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WEAK ATTRACTORS AND LYAPUNOV-LIKE FUNCTIONS

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ABSTRACT. Recently Hurley [3] proved that if A is a weak attractor of a discrete dynamical system f then there exists a Lyapunov-like function for A. The purpose of this note is to study whether the converse of the above result does hold or not.

Let (f^t) be a dynamical system, i.e., a group or semigroup of maps $M \to M$ parametrized by a discrete or continuous time t. We may assume that M has a topological or differentiable structure, and that the f^t are continuous or differentiable. There often exist subsets A of M which attract neighboring points x, this means that $f^t(x)$ tends to A when $t \to \infty$. Such subsets A are called attractors.

A study of qualitative behavior of dynamical system inevitably involves the discussion of invariant sets called attractors. They are of interest for the discription of the asymptotic behavior of phsyical systems (or the long term behavior of all kinds of natural phenomena).

However the definitions of attractors are various, and we can find the various definitions of attractors in the paper of Milnor [6]. Along with a variety of definitions for the term "attractor", the semantically similiar phrase "attracting set" is often encountered with either more or fewer restriction on its definition.

Surprisingly, in many literature, the dynamics of attractors (or attracting sets) in compact spaces are mainly considered, even if it is valuable to study the dynamics of attractors (or attracting sets) in noncompact spaces.

In this note, we will adapt the definition of attractor in a compact space in the sense of Conley [1], and then extend this concept to the

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dynamical system generated by a continuous map f on a noncompact space, which will be called the weak attractor of f.

In this direction, there were some results by Hurley [2,3] and Lee [4]. In particular, Hurley obtained the following result (see Lemmas 8.3, 8.4, 8.5 in [2] and part 1 of Theorem 2 in [3]).

THEOREM 1. *Suppose that X is a locally compact, second countable metric space, and that $f : X \to X$ is continuous. If A is a weak attractor of f, then there exists a continuous function $h : X \to [0,2]$ such that*

(1) $h^{-1}(0) = A$,
(2) $h^{-1}(2) = X - B(A)$,
(3) $0 < h(f(x)) < h(x) < 2$ *for* $x \in B(A) - A$,

where $B(A)$ is the basin of A.

The function h obtained in the above theorem is called a *Lyapunov-like function* for the weak attractor A of f (see [3]).

The purpose of this note is to investigate whether the converse of the above theorem does hold or not, in the following sense.

QUESTION. *Let X be a locally compact, second countable metric space, $f : X \to X$ a continuous map, and A a nonempty, closed, f-invariant subset of X. Suppose that there exist a neighborhood W of A and a continuous function $h : X \to [0,2]$ such that*

(1) $h^{-1}(0) = A$,
(2) $h^{-1}(2) = X - W$,
(3) $0 < h(f(x)) < h(x) < 2$ *for* $x \in W - A$.

Is the set A a weak attractor of f?

Prior to give an answer to the above question, we need some definitions and examples. First we consider the definition of an attractor which is given by McGehee [5] and Norton [8] for a continuous function f on a compact metric space M as follow.

DEFINITION 1. A nonempty compact subset A of M is called an *attractor* of f if there exists a neighborhood U of A such that $\omega(U) = A$, where $\omega(U) = \bigcap_{m \geq 0}(\overline{\bigcup_{n \geq m} f^n(U)})$.

The following result is due to Norton[8].

THEOREM 2. *Let f be a continuous map on a compact metric space M. Then a nonempty compact subset A of M is an attractor of f if and only if there exist a neighborhood U of A and two constants $\varepsilon > 0$ and $N > 0$ such that*

(1) $B_\varepsilon(f^n(x)) \subset U$ *for each* $x \in U$ *and* $n > N$,
(2) $A = \bigcap_{n \geq 0} f^n(U)$,

where $B_\varepsilon(p)$ denotes the ball of radius ε centered at p.

We can suggest two definitions for the concepts of attractors of a dynamical system $f : X \to X$ generated by a continuous map f on a noncompact metric space X, which are motivated by Theorem 2 as follows.

DEFINITION 3. A nonempty closed subset A of X is said to be an *attractor* of f if there exist a neighborhood U of A and two constants $\varepsilon > 0$ and $N > 0$ with the properties (1) and (2) in Theorem 2.

DEFINITION 4. A nonempty closed subset A of X is called a *weak attractor* of f if there exist a neighborhood V of A, a constant $N > 0$ and a continuous map $\varepsilon : X \to \mathbb{R}^+$ such that

(1) $B_{\varepsilon(f^n(x))}(f^n(x)) \subset V$ for any $x \in V$ and $n > N$,
(2) $A = \bigcap_{n \geq 0} f^n(V)$.

We say that the set V is a *weakly absorbing neighborhood* of A, and the set $B(A, U) = \cup_{n \geq 0} f^{-n}(V)$ will be called the *relative basin* of A to V.

Every attractor is a weak attractor, but we can see in [4] that some weak attractor need not be an attractor. However it is clear that the above two concepts of attractors and weak attractors are pairwise equivalent whenever the phase space X is compact.

