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A POISSON EQUATION ASSOCIATED WITH AN INTEGRAL KERNEL OPERATOR

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Abstract. Suppose the kernel function κ belongs to $\mathcal{S}(\mathbb{R}^2)$ and is symmetric such that $< x \otimes x, \kappa > \geq 0$ for all $x \in \mathcal{S}'(\mathbb{R})$. Let $\mathcal{A}$ be the class of functions f such that the function f is measurable on $\mathcal{S}'(\mathbb{R})$ with $\int_{\mathcal{S}'(\mathbb{R})} |f((I+tK)^{\frac{1}{2}}x)|^2 d\mu(x) < M$ for some $M > 0$ and for all $t > 0$, where K is the integral operator with kernel function κ. We show that the λ-potential $G_K f$ of f is a weak solution of $(\lambda I - \frac{1}{2}\tilde{\Xi}_{0,2}(\kappa))u = f$.

1. Introduction

Several generalizations of Schwartz distribution theory to infinite dimensional spaces have appeared in the recent years since the white noise calculus has launched out by Hida in 1975.

In the infinite dimensional space the Lebesgue measure as one in finite dimensional space doesn't exist. So we can't expect the same results as ones in finite dimensional case. Gross[2] has developed a measure theoretic structure well-suited to the study of potential theory on an arbitrary separable Banach space. It is used to extend the potential theory in $\mathbb{R}^n$ to an infinite dimensional spaces.

Gross[2] has studied the infinite dimensional analogue of the heat equation $\frac{\partial u(t,x)}{\partial t} = \frac{1}{2}\Delta u(t,x)$ and Poisson equation $\Delta u = f$ on a abstract Wiener space, where Δu is the Gross Laplacian. Kuo[6], Piech[12] and Lee[9,10] have proved some theorems in potential theory with respect to Ornstein-Uhlenbeck process in abstract Wiener space. Kuo[7], Chung

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and Ji[1] proved some similar theorems associated with the Gross Laplacian and Volterra Laplacian in white noise setting. Kang[4] has proved that the solutions of generalized Poisson equations associated with number operator are represented by the $\lambda-$ potentials in white noise setting.

Gross Laplacian is defined on the space of test functionals $(\mathcal{S})$ and it has no extension to $(\mathcal{S})^*$. We know that the Gross Laplacian is a singular operator given by the trace kernel function. Furthermore, the Hida distribution has no compact support and so we can not use the same method that Gross has used to find the solution of Poisson equation.

In this paper we consider the integral kernel operator with a kernel function as a symmetric function in $\mathcal{S}(\mathbb{R}^2)$. We show that the solution of the Poisson equation associated the integral kernel operator, $(\lambda I - \frac{1}{2}\tilde{\Xi}_{0,2}(\kappa))u = f$, is represented by the λ- potential.

2. White noise calculus

We shall shortly recall some necessary facts from white noise analysis [3,7,11].

Let us consider the real Schwartz space $\mathcal{S}(\mathbb{R})$ and its dual space $\mathcal{S}'(\mathbb{R})$ is equipped with the Borel σ-algebra $\mathcal{B}$ of weak topology and with the white noise measure μ given by

$$\int_{\mathcal{S}'(\mathbb{R})} e^{i<x,\xi>} d\mu(x) = e^{-\frac{1}{2}|\xi|_2^2} \quad \xi \in \mathcal{S}(\mathbb{R}),$$

where $|\cdot|_2$ denotes the norm on $L^2(\mathbb{R})$ and $<\cdot,\cdot>$ is the dual pairing between $\mathcal{S}'(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$. Then $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})$ is a Gelfand triple.

