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LINEAR SYSTEM ON THE FANO THREEFOLD

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ABSTRACT. Let X be a smooth projective threefold whose anticanonical divisor $-K_X$ is ample, i.e., Fano threefold. In this paper, we studied the linear system $|-nK_X|$ for a positive integer n. In Theorem 4, we studied the cases that $|-nK_X|$ has no base-points and the cases that $|-nK_X|$ generates the birational map. In Proposition 5, we studied the possible exceptional cases given in Theorem 4. Some results in this paper are already known, but we have gave brief proofs for those results.

Throughout this paper, we are working over the complex field $\mathbf{C}$.

The aim of this paper is to study the anticanonical linear system $|-nK_X|$ on the Fano threefold. We have investigated the base locus and the birationality of the complete linear system $|-nK_X|$ in Theorem 4. In Proposition 5, we have studied the possible exceptional cases given in the Theorem 4. Some part of the results in this paper is already studied in [2]. The proofs in [2] heavily have used the fact that the general section of the linear system is K3 surface or Del Pezzo surface and the properties of the linear systems on those surfaces. But we have used Theorem 2 introduced by Reider. Reider's theorem is quite simple to apply because it gives only numerical conditions which is easy, in some sense, to compute.

Let X be a smooth projective variety. Let's denote by K_X a canonical divisor on X. Let D be a divisor on X. Denote by $\Phi_{|D|}$ the rational map associated to the complete linear system $|D|$. Let $\mathrm{Bs}|D|$ mean the base locus of $|D|$. Let's denote by $\sim$ the linear equivalence.

In here, Fano threefold means a smooth projective threefold whose anticanonical divisor $-K_X$ is ample. From now on, let's denote by X a Fano threefold. The index r of X means the greatest integer≥ 1 such that $rH \sim -K_X$ for some divisor H on X. Clearly, H must be ample.

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THEOREM 1. (Kawamata-Viehweg vanishing theorem) *Let X be a nonsingular projective variety and D a divisor on X. If D is nef and big, then for all $i \geq 1$*

$$H^i(X, \mathcal{O}_X(K_X + D)) = 0.$$

For a proof, see Kawamata [3].

LEMMA. *Let X be Fano threefold. Then we have the following.*

(a) *For* $n \geq 1, h^0(X, \mathcal{O}_X(-nK_X)) = \frac{n(n+1)(2n+1)}{12}(-K_X{}^3) + 2n + 1$.

(b) $K_X{}^3$ *is even.*

PROOF. (a) comes from Riemann - Roch theorem and Kawamata - Viehweg vanishing theorem.

For (b), $K_X{}^3$ is even clearly since $h^0(X, \mathcal{O}_X(-K_X)) = -K_X{}^3/2+3 \in \mathbf{Z}$. □

THEOREM 2. *Let S be a nonsingular projective surface and let L be a nef divisor.*

(a) *If $L^2 \geq 5$ and p is a base point of $|K_S + L|$, then there exists an effective divisor E passing through p such that*

$$\text{either (i) } L \cdot E = 0,\ E^2 = -1 \text{ or } -2,$$
$$\text{or (ii) } L \cdot E = 1,\ E^2 = 0.$$

(b) *If $L^2 \geq 10$ and the points p, q are not separated by $|K_S + L|$, then there exists an effective divisor E on S passing through p and q such that*

$$\text{either (i) } L \cdot E = 0,\ E^2 = -1 \text{ or } -2,$$
$$\text{or (ii) } L \cdot E = 1,\ E^2 = -1 \text{ or } 0,$$
$$\text{or (iii) } L \cdot E = 2,\ E^2 = 0.$$

For a proof, see Reider [4].

THEOREM 3. *The general member of $|H|$ is smooth when $rH \sim -K_X$.*

For a proof, see Iskovskikh [2].

THEOREM 4. *Let X be a Fano threefold of index r.*

(a) *$Bs|-nK_X| = \emptyset$ for $n \geq 2$. In particular, $Bs|-K_X| = \emptyset$ when $r \geq 2$.*

(b) *$\Phi_{|-nK_X|}$ is birational for $n \geq 3$. In particular, $\Phi_{|-nK_X|}$ is birational for $n \geq 1$ when $r \geq 3$, or $r = 2$ and $-K_X{}^3 > 8$.*

PROOF. Let S be a general member of $|H|$, where $rH \sim -K_X$. By Theorem 3, S is smooth. Let n be a positive integer.

