int_0^t \triangle_{S^{d-1}} g_2(\tilde{X}(s))ds + \int_0^t \triangle_{Sd-1} g_2 \cdot (\mathbf{n}+\mathbf{q})(\tilde{X}(s))d\tilde{L}(s). \tag{8}$$

By (6) and (7), $g_1(Y(t))$ and $g_2(\tilde{X}(t))$ are semimartingales. Therefore by intergration by parts, we have

$$\begin{aligned} (9)\qquad & g_1(Y(t))g_2(\tilde{X}(t)) \\ &= \int_0^t g_1(Y(s))dg_2(\tilde{X}(s)) + \int_0^t g_2(\tilde{X}(s))dg6_1(Y(s)) \\ &= \int_0^t g_1(Y(s))(\nabla_{S^{d-1}} g_2 \cdot (\mathbf{n}+\mathbf{q})(\tilde{X}(s))d\tilde{L}(s) \\ &+ \int_0^t g_1(Y(s))\frac{1}{2}\triangle_{S^{d-1}} g_2(\tilde{X}(s))ds \\ &+ \int_0^t g_2(\tilde{X}(s))\frac{1}{2}[\frac{\partial^2}{\partial y^2}g_1(Y(s)) + (d-2)\frac{\partial}{\partial y}g_1(Y(s))]ds \\ &+ \int_0^t g_2(\tilde{X}(s))[\frac{\partial}{\partial y}g_1(Y(s))v_r(\tilde{X}(s))d\tilde{L}(s)] \\ &+ \int_0^t g_2(\tilde{X}(s))g_1'(Y(s))bB^1(s) \\ &= \mathrm{I} + II + III + IV + V. \end{aligned}$$

Now V is a martingale and let

$$A = \frac{1}{2}[\triangle_{S^{d-1}} + \frac{\partial}{\partial y^2} + (d-2)\frac{\partial}{\partial y}] \tag{10}$$

then $II + III = \int_0^t Ag_1(Y(s))g_2(\tilde{X}(s))ds$ and (9) is

$$\begin{aligned} (11) \qquad & g_1(Y(t))g_2(\tilde{X}(t)) \\ &= \int_0^t Ag_1(Y(s))g_2(\tilde{X}(s)) \\ &+ \int_0^t (\frac{\partial}{\partial y}, \triangle_{S^{d-1}} g_1(Y(s))g_2(\tilde{X}(s)) \cdot (v_r, \mathbf{n}+\mathbf{q})d\tilde{L}(s) \\ &+ \text{martingale.} \end{aligned}$$

Let $Q_{y,\tilde{x}}$ be the probability measure on $(S, \mathcal{G})$ associated with $(Y, \tilde{X})$ starting from $(y, \tilde{x}) \in R^1 \times \overline{\Omega}$. Next define $A(t) = \int_0^t e^{2Y(s)}ds$ for all $t \geq 0$ and $A_\infty = \int_0^\infty e^{2Y(s)}ds$. A^{-1} denotes the functional inverse of A with $A^{-1}(t) \equiv \infty$ if $t \geq A_\infty$. For each $t \in [0, A_\infty)$, define

$$R(t) = e^{Y(A^{-1}(t))}, \quad \Theta(t) = \tilde{X}(A^{-1}(t)).$$

Further define $R(t) = 0$ for all $t \geq A_\infty$. Now let

$$X^* = (R(t), \Theta(t)). \tag{12}$$

Then $X^*(t)$ has continuous paths in $\overline{G}$ absorbed at the origin. Thus it suffices to verify that property (2.2) of the chracterization of X^0 holds for X^* for each starting point $x \in \overline{G} \setminus \{0\}$.