Let A denote the self-adjoint operator $-\frac{d^2}{dt^2} + 1 + t^2$ on $L^2(\mathbb{R})$. For each $p \geq 0$, let $\mathcal{S}_p(\mathbb{R})$ be the $L^2(\mathbb{R})$-domain of A^p and the norm on it be defined by

$$|f|_{2,p} = |A^p f|_2. \tag{2-1}$$

Then $\mathcal{S}_p(\mathbb{R})$ is a Hilbert space and the dual of $\mathcal{S}_p(\mathbb{R})$ is denoted by $\mathcal{S}_{-p}(\mathbb{R})$. Moreover, we have $\mathcal{S}_q(\mathbb{R}) \subset \mathcal{S}_p(\mathbb{R})$ for $p < q$ and

$$\mathcal{S}(\mathbb{R}) = \bigcap_{p\geq 0} \mathcal{S}_p(\mathbb{R}), \qquad \mathcal{S}'(\mathbb{R}) = \bigcup_{p\geq 0} \mathcal{S}_{-p}(\mathbb{R}).$$

The space $\mathcal{S}(\mathbb{R})$ is often regarded as a nuclear space with the family $\{|\cdot|_{2,p} : p \geq 0\}$ of norms and $(L^2(\mathbb{R}), \mathcal{S}_p(\mathbb{R}))$ is an abstract Wiener space for any $p \geq \frac{1}{2}$.

For $p \leq 0$, we can still use (2-1) to define the norm $|\cdot|_{2,p}$ on $\mathcal{S}_p(\mathbb{R})$.

For $n \geq 0$, we let

$$\mathcal{S}_p(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : |f|_{2,p} = |(A^p)^{\otimes n} f|_2 < \infty\}.$$

Let (L^2) be the Hilbert space of complex- valued μ-square integrable functions with norm denoted by $\|\cdot\|_2$. The well-known Wiener-Ito theorem states that the space (L^2) has the following orthogonal decomposition:

$$(L^2) = \bigoplus_{n=0}^{\infty} K_n,$$

where K_n consists of n-fold Wiener integrals $I_n(f)$, $f \in \hat{L}_c^2(\mathbb{R}^n)$, the symmetric complex-valued L^2-functions on $\mathbb{R}^n$. It is well-known that $I_n f(x) =<: x^{\otimes n} :, f >$, where $: x^{\otimes n} :$ is the Wick ordering.

Each $\varphi \in (L^2)$ can be represented uniquely by

$$\begin{aligned} \text{(2–2)} \qquad \varphi(x) &= \sum_{n=0}^{\infty} I_n f_n(x) \\ &= \sum_{n=0}^{\infty} <: x^{\otimes n} :, f_n > \qquad f_n \in \hat{L}_c^2(\mathbb{R}^n) \end{aligned}$$

and, moreover, we have

$$\| \varphi \|_2^2 = \sum_{n=0}^{\infty} n! |f_n|^2_{L_c^2(\mathbb{R}^n)}.$$

The second quantization $\Gamma(A)$ of A is densely defined on (L^2) as follows. For φ given as in (2-2), we define

$$\begin{aligned} (\Gamma(A)\varphi)(x) &= \sum_{n=0}^{\infty} I_n(A^{\otimes n} f_n)(x) \\ &= \sum_{n=0}^{\infty} <: x^{\otimes n} :, A^{\otimes n} f_n > . \end{aligned}$$

For $p \in \mathbb{R}$, let $\| \varphi \|_{2,p} \equiv \| \Gamma(A)^p \varphi \|_2$. Then we have $\| \varphi \|_2 \leq \| \varphi \|_{2,p}$ for $p \geq 0$. Now we can define the Sobolev space $(\mathcal{S})_p$ for $p \geq 0$ by

$$(\mathcal{S})_p = \{\varphi \in (L^2); \| \varphi \|_{2,p} < \infty\}.$$

Note that $(\mathcal{S})_p$ is a complex Hilbert space with the norm $\| \cdot \|_{2,p}$. For $p < 0$, let $(\mathcal{S})_p$ be the completion of (L^2) with respect to $\| \varphi \|_{2,p}$. The dual space $(\mathcal{S})_p^*$ of $(\mathcal{S})_p$ is $(\mathcal{S})_{-p}, p \geq 0$.