Consider the following exact sequence:

$$0 \to \mathcal{O}_X(-nK_X - S) \to \mathcal{O}_X(-nK_X) \to \mathcal{O}_S(-nK_X|_S) \to 0.$$

Since $rH \sim -K_X$, $-(n+1)K_X - S \sim (nr + r - 1)H$. Hence $-(n+1)K_X - S$ is ample since H is ample. Hence we have

$$h^1(X, \mathcal{O}_X(-nK_X - S)) = h^1(X, \mathcal{O}_X(K_X - (n+1)K_X - S)) = 0.$$

It means that the restriction map

$$H^0(X, \mathcal{O}_X(-nK_X)) \to H^0(S, \mathcal{O}_S(-nK_X|_S))$$

is surjective. So we have $\Phi_{|-nK_X|}|_S = \Phi_{|-nK_X|_S|}$. Let $R = H|_S$.

$$-nK_X|_S = (K_X + S + (rn + r - 1)H)|_S = K_S + (rn + r - 1)R.$$

Hence $\Phi_{|-nK_X|_S|} = \Phi_{|K_S + (rn+r-1)R|}$.

For (a), if $\mathrm{Bs}|-nK_X| \neq \emptyset$, then $\mathrm{Bs}|K_S + (rn + r - 1)R| \neq \emptyset$ since S contains $\mathrm{Bs}|-nK_X|$ as set. We have that $((rn + r - 1)R)^2 \geq 5$ for any positive integer n when $r \geq 2$. When $r = 1$, $(nR)^2 \geq 5$ for $n \geq 2$, since $R^2 = S^3 = (-K_X)^3$ is even.

If $\mathrm{Bs}|K_S + (rn+r-1)R| \neq \emptyset$, then Theorem 2 guarantees the existence of an effective divisor E on S with $(rn + r - 1)R \cdot E \leq 1$. But if $r \geq 2$ or $n \geq 2$, then $(rn + r - 1)R \cdot E \geq 2$. It is impossible. Therefore $\mathrm{Bs}|K_S + (rn + r - 1)R|$ must be empty. Thus we have $\mathrm{Bs}|-nK_X| = \emptyset$.

For (b), we are going to prove the following claim.

CLAIM. *If* $\Phi_{|K_S+(rn+r-1)R|}$ *is birational, then* $\Phi_{|-nK_X|}$ *is birational.*

PROOF. Since $-nK_X \sim (rn-1)H+S$, choose a section $t \in H^0(X, \mathcal{O}_X((rn-1)H))$ which determines $(rn-1)H$. If $\Phi_{|-nK_X|}$ is not birational, there exists a nonempty Zariski open set U in X such that $U \cap (rn-1)H = \emptyset$ and the base locus of $|-nK_X|$ is disjoint from U and such that for any $x \in U$, there is some $y \in U$ distinct from x with $\Phi_{|-nK_X|}(x) = \Phi_{|-nK_X|}(y)$. We may also assume that $U \cap S \neq \emptyset$ because S is a general member of $|H|$. So take a section $s \in H^0(X, \mathcal{O}_X(H))$ which determines S. For any $x \in U \cap S$, there exists $y \in U$ distinct from x with $\Phi_{|-nK_X|}(x) = \Phi_{|-nK_X|}(y)$. Since $ts \in H^0(X, \mathcal{O}_X((rn-1)H+S)) = H^0(X, \mathcal{O}_X(-nK_X))$, there exists $\alpha \in \mathbf{C} \setminus \{0\}$ such that $t(x)s(x) = \alpha t(y)s(y)$. Since we have $U \cap (rn-1)H = \emptyset$, we have $t(y) \neq 0$ and $s(x) = 0$. Therefore $s(y) = 0$. It means $y \in S$. In other words, $\Phi_{|-nK_X|}|_S = \Phi_{|K_S+(rn+r-1)R|}$ is not birational. □

If $((nr+r-1)R)^2 \geq 10$ and $nr+r-1 \geq 3$, then $\Phi_{|K_S+(rn+r-1)R|}$ is birational. If not, then by Theorem 2, there is an effective divisor E on S with $(nr+r-1)R \cdot E \leq 2$. But it is impossible because $nr+r-1 \geq 3$ and R is ample. Hence, it is enough to show that $((nr+r-1)R)^2 \geq 10$ and $nr+r-1 \geq 3$.