THEOREM 2.1. *$X^* = (R, \Theta)$ in (12) is a representation for X^0, i.e., X^* is equivalent in law to X^0.*

PROOF. To verify the property (2.2), it is enough to show that for each $x \in \overline{G}\{0\}$, $0 < \varepsilon < 1$ and for $f \in C_c^2(\overline{G})$ such that $\nabla_{R^d} f \cdot \mathbf{v} \geq 0$,

$$\{f(X^*(t \wedge \tau_\varepsilon^*)) - \frac{1}{2}\int_0^{t\wedge\tau_\varepsilon^*} \triangle_{R^d} f(X^*(s))ds, \qquad \mathcal{F}^*_{t\wedge\tau_\varepsilon^*}, \quad t \geq 0\} \tag{13}$$

is a $Q_{y,\tilde{x}}$-submartingale, where $(y,\tilde{x}) = (ln|x|, x/|x|) \in R^d \times \overline{\Omega}$, $\tau_\varepsilon^* = \inf\{s \geq 0 : |X^*(s)| \leq \varepsilon \quad \text{or} \quad |X^*(s)| \geq \varepsilon^{-1}\}$ and $\mathcal{F}_t^* = \sigma\{X^*(s) : 0 \leq s \leq t\}$. Given $0 < \varepsilon < 1$ and $f(r,\theta) \in C_c^2(\overline{G})$, let

$$\tau_\varepsilon^Y = \inf\{s \geq 0 : Y(s) \leq ln\varepsilon, \quad \text{or} \quad Y(s) \geq ln\varepsilon^{-1}\}$$

and define $g \in C_c^2(R^1 \times \overline{\Omega})$ by $g(y,\tilde{x}) = f(e^y, \tilde{x})$ for all $(y,\tilde{x}) \in R^1 \times \overline{\Omega}$. Moreover it is enough to show (13) for f of the type $f(r,\theta) = f_1(r)g_2(\theta)$ such that $f_1 \in C_c^2([0,\infty))$ and $g_2 \in C^2(\overline{\Omega})$. Define $g_1(y) = f_1(e^y)$, then $g_1 \in C_c^2(R^1)$ and

$$\begin{aligned} f(X^*(t)) = f_1(R(t))g_2(\Theta(t)) &= f_1(e^{Y(A^{-1}(t))})g_2(\tilde{X}(A^{-1}(t))) \\ &= g_1(Y(A^{-1}(t)))g_2(\tilde{X}(A^{-1}(t))). \end{aligned}$$

Then by (11) and Doob's stopping theorem since $E^{Q_{y,\tilde{x}}}(\tau_\varepsilon^Y) < \infty$, we have

(14)

$$\begin{aligned} &g_1(Y(A^{-1}(t) \wedge \tau_\varepsilon^Y))g_2(\tilde{X}(A^{-1}(t) \wedge \tau_\varepsilon^Y)) \\ &= \text{martingale} + \int_0^{A^{-1}(t)\wedge\tau_\varepsilon^Y} Ag_1(Y(s))g_2(\tilde{X}(s))ds \\ &\quad + \int_0^{A^{-1}(t)\wedge\tau_\varepsilon^Y} (\frac{\partial}{\partial y}, \nabla_{S^{d-1}})g_1(Y(s))g_2(\tilde{X}(s)) \cdot (v_r, \mathbf{n} + \mathbf{q})d\tilde{L}(s). \end{aligned}$$

Here

(15)

$$\begin{aligned} Ag_1(y)g_2(\tilde{x}) &= \frac{1}{2}[\frac{\partial^2}{\partial y^2} + (d-2)\frac{\partial}{\partial y} + \triangle_{S^{d-1}}]f_1(e^y)g_2(\tilde{x}) \\ &= \frac{1}{2}[e^{2y}\frac{\partial^2}{\partial r^2} + (d-2)e^y\frac{\partial}{\partial r} + \triangle_{S^{d-1}}]f_1(e^y)g_2(\tilde{x}) \end{aligned}$$

since if we let $r = e^y$,

$$\frac{\partial}{\partial y}f_1(e^y) = \frac{\partial}{\partial r}f_1(e^y)\frac{\partial r}{\partial y} = \frac{\partial}{\partial r}[f_1(e^y)]e^y,$$

$$\frac{\partial^2}{\partial y^2} = \frac{\partial}{\partial y}[\frac{\partial}{\partial r} f_1(e^y)]e^y + [\frac{\partial}{\partial r} f_1(e^y)]e^y,$$
$$= \frac{\partial^2}{\partial r^2} f_1(e^y)e^{2y} + [\frac{\partial}{\partial r} f_1(e^y)]e^y.$$