Let $(\mathcal{S})$ be the projective limit of $\{(\mathcal{S})_p; p \geq 0\}$. Then $(\mathcal{S})$ is a Fréchet algebra and the dual space $(\mathcal{S})^*$ of $(\mathcal{S})$ is the union of $\{(\mathcal{S})_p^*; p \geq 0\}$ and we get the following continuous inclusions

$$(\mathcal{S}) \subset (\mathcal{S})_p \subset (L^2) \equiv (L^2)^* \subset (\mathcal{S})_p^* \subset (\mathcal{S})^*, \quad p \geq 0.$$

The elements of $(\mathcal{S})$ and $(\mathcal{S})^*$ are called the test functionals and Hida distributions, respectively. The dual pairing of $(\mathcal{S})^*$ and $(\mathcal{S})$ is denoted by $\ll \cdot, \cdot \gg$, which is associated with the Gaussian measure μ.

For each $x \in \mathcal{S}'(\mathbb{R})$ the Gateaux differentiation D_x in the direction x is a continuous linear operator from $(\mathcal{S})$ into itself. Its adjoint D_x^* is continuous from $(\mathcal{S})^*$ into itself. In particular, the *white noise differentiation* $\partial_t \equiv D_{\delta_t}$ is continuous from $(\mathcal{S})$ into itself. Its adjoint operator ∂_t^* is continuous from $(\mathcal{S})^*$ into itself. If $\xi \in \mathcal{S}(\mathbb{R})$, then D_ξ extends by continuity to a continuous linear operator $\tilde{D}_\xi$ from $(\mathcal{S})^*$ into itself.

Let (H, B) be an abstract Wiener space. Suppose φ is a twice Gross differentiable function defined on B such that $\varphi''(x)$ is a trace class operator of H. Then the Gross Laplacian[7] $\Delta_G \varphi$ of φ is defined by:

$$(\Delta_G \varphi)(x) = \text{trace}_H \varphi''(x).$$

Note that $(L^2(\mathbb{R}), \mathcal{S}_p'(\mathbb{R}))$ is an abstract Wiener space for any $p \geq \frac{1}{2}$. Therefore, we can define Δ_G acting on ordinary Brownian functionals. The set $\{x(t) : t \in \mathbb{R}\}$ is taken as a coordinate system in white noise calculus. Thus it is desirable to express the Gross Laplacian Δ_G in terms of the Hida differentiation ∂_t and its adjoint ∂_t^*. We note [11] that the Gross Laplacian and its adjoint are identified as

$$\Delta_G \varphi = \int_{\mathbb{R}} \partial_t^2 \varphi dt, \qquad \varphi \in (\mathcal{S})$$

and

$$\Delta_G^*\Phi = \int_{\mathbb{R}} (\partial_t^*)^2 \Phi dt, \qquad \Phi \in (\mathcal{S})^*.$$

Let $\theta(\mathbf{s},\mathbf{t}) \in \mathcal{S}'(\mathbb{R}^{j+k}), \mathbf{s} = (s_1, s_2, ..., s_j), \mathbf{t} = (t_1, t_2, ... t_k)$. The following integral kernel operator is introduced in [3,11],

$$\Xi_{j,k}(\theta) = \int_{\mathbb{R}^{j+k}} \theta(\mathbf{s},\mathbf{t}) \partial_{s_1}^* \cdots \partial_{s_j}^* \partial_{t_1} \cdots \partial_{t_k} d\mathbf{s} d\mathbf{t}.$$

The operator $\Xi_{j,k}(\theta)$ is continuous from $(\mathcal{S})$ into $(\mathcal{S})^*$. For example, let τ be the element in $\mathcal{S}'(\mathbb{R}^2)$, called *trace*, defined by

$$< \tau, f >= \int_{\mathbb{R}} f(t,t)dt, \quad f \in \mathcal{S}(\mathbb{R}^2).$$