If $r \geq 3$, $3n+3-1 \geq 4$ for any positive integer $n \geq 1$. Hence $((3n+3-1)R)^2 \geq 10$.

If $r = 2$, $2n+2-1 \geq 3$ for any positive integer $n \geq 1$ and $((2n+2-1)R)^2 \geq 10$ for $n \geq 2$ or $R^2 \geq 2$. Since $R^2 = H^3$, we have $R^2 \geq 2$ for $H^3 \geq 2$. Recall that $-K_X{}^3 = 8H^3$ when $r = 2$. Thus $R^2 \geq 2$ if $-K_X{}^3 > 8$.

If $r = 1$, then $(n+1-1) \geq 3$ for $n \geq 3$. Since $R^2 = H^3 = -K_X{}^3$, R^2 is even. Thus $(nR)^2 \geq 10$ for $n \geq 3$.

Hence $\Phi_{|K_S+(rn+r-1)R|}$ is birational. Thus by claim, $\Phi_{|-nK_X|}$ is birational for the cases given in the theorem. □

Let's investigate the possible exceptional cases given in Theorem 4, i. e., when $r = 2$, $n = 1$ and $r = 1$, $n = 2$ or 1. But in this paper, we are going to look at only two cases $r = 2$, $n = 1$ and $r = 1$, $n = 2$.

PROPOSITION 5. *Let* X *be a Fano threefold of index* r.

(a) *Let* $r = 2$. *If* $\Phi_{|-K_X|}$ *is not birational,* $\Phi_{|-K_X|}$ *is a morphism of* *degree 2.*

(b) *Let* $r = 1$. *If* $\Phi_{|-2K_X|}$ *is not birational,* $\Phi_{|-2K_X|}$ *is a morphism of degree 2 or 3. In particular, the degree of* $\Phi_{|-2K_X|}$ *is 2 if* $-K_X{}^3 > 4$.

PROOF. We are going to use the same notations in Theorem 4.

For (a), $|-K_X|$ is base-point-free by Theorem 4 since $r = 2$. Hence the image W of $\Phi_{|-K_X|}$ has the dimension 3. Hence we have

$$\begin{aligned} -K_X{}^3 &\geq \deg W \deg\Phi_{|-K_X|} \\ &\geq (h^0(X, \mathcal{O}_X(-K_X)) - 3)\deg\Phi_{|-K_X|} \\ &\geq 1/2(-K_X{}^3)\deg\Phi_{|-K_X|}. \end{aligned} \tag{1}$$

Thus we have $\deg\Phi_{|-K_X|} \leq 2$.

For (b), $|-2K_X|$ is base-point-free by Theorem 4 since $r = 1$ and $n = 2$. Hence, by the same arguments, we have $\deg\Phi_{|-2K_X|} \leq 3$. Restrict $|-2K_X|$ to the general member S of $|-2K_X|$. Then we have $-2K_X|_S = 2(K_X - 2K_X)|_S = 2K_S$. From the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(-2K_X) \to \mathcal{O}_S(2K_S) \to 0.$$

Since $h^1(X, \mathcal{O}_X) = 0$, we have $\Phi_{|-2K_X|}|_S = \Phi_{|2K_S|}$. Since $K_S = -K_X|_S$, K_S is ample, so S is the minimal surface of the general type. It is known that the degree of $\Phi_{|2K_S|}$ is less than or equal to 2 if $K_S^2 \geq 10$. Since we have $K_S^2 = -2K_X{}^3$, the degree of $\Phi_{|-2K_X|}$ is 2 if $-K_X{}^3 > 4$. $\square$

REMARK. Actually, in (a) of Proposition 5, when $\deg\Phi_{|-K_X|} = 2$, we have the equality in (1). It is well known about this case. Hence we may have more information from this fact, but we are not going to talk about. This case was well treated in [2].

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