And

$$(\frac{\partial}{\partial y}, \nabla_{S^{d-1}})g_1(y)g_2(\tilde{x}) = (e^y \frac{\partial}{\partial r} f_1(e^y), \nabla_{S^{d-1}} g_2(\tilde{x})).$$

Hence

$$\begin{aligned} & r[\frac{\partial}{\partial r}\mathbf{e}_r + \frac{1}{r}\nabla_{S^{d-1}}][f_1 g_2](e^y, \tilde{x}) \cdot \mathbf{v}(e^y, \tilde{x}) \\ &= r[\nabla_{R^d}(f_1 g_2)(e^y, \tilde{x})] \cdot \mathbf{v}(e^y, \tilde{x}) \\ &= r[\frac{\partial}{\partial y} f_1(r) \cdot v_r + \frac{1}{r}\nabla_{S^{d-1}} g_2(\tilde{x})] \cdot (\mathbf{n} + \mathbf{q}) = r[\nabla_{R^d} f_1 g_2 \cdot \mathbf{v}](r, \tilde{x}). \end{aligned}$$

Hence if $\nabla f_1 g_2 \cdot \mathbf{v} \geq 0$, we have

$$g_1(Y(A^{-1}(t) \wedge \tau_\varepsilon^Y))g_2(\tilde{X}(A^{-1}(t) \wedge \tau_\varepsilon^Y)) - \int_0^{A^{-1}(t)\wedge\tau_\varepsilon^Y} Ag_1(Y(s))g_2(\tilde{X}(s))ds \tag{16}$$

is a $Q_{y,\tilde{x}}$-submartingale for each $(y, \tilde{x}) \in R^1 \times \Omega$ where A is in (10). Also, $A^{-1}(t) \wedge \tau_\varepsilon^Y = A^{-1}(t \wedge \tau_\varepsilon^*)$. By substituting the above in (15) and changing the variable of integration there to $u = A(s)$ (so that $du = e^{2Y(s)}ds$), we obtain

$$\begin{aligned} &\{f_1(exp(Y(A^{-1\wedge\tau_\varepsilon^*}))g_2(\tilde{X}(A^{-1}(t \wedge \tau_\varepsilon^*))) \\ &- \int_0^{t\wedge\tau_\varepsilon^*} \frac{1}{2}\triangle_{R^d} f_1(exp(Y(A^{-1}(u)))g_2(\tilde{X}(A^{-1}(u))), \quad \mathcal{G}_{A^{-1}(t\wedge\tau_\varepsilon^*)}, t \quad 0\} \end{aligned}$$

is a $Q_{y,\tilde{x}}$-submartingale since $\triangle_{R^d} = \frac{\partial^2}{\partial r^2} + \frac{(d-2)}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\triangle_{S^{d-1}}$. Recalling the definition of X^* and noting that $\mathcal{F}^*_{t\wedge\tau_\varepsilon^*} \subset \mathcal{G}_{A^{-1}(t\wedge\tau_\varepsilon^*)}$, we have (13) for $f \in C_c^2(\overline{G})$ such that $\nabla f \cdot \mathbf{v} \geq 0$ and $f(r, \theta) = f_1(r)g_2(\theta)$ where $f_1 \in C_c^2([0, \infty))$, $g_2 \in C^2(\overline{\Omega})$. Now by approximation, (13) holds for all $f \in C_c^2(\overline{G})$ such that $\nabla f \cdot \mathbf{v} \geq 0$. Hence we are done.

REMARK 1. In [W], Williams showed properties of sample paths of X^0, a diffusion related to the operator that has the singular point at 0 by the twist product representation. Here in a similar way we may show properties of sample paths of X^0, reflected Brownian motion using our twist product representation.

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