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ON THE PROJECTIVELY FLAT FINSLER SPACE WITH A SPECIAL (α, β)-METRIC

BYUNG-DOO KIM

ABSTRACT. The (α, β)-metric is a Finsler metric which is constructed from a Riemannian metric α and a differential 1-form β; it has been sometimes treat in theoretical physics. In particular, the projective flatness of Finsler space with a metric $L^2 = 2\alpha\beta$ is considered in detail.

1. Introduction

Let $F^n = (M^n, L)$ be an n-dimensional Finsler space, that is, an n-dimensional differential manifold M^n equipped with a fundamental function $L(x, y)$. A Finsler metric $L(x, y)$ is called an (α, β)-metric if L is a positive homogeneous function of α and β of degree one, where $\alpha^2 = a_{ij}(x)y^iy^j$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form. We have specially interesting examples of an (α, β)- metric, for instance, $L = \alpha + \beta$ (Randers metric), $L = \alpha^2/\beta$ (Kropina metric) and $L = (\alpha^2\beta)^{1/3}$ (a decomposition of cubic metric). In particular the Randers metric and the Kropina metric are of special intrest in physics [1]. Moreover, in 1992 Hojo [3] introduced a Finsler metric $L^2 = 2\alpha\beta$. In this paper, we consider a Finsler space with a $L^2 = 2\alpha\beta$.

A Finsler spaces is called projectively flat if it is projective to a locally Minkowski space. In the previous paper [6], Matsumoto studied the projective flatness of the Randers space, the Kropina space and a special generalized Kropina space.

The purpose of the present paper is to give the condition that the Finsler space with a metric $L^2 = 2\alpha\beta$ is projectively flat.

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Throughout the terminology and notations are refered to Matsumoto's monograph [5].

2. A Berwald space

If we put $F = L^2/2$, the metric tensor and the Cartan's C-tensor are given by

$$g_{ij} = \dot{\partial}_i \dot{\partial}_j F, \; C_{ijk} = \dot{\partial}_k g_{ij}/2, \tag{2.1}$$

where we put $\dot{\partial}_k = \partial/\partial y^k$. Then we can introduce in F^n the Cartan connection $C\Gamma = (\Gamma^{*i}_{j\ k}, G^i_j, C_j{}^i{}_k)$. The covariant derivative of a vector $X^i(x,y)$ is given by

$$X^i_{|j} = \delta_j X^i + \Gamma^{*i}_{mj} X^m, \tag{2.2}$$

where we put $\delta_j = \partial_j - \dot{\partial}_m G^m_j$ and $\partial_j = \partial/\partial x$.

An affinely connected space defined by L. Berwald is also called a Berwald space which is defined as the Finsler space such that Berwald's connection coefficients depend on position alone. If we obey the Cartan connection, such a space is also the one in which Cartan's connection coefficients $\Gamma^{*i}_{j\ k}$ depend on position alone, and it is characterized by the well-known condition $C_{ijk|h} = 0$.

In this section, the Kikuchi's method of [4] will now be applied to find the condition that F^n be a Berwald space. We consider the Finsler space whose metric function is given by $L^2 = 2\alpha\beta$.

Differentiating this covariantly with respect to the Cartan connection $C\Gamma$, we have

$$\beta a_{ij|k} y^i y^j + 2\alpha^2 b_{h|k} y^h = 0. \tag{2.3}$$

We assume that F^n is a Berwald space, since the quadratic form $a_{ij}y^iy^j$ is positive definite, there exists the covariant vector $\lambda_j(x)$ such that

$$a)\ a_{ij|k} = \lambda_k a_{ij}, \qquad b)\ b_{i|k} = -\lambda_k b_k/2. \tag{2.4}$$

From (2.4)a), we have

$$\Gamma^{*}{}_{j}{}^{i}{}_{k} = \gamma_j{}^i{}_k - (\lambda_k \delta^i_j + \lambda_j \delta^i_k - \lambda^i a_{jk})/2, \tag{2.5}$$

where $\lambda^i = a^{ij}\lambda_j$ and $\gamma_j{}^i{}_k$ are the Christoffel symbols of a_{ij}. Substituting (2.5) in (2.4)b), we obtain

$$\nabla_k b_i = -(2\lambda_k b_i + \lambda_i b_k - \lambda^m b_m a_{ki})/2, \tag{2.6}$$

where ∇_k is the covariant derivative with respect to $\gamma_j{}^i{}_k$.

