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SEIBERG-WITTEN INVARIANTS ON CONNECTED SUMS OF 4-MANIFOLDS

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ABSTRACT. We construct closed orientable four-manifolds which have nontrivial Seiberg-Witten invariants, but do not admit any symplectic structure.

1. Introduction

Let Y be an oriented, closed Riemannian 4-manifold. There is an integral cohomology class which reduces to the second Stieffel-Whiney class $w_2(Y)$ mod (2). This integral cohomology class induces a $Spin^c$-structure on Y. Seiberg and Witten in [10] introduced a new invariant on Y which is a differential-topological invariant. Taubes in [9] proved that every closed symplectic 4-manifold has a non-trivial Seiberg-Witten invariant. The Seiberg-Witten invariants of connected sums of 4-manifolds with $b_2^+ > 0$ identically vanish. Kotschick, Morgan and Taubes in [8] showed that there are closed oriented 4-manifolds with nontrivial Seiberg-Witten invariants which do not admit symplectic structures. They considered the case which is the first Betti number $b_1(N) = 0$. We would like to generalize their theorem by giving a certain condition instead of $b_1(N) = 0$, of course our case will cover their case. We introduce our theorem :

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THEOREM 1.1. *Let Y be a closed oriented 4-manifold with a nontrivial Seiberg Witten invariant and $b_2^+(Y) > 1$, and let N be a closed oriented 4-manifold with $b_2^+(N) = 0$. If there are even integers λ_i, $i = 1, \cdots, n$ such that $4b_1(N) = 2\lambda_1 + \cdots + 2\lambda_n + \lambda_1^2 + \cdots \lambda_n^2$ where $n = b_2(N)$, and the fundamental group of N has a nontrivial finite quotient. Then the connected sum $Y\sharp N$ has a nontrivial Seiberg-Witten invariant but does not admit any symplectic structure. The Seiberg-Witten invariants of $Y\sharp N$ and Y are same up to sign $(-1)^{b_1(N)}$.*

2. Characteristic Elements

Let X be a closed symplectic 4-manifold and let $X = Y\sharp N$ be a smooth connected sum decomposition. By the vanishing theorem of Seiberg-Witten invariants and non-trivial Seiberg-Witten invariants for symplectic manifolds, one of the summands, say it N, has a negative definite intersection form. By Donaldson's Theorem in [5] there is a basis $\{e_1, \cdots, e_n\}$ of the free part of $H^2(N, \mathbb{Z})$ such that in this basis the intersection form of N is diagonal, where n is the rank of $H^2(N, \mathbb{Z})$. An element $\alpha \in H^2(N, \mathbb{Z})$ is said to be characteristic if the intersection number $\alpha \cdot x = x \cdot x \mod (2)$ for any $x \in H^2(N, \mathbb{Z})$. If α is characteristic, then $\alpha \equiv w_2(N)$ modulo 2.

LEMMA 2.1. *Let N be a closed oriented Riemannian 4-manifold with $b_2^+(N) = 0$ and let $\{e_1, \cdots, e_n\}$ is a basis for the free part of $H^2(N, \mathbb{Z})$ such that $e_i \cdot e_j = -\delta_{ij}$. Then*

(1) *$e = e_1 + \cdots + e_n$ is characteristic*
(2) *$\alpha = (1 + \lambda_1)e_1 + \cdots + (1 + \lambda_n)e_n$ is characterisitc if and only if the λ_i are even.*

PROOF. It is sufficient to consider the free elements in the proof because the intersection numbers with torsion elements are zero. Let $x = x_1e_1 + \cdots x_ne_n \in H^2(N, \mathbb{Z})$ where the x_i are integers $i = 1, \cdots, n$. Then

$$\alpha \cdot x = -(1 + \lambda_1)x_1 - \cdots - (1 + \lambda_n)x_n \quad \text{and}$$
$$x \cdot x = -x_1^2 - \cdots - x_n^2.$$

$\alpha \cdot x = x \cdot x \mod (2)$ for all $x \in H^2(N, \mathbb{Z})$, if and only if

$-(1 + \lambda_1)x_1 - \cdots - (1 + \lambda_n)x_n = -x_1^2 - \cdots - x_n^2 \mod (2)$ for all $x_1, \cdots x_n$, if and only if

$\lambda_1 x_1 + \cdots + \lambda_n x_n = 0 \mod (2)$ for all $x_1, \cdots x_n$, if and only if

$\lambda_1, \cdots, \lambda_n$ are even.