REMARK 5. It is easy to show that $x \in B(A, V)$ if and only if $\omega(x) \subset A$ whenever $\omega(x) \neq \emptyset$, where $\omega(x) = \omega(\{x\})$. But if $\omega(x) = \emptyset$ then the fact $\omega(x) \subset A$ does not imply that $x \in B(A, V)$. Moreover any two different weakly absorbing neighborhoods of A may induce the different relative domains. In fact, we let X denote the disjoint union of two copies of the reals $\mathbb{R}_1$ and $\mathbb{R}_2$. Let ϕ be the flow on X generated by the differential equations:

$$x' = x \text{ on } \mathbb{R}_1, \text{ and } x' = -x \text{ on } \mathbb{R}_2.$$

Let f be the time one map of the flow ϕ, i.e., $f(x) = \phi(x, 1)$. We will use subscripts to indicate whether points belong to $\mathbb{R}_1$ or $\mathbb{R}_2$. Clearly $A = \{0_2\}$ is a weak attractor of f, and $U = (-1, 1)_2$ and $V = U \cup (1, \infty)_1$ are weakly absorbing neighborhoods of A. But we have $B(A, U) \neq B(A, V)$.

Hence we define the *basin* of A to be the union of the relative basins $B(A, U)$ as U varies over all the weakly absorbing neighborhoods of A.

When the phase space X is compact, the definition of the relative basin $B(A, U)$ is independent of the choice of U in the sense that if V is another weakly absorbing neighborhood of A then we have $B(A, U) = B(A, V)$.

Now we will give an example to show that the answer to our question is negative, and then claim that the answer to our question is positive if the invariant set A is compact.

EXAMPLE 6. Let us consider a homeomorphism f on the plane $\mathbb{R}^2$ given by $f(x, y) = (x + 1, y)$, and let $A = \{(x, 0) : x \in \mathbb{R}\}$. Then we can see that the set A is a closed, f-invariant subset of $\mathbb{R}^2$. Define a map $h : \mathbb{R}^2 \to [0, 2]$ by

$$h(x, y) = (1 + \frac{2}{\pi} \tan^{-1}(-x)) \times (\frac{2}{\pi} \tan^{-1}(|y|)).$$

Then h is a continuous map which satisfy the following properties:

(1) $h^{-1}(0) = A$,

(2) $0 < h(f(p)) < h(p) < 2$ for $p \in \mathbb{R}^2 - A$.

Hence the pair (A, h) satisfy the assumptions of our question, but it is clear that the set A is not a weak attractor of f.

THEOREM 7. *Let X be a locally compact metric space, $f : X \to X$ a continuous map, and A a nonempty, compact, f-invariant subset of X. Suppose that there exist a neighborhood W of A and a continuous map $h : X \to [0, 2]$ such that*

(1) $h^{-1}(0) = A$,

(2) $0 < h(f(x)) < h(x)$ *for* $x \in W - A$.

Then the set A is a weak attractor of f.

PROOF. Let V_0 be a neighborhood of A such that $\overline{V_0}$ is compact and $\overline{V_0} \subset W$. Choose a number $\varepsilon > 0$ with $h^{-1}([0,\varepsilon)) \subset V_0$, and let $V = h^{-1}([0,\frac{\varepsilon}{2}))$. Then V is a neighborhood of A and forward f–invariant. In fact, if $x \in V$ then $x \in V - A$ or $x \in A$. By assumption, we have $h(f(x)) < h(x) < \frac{\varepsilon}{2}$ or $h(x) = 0$, and so $f(x) \in h^{-1}([0,\frac{\varepsilon}{2})) = V$. Let U be any neighborhood of A. First we will show that there exists a number $k = k(U) > 0$ such that $\overline{f^n(V)} \subset U$ whenever $n > k$. Since $\overline{V}$ is compact, $h(\overline{f^n(V)}) = h(f^n(\overline{V}))$ is also compact in $[0,2]$, for each $n = 1, 2, ...$ Hence we can define

$$a_n = \max\{h(x) \ : \ x \in \overline{f^n(V)}\}.$$

Then we have: either $a_n = 0$, or else $a_{n+1} < a_n$ for each $n \geq 1$. If $a_n = 0$ then $\overline{f^n(V)} = A$, and so $\overline{f^n(V)} \subset U$. Hence we suppose $a_{n+1} < a_n$ for each $n \geq 0$. Then we can see that the sequence $\{a_n\}$ converges to 0. Since U is a neighborhood of A, we can choose a number $\delta > 0$ sastisfying $h^{-1}([0,\delta)) \subset U$. Select a number $k = k(\delta) > 0$ such that $n > k$ implies $a_n < \delta$. For each $n > k$, we have

$$h(\overline{f^n(V)}) \subset [0, a_n] \subset [0,\delta), \text{ and so } \overline{f^n(V)} \subset U.$$

Consequently we get $\bigcap_{i \geq 0} \overline{f^i(V)} = A$.

On the other hand, let G be a neighborhood of A such that $\overline{G}$ is compact and $\overline{G} \subset V$. Choose a number $N = N(G) > 0$ such that $n \geq N$ implies $\overline{f^n(V)} \subset G$. Define a map $\varepsilon : X \to \mathbb{R}^+$ by

$$\varepsilon(x) = \frac{1}{3}\{d(x,\overline{G}) \ + \ d(x, X - V)\}.$$

To show that the ε-ball $B_{\varepsilon(f^n(x))}(f^n(x))$ is a subset of V for any $x \in V$ and $n \geq N$, we suppose $B_{\varepsilon(f^n(x))}(f^n(x)) \cap (X - V) \neq \emptyset$ for some $x \in V$ and $n \geq N$. Then we can select $y \in X - V$ satisfying $d(f^n(x), y) < \varepsilon(f^n(x))$. Since $y \notin V$ and $f^n(x) \in \overline{G}$, we have

$$d(f^n(x), y) \geq d(f^n(x), X - V) > \varepsilon(f^n(x)).$$

The contradiction shows that V is a weakly absorbing neighborhood of A, and so completes the proof.

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