Conversely if there exists the covariant vector $\lambda_j(x)$ satisfying (2.6), we put

$$\Gamma_{jk} = \gamma_j{}^i{}_k - (\lambda_k \delta^i_j + \lambda_j \delta^i_k - \lambda^i a_{jk})/2. \tag{2.7}$$

Then for the Finsler connection $\Gamma_j{}^i{}_k$, $L_{|k} = 0$ by virtue of (2.4), hence by Hashiguchi-Ichijyō theorom [2], the space is a Berwald space. Thus we have

THEOREM 2.1. *The Finsler space with a metric $L^2 = 2\alpha\beta$ is a Berwald space if and only if there exists the covariant vector $\lambda_k(x)$ such that (2.6) holds good.*

3. Projectively flat Finsler space with a metric $L^2 = 2\alpha\beta$

The condition for a Finsler space with an (α,β)-metric $L(\alpha,\beta)$ to be projectively flat was given by Matsumoto[6]. We shall here discuss that a Finsler space with an (α,β)-metric $L^2 = 2\alpha\beta$ be projectively flat.

From the fact the differential one-form $\beta = b_i(x)y^i$, we define

$$2r_{ij} = \nabla_j b_i + \nabla_i b_j, \quad 2s_{ij} = \nabla_j b_i - \nabla_i b_j,$$
$$s^i_j = a^{ir}s_{rj}, \quad b^i = a^{ir}b_r, \quad s_i = b^r s_{ri},$$
$$b^2 = a^{rs}b_r b_s, \quad \gamma_{jhk} = a_{hr}\gamma_j{}^r{}_k.$$

In the following we denote by the subscript 0 the transvection by y^i and by subscripts α and β of L the partial differentiations by α and β, respectively. In the paper [6], Matsumoto treated Finsler spaces with (α,β) of several types, and obtained the condition that such a space of each type be projectively flat. The condition is given as follows:

THEOREM 3.1[6]. *A Finsler space with an (α,β)-metric $L(\alpha,\beta)$ is projectively flat if and only if the space is covered by coordinate neighbourhoods in which the following equation is satisfied:*

$$(3.1)\quad (\gamma_0{}^i{}_0 - \gamma_{000}y^i/\alpha^2)/2 + (\alpha L_\beta/L_\alpha)s^i_0 + (L_{\alpha\alpha}/L_\alpha)(C + \alpha r_{00}/2\beta)(\alpha^2 b^i/\beta - y^i) = 0,$$

where quantities C is given by

$$(3.2)\quad \{1 + (\alpha L_{\alpha\alpha}/\beta^2 L_\alpha)(\alpha^2 b^2 - \beta^2)\}(C + \alpha r_{00}/2\beta) = (\alpha/2\beta)\{r_{00} - (2\alpha L_\beta/L_\alpha)s_0\}.$$

Now we consider a special (α,β)-metric $L^2 = 2\alpha\beta$. This metric, a kind of (α,β)-metric, was investigated by Hojo [3]. In this case we have

$$(3.3)\quad L_\alpha = \beta/L,\ \ L_\beta = \alpha/L,\ \ L_{\alpha\alpha} = -\beta^2/L^3.$$

Eliminating $C + \alpha r_{00}/2\beta$ from (3.1) and (3.2) we get

$$(3.4)\quad \{(\alpha^2\gamma_0{}^i{}_0 - \gamma_{000}y^i)LL_\alpha + 2\alpha^3 s^i_0 LL_\beta\}\{\beta^2 L^3 L_\alpha + \alpha L^3 L_{\alpha\alpha}(b^2\alpha^2 - \beta^2)\} + \alpha^3 L^3 L_{\alpha\alpha}(b^i\alpha^2 - \beta y^i)(r_{00}LL_\alpha - 2s_0\alpha LL_\beta) = 0.$$

Substituting (3.3) in (3.4), we get

$$(3.5)\quad (3\beta^2 - \alpha^2 b^2)\{\beta(\alpha^2\gamma_0{}^i{}_0 - \gamma_{000}y^i) + 2\alpha^4 s^i_0\} = \alpha^2(\alpha^2 b^i - \beta y^i)(\beta r_{00} - 2\alpha^2 s_0).$$

It is first observed in (3.5) that the term $2\alpha^6(s_0 b^i - b^2 s^i_0)$ must have a factor β, that is, we must have $\lambda^i(x)$ satisfying $s_0 b^i - b^2 s^i_0 = \beta\lambda^i$. This is written in the form $b^2 s_{ij} = s_j b_i - b_j\lambda_i$, where $\lambda_i = a_{ik}\lambda^k$. Since s_{ij} is skew-symmetric, λ_i is must be equal to s_i, as is easily seen. Thus we get

$$(3.6)\quad b^2 s_{ij} = s_j b_i - s_i b_j.$$

Then the equation (3.5) is written in the form

$$(3.7)\quad p_0\alpha^6 + p_1\alpha^4 + p_2\alpha^2 + p_3 = 0,$$

where we put

$$\begin{aligned}p_0 &= 2(s_0 b^i - b^2 s^i_0),\\ p_1 &= \beta(6\beta s^i_0 - b^2 \gamma_0{}^i{}_0 + r_{00} b^i + 2 s_0 y^i),\\ p_2 &= \beta(b^2 \gamma_{000} y^i + \beta r_{00} y^i + 3\beta^2 \gamma_0{}^i{}_0),\\ p_3 &= -3\beta^3 \gamma_{000} y^i.\end{aligned}$$