If the fundamental group $\pi_1(N)$ of N has a nontrivial quotient, then there is a connected convering of N with the cardinarity of fiber > 1 and so is a connected sum with N.

LEMMA 2.2[8]. *Let $X = Y \sharp N$ be a closed symplectic 4-manifold which decomposes as a connected sum. If N has a negative definite intersection form then its fundamental group does not admit nontrivial finite quotient.*

3. Seiberg-Witten Invariants

We recall briefly the Seiberg-Witten invariants for a compact, oriented Riemannian 4-manifold X with $b_2^+(Y) > 1$.

Let $e \in H^2(Y, \mathbb{Z})$ with $e \equiv w_2(Y) \mod (2)$.

The cohomology class e defines a $Spin^c$-structure on Y. Let $W^+(W^-) \to Y$ be the positive (negative respectively) spinor bundle on Y and $L = \det(W^+)$ the determinant line bundle of W^+. Let $\tau : End(W^+) \to \Lambda^+(T^*Y)\otimes\mathbb{C}$ be the adjoint of Clifford multiplication. A connection A on L with the Levi-Civita connection on T^*Y defines a covariant derivative $\nabla_A : \Gamma(W^+) \to \Gamma(W^+ \otimes T^*Y)$. The composition of ∇_A and Clifford multiplication define a Dirac operator

$$D_A : \Gamma(W^+) \to \Gamma(W^-).$$

For each connection on L $A \in \mathcal{A}(L)$ and $\phi \in \Gamma(W^+)$, the equations

$$\begin{cases} D_A\phi &= 0 \\ F_A^+ &= \frac{1}{4}\tau(\phi \otimes \phi^*) \end{cases}$$

are called the Seiberg-Witten monopole equations. The gauge group $C^\infty(Y, U(1))$ of the complex line bundle L acts on the space of solutions of the monopole equations. The moduli space $\mathfrak{M}(Y,e)$ is the quotient of the space of solutions by the gauge group. Then the moduli space is generically a compact smooth manifold with its dimension $-\frac{1}{4}(2\chi(Y)+3\sigma(Y))+\frac{1}{4}c_1(L)^2$ and defines cannonically an invariant which is so called the Seiberg-Witten invariants. For details see [7].

Let Y and N be compact oriented 4-manifolds. Let $\alpha\in H^2(Y,\mathbb{Z})$ and $\beta\in H^2(N,\mathbb{Z})$ such that $\alpha\equiv w_2(Y) \mod (2)$, $\beta\equiv w_2(N) \mod (2)$. Let $X=Y\sharp N$, then $\alpha+\beta\equiv w_2(X) \mod (2)$. Let the complex line bundles $L_\alpha\to Y$, $L_\beta\to N$, $L_{\alpha+\beta}\to X$ with their Chern classes $c_1(L_\alpha)=\alpha$, $c_1(L_\beta)=\beta$ and $c_1(L_{\alpha+\beta})=\alpha+\beta$ respectively. We can easily calculate the virtual dimensions of the moduli spaces.

LEMMA 3.1. $\dim\mathfrak{M}(X,\alpha+\beta)=\dim\mathfrak{M}(Y,\alpha)+\dim\mathfrak{M}(N,\beta)+1$.

PROOF. The Euler characteristic is $\chi(X)=\chi(Y)+\chi(N)-2$. The signature is $\sigma(X)=\sigma(Y)+\sigma(N)$. The first Chern classes are $c_1(L_{\alpha+\beta})=c_1(L_\alpha)+c_1(L_\beta)$ and $\alpha\cdot\beta=0$. Thus

$$\begin{aligned}\dim\mathfrak{M}(X,\alpha+\beta)&=-\frac{1}{4}(2\chi(X)+3\sigma(X))+\frac{1}{4}c_1(L_{\alpha+\beta})^2\\&=\left[-\frac{1}{4}(2\chi(Y)+3\sigma(Y))+\frac{1}{4}c_1(L_\alpha)^2\right]\\&\quad+\left[-\frac{1}{4}(2\chi(N)+3\sigma(N))+\frac{1}{4}c_1(L_\beta)^2\right]+1\\&=\dim\mathfrak{M}(Y,\alpha)+\dim\mathfrak{M}(N,\beta)+1.\end{aligned}$$

Let N have a negative definite intersection form.

As in Lemma 2.1, let $\{e_1,\cdots,e_n\}$ be a basis of the free part of $H^2(N,\mathbb{Z})$. If $\alpha=(1+\lambda_1)e_1+\cdots+(1+\lambda_n)e_n$ and the λ_i are even, then α is characteristic.

LEMMA 3.2. *If* $4b_1(N)=2\lambda_1+\cdots+2\lambda_n+\lambda_1^2+\cdots+\lambda_n^2$, *then* $\dim\mathfrak{M}(N,\alpha)=-1$.

COROLLARY 3.3. *If Y is a symplectic manifold and K is the canonical line bundle on Y,* $4b_1(N)=2\lambda_1+\cdots+2\lambda_n+\lambda_1^2+\cdots+\lambda_n^2$, *and* $X=Y\sharp N$, *then* $\dim\mathfrak{M}(X,c_1(K)+\alpha)=\dim\mathfrak{M}(Y,c_1(K))=0$

PROOF. For the proof use $c_1(K)^2 = 2\chi + 3\sigma$, Lemma 3.1 and Lemma 3.2.

PROOF OF LEMMA 3.2. The virtual dimension of the moduli space is

$$\begin{aligned}\dim \mathfrak{M}(N,\alpha) &= -\frac{1}{4}(2\chi(N) + 3\sigma(N)) + \frac{1}{4}\alpha^2 \\ &= -\frac{1}{4}\{2(2 - 2b_1(N) + b_2(N)) + 3(-b_2(N))\} \\ &\quad + \frac{1}{4}[(1+\lambda_1)e_1 + \cdots + (1+\lambda_n)e_n]^2 \\ &= -\frac{1}{4}[4 - 4b_1(N) - b_2(N)] + \frac{1}{4}[-(1+\lambda_1)^2 - \cdots - (1+\lambda_n)^2] \\ &= -\frac{1}{4}[4 - 4b_1(N) + 2\lambda_1 + \lambda_1^2 + \cdots + 2\lambda_n + \lambda_n^2] \\ &= -1, \quad \text{since} \quad 4b_1(N) = 2\lambda_1 + \cdots 2\lambda_n + \lambda_1^2 + \cdots + \lambda_n^2.\end{aligned}$$

REMARK 3.4. For the equation $4b_1(N) = 2\lambda_1 + \cdots + 2\lambda_n + \lambda_1^2 + \cdots + \lambda_n^2$,

1. If $\lambda_2 = \cdots = \lambda_n = 0, b_1(N) = 6$ and $\lambda_1 = 4$ or -6, then the equation holds.
2. If $\lambda_1 = \cdots = \lambda_n = 0 = b_1(N)$, then the equation also holds.

4. Seiberg-Witten Invariants on Non-Symplectic 4-Manifolds

In this section we would like to construct non-symplectic 4-manifolds which have non-trivial Seiberg-Witten invariants.