Secondly, we observe in (3.7) that p_3 must have a factor α^2. Then p_3 shows

$$\gamma_{000} = \alpha^2 \nu_0, \tag{3.8}$$

where $\nu_0 = \nu_i(x) y^i$. Substituting (3.6) and (3.8) in (3.5), we have

$$p_4 \alpha^2 + p_5 = 0, \tag{3.9}$$

where we put

$$\begin{aligned}p_4 &= b^2(\gamma_0{}^i{}_0 - \nu_0 y^i) + (r_{00} - 6\beta s_0/b^2) b^i - 2(\alpha^2 - 3\beta^2/b^2) s^i + 2 s_0 y^i,\\ p_5 &= -\beta\{3\beta(\gamma_0{}^i{}_0 - \nu_0 y^i) + r_{00} y^i\}.\end{aligned}$$

Consequently p_5 must have a factor α^2, that is, we must have $\xi^i_0 = \xi^i_j(x) y^j$ satisfying

$$3\beta(\gamma_0{}^i{}_0 - \nu_0 y^i) + r_{00} y^i = \alpha^2 \xi^i_0. \tag{3.10}$$

Transvecting this by y_i, the equation (3.8) leads us to

$$r_{00} = \xi^i_0 y_i. \tag{3.11}$$

Substituting (3.10) in (3.9), we obtain

$$\begin{aligned}b^2(\gamma_0{}^i{}_0 - \nu_0 y^i) + (r_{00} - 6\beta s_0/b^2) b^i - 2(\alpha^2 - 3\beta^2/b^2) s^i&\\ + 2 s_0 y^i = \beta \xi^i_0.&\end{aligned} \tag{3.12}$$

Multiplying this by 3β and substituting from (3.10), we are led to

$$(b^2\alpha^2 - 3\beta^2)(b^2\xi_{i0} - 6\beta s_i) = (b^2\xi_{00} - 6\beta s_0)(b^2 y_i - 3\beta b_i), \tag{3.13}$$

where we put $\xi_{ij} = a_{ir}\xi^r_j$. If we define the tensors

$$E_{ij} = b^2 a_{ij} - 3b_i b_j, \ \ Z_{ij} = b^2\xi_{ij} - 6s_i b_j,$$

then (3.13) is written in the form

$$E_{hj}Z_{ik} + E_{jk}Z_{ih} + E_{kh}Z_{ij} = A_{jk}E_{ih} + A_{kh}E_{ij} + A_{hj}E_{ik} \tag{3.14}$$

where we put $A_{jk} = (Z_{jk} + Z_{kj})/2$. It is easy to show that the tensor E_{ij} has the reciprocal $E^{ij} = (a^{ij} - 3b^i b^j/2b^2)/b^2$. Thus, transvecting (3.14) by E^{hj}, we get

$$(n+1)Z_{ik} = Z_{ki} + nZE_{ik}, \ \ nZ = E^{hj}Z_{hj},$$

which give $Z_{ik} = Z_{ki} = ZE_{ik}$. Therefore, we have

$$b^2\xi_{ik} = Z(b^2 a_{ik} - 3b_i b_k) + 6s_i b_k, \tag{3.15}$$

and (3.11) is rewritten as

$$b^2 r_{00} = Z(b^2\alpha^2 - 3\beta^2) + 6s_0\beta \tag{3.16}$$

Substituting (3.15) and (3.16) in (3.12), we have

$$b^2(\gamma_0{}^i{}_0 - \nu_0 y^i) = (Z\beta - 2s_0)y^i - Z\alpha^2 b^i + 2\alpha^2 s^i. \tag{3.17}$$

Conversely, it is easily verified that (3.5) is the consequence of (3.15) and (3.16). These equations (3.15) and (3.16) may be written, respectively, in the forms

$$b^2 r_{ij} = Z(b^2 a_{ij} - 3b_i b_j) + 3(s_i b_j + s_j b_i), \tag{3.18}$$

$$\gamma_j{}^i{}_k = \sigma_j\delta^i_k + \sigma_k\delta^i_j - a_{jk}(Zb^i - 2s^i)/b^2, \tag{3.19}$$

where $\sigma_j = (b^2\nu_j + Zb_j - 2s_j)/2b^2$. Thus we have

THEOREM 3.2. *An n-dimensional Finsler space F^n with a metric $L^2 = 2\alpha\beta$ is projectively flat if and only if we have (3.6) and (3.18), and the space is covered by coordinate neighbourhoods in which the Christoffel symbols of the associated Riemannian space with the metric α are written in the form (3.19).*

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Department of Mathematics
Kyungpook Sanup University
Kyungsan 712-701, Korea