THEOREM 4.1. *Let Y have a nontrivial Seiberg-Witten invariant and let N have a negative definite intersection form. If there are even integers $\lambda_i, i = 1 \cdots n$ such that $4b_1(N) = 2\lambda_1 + \cdots + 2\lambda_n + \lambda_1^2 + \cdots + \lambda_n^2$, then the connected sum $X = Y \sharp N$ has a nontrivial Seiberg-Witten invariant.*

PROOF. Suppose N has a negative definite intersection form. As in Lemma 4, choose $\alpha = (1+\lambda_1)e_1 + \cdots + (1+\lambda_n)e_n$ such that $4b_1(N) = 2\lambda_1 + \cdots + 2\lambda_n + \lambda_1^2 + \cdots + \lambda_n^2$ and the λ_i are even. Then α is characteristic by Lemma 1 and there is a $Spin^c$-structure on N with first Chern class α. The Seiberg-Witten monopole equation is

$$\begin{cases} D_A\psi &= 0 \\ F_A^+ &= \frac{1}{4}\tau(\psi \otimes \psi^*). \end{cases}$$

For a generic metric on N there is no irreducible solution of the equations since $\dim \mathfrak{M}(N,\alpha) = -1$. We have a unique reducible solution $(A_\alpha, 0)$ given by the zero section of the positive spinor bundle and a connection A_α whose curvature is the harmonic form representing $\alpha = \frac{i}{2\pi}F_{A_\alpha} \in H^2(N,\mathbb{R})$. The given $Spin^c$-structure $e \in H^2(Y,\mathbb{Z})$ on Y and α induce a $Spin^c$-structure on X. By choosing generic metrics on $[Y \setminus D^4] \cup [0,\infty) \times S^3$ and $[N \setminus D^4] \cup [0,\infty) \times S^3$, and product metric on the cylinder $S^3 \times \mathbb{R}$ and connecting them, we have a Riemannian metric on $X = Y \sharp N$. The solutions of the Seiberg-Witten equations in $\mathfrak{M}(X, e+\alpha)$ are given by gluing the solutions in $\mathfrak{M}(Y,e)$ on Y to the unique solution $(A_\alpha, 0)$ in $\mathfrak{M}(N,\alpha)$ on N.
In particular, $\dim \mathfrak{M}(X, e+\alpha) = \dim \mathfrak{M}(Y,e)$.

By combining previous Lemmas to Theorems we have the following Theorem.

THEOREM 4.2. *Let Y be a manifold with a nontrivial Seiberg-Witten invariant defined by $e \in H^2(Y,\mathbb{Z})(b_2^+(Y) > 1)$, and let N be a manifold with negative definite intersection form. If there are even integers $\lambda_i, i = 1 \cdots n$ such that $4b_1(N) = 2\lambda_1 + \cdots + 2\lambda_n + \lambda_1^2 + \cdots + \lambda_n^2$ and that the fundamental group of N has a nontrivial finite quotient, then the connected sum $Y \sharp N$ has a nontivial Seiberg-Witten invariant but does not admit any symplectic structure.*

As in Theorem 4.2, let $e \in H^2(Y,\mathbb{Z})$ define a non-zero Seiberg-Witten invariant. Let N be a closed oriented 4-manifold with negative definite intersection form and with the second Betti number $b_2(N) = n$. $\alpha = (1+\lambda_1)e_1 + \cdots + (1+\lambda_n)e_n$. The orientation of the moduli space

$\mathfrak{M}(Y\sharp N, e+\alpha)$ of the solutions of the Seiberg-Witten equations for the $Spin^c$-structure defined by $e+\alpha$ in $Y\sharp N$ is a choice of orientation for the line

$$\det H^0(Y\sharp N)\otimes \det H^1(Y\sharp N)\otimes \det H^{2,+}(Y\sharp N)$$
$$=\det H^0(Y)\otimes \det H^1(Y)\otimes \det H^{2,+}(Y)\otimes \det H^1(N),$$

where the cohomology groups have the real coefficients. Since the moduli spaces $\mathfrak{M}(Y\sharp N, e+\alpha)$ and $\mathfrak{M}(Y,e)$ are in a one-to-one correspondence, we have the following theorem.

THEOREM 4.3. *Under the assumptions of Theorem 4.2, the Seiberg-Witten invariants are related by the equality:*

$$SW(Y\sharp N, e+\alpha)=(-1)^{b_1(N)}SW(Y,e)$$

where $b_1(N)$ is the first Betti number of N.

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