Then the associated operator $\Xi_{0,2}(\tau)$ is continuous from $(\mathcal{S})$ into itself and $\Xi_{2,0}(\tau)$ is continuous from $(\mathcal{S})^*$ into $(\mathcal{S})^*$. These operators can be identified as the Gross Laplacian and its adjoint, i.e.,

$$\Delta_G = \Xi_{0,2}(\tau) = \int_{\mathbb{R}^2} \tau(s,t) \partial_s \partial_t ds dt$$

$$\Delta_G^* = \Xi_{2,0}(\tau) = \int_{\mathbb{R}^2} \tau(s,t) \partial_s^* \partial_t^* ds dt$$

In particular, the Gross Laplacian is a singular operator in sense that the kernel function τ is a delta function.

3. Poisson equation with the integral kernel operator

Suppose the kernel function κ belongs to $\mathcal{S}(\mathbb{R}^2)$ and is symmetric such that $< x \otimes x, \kappa > \geq 0$ for all $x \in \mathcal{S}'(\mathbb{R})$. Let $\mathcal{A}$ be the class of functions f such that the function f is measurable on $\mathcal{S}'(\mathbb{R})$ with $\int_{\mathcal{S}'(\mathbb{R})} |f((I + tK)^{\frac{1}{2}} x)|^2 d\mu(x) < M$ for some $M > 0$ and for all $t > 0$, where K is the integral operator with kernel function κ.

Note that the integral kernel operator $\Xi_{0,2}(\kappa) = \int_{\mathbb{R}^2} \kappa(s_1, s_2) \partial_{s_1} \partial_{s_2} ds_1 ds_2$ is continuous from $(\mathcal{S})$ into itself. Moreover, $\Xi_{0,2}(\kappa)$ extends to a continuous linear operator $\tilde{\Xi}_{0,2}(\kappa)$ from $(\mathcal{S})^*$ into itself[8].

LEMMA 3.1[8]. *Let K be a continuous linear operator from $\mathcal{S}'(\mathbb{R})$ into $\mathcal{S}(\mathbb{R})$ such that it is positive and self-adjoint on $L^2(\mathbb{R})$. Let f be a measurable function on $\mathcal{S}'(\mathbb{R})$ with $\int_{\mathcal{S}'(\mathbb{R})} |f((I+tK)^{\frac{1}{2}}x)|^2 d\mu(x) < \infty$ for all $t > 0$. Then we have*

$$\int_{\mathcal{S}'(\mathbb{R})} \int_{\mathcal{S}'(\mathbb{R})} |f(x+(tK)^{\frac{1}{2}}y)|^2 d\mu(x) d\mu(y) = \int_{\mathcal{S}'(\mathbb{R})} |f((I+tK)^{\frac{1}{2}}x)|^2 d\mu(x).$$

Let $f \in \mathcal{A}$. We define $P_{t,K}f$ as following;

(3–1)

$$\ll P_{t,K}f, \phi \gg = \int_{\mathcal{S}'(\mathbb{R})} \int_{\mathcal{S}'(\mathbb{R})} \phi(x) f(x+(tK)^{\frac{1}{2}}y) d\mu(y) d\mu(x), \quad \phi \in (\mathcal{S})$$

LEMMA 3.2. *$P_{t,K}f$ is well-defined and is a continuous linear functional on $(\mathcal{S})$, i.e. $P_{t,K}f \in (\mathcal{S})^*$.*

PROOF. Note that for $\phi \in (\mathcal{S})$,

$$\begin{aligned} | \ll P_{t,K}f, \phi \gg | &= \left| \int_{\mathcal{S}'(\mathbb{R})} \int_{\mathcal{S}'(\mathbb{R})} \phi(x) f(x+(tK)^{\frac{1}{2}}y) d\mu(x) d\mu(y) \right| \\ &= \int_{\mathcal{S}'(\mathbb{R})} \int_{\mathcal{S}'(\mathbb{R})} |\phi(x) f(x+(tK)^{\frac{1}{2}}y)| d\mu(x) d\mu(y) \\ &\leq \left(\int_{\mathcal{S}'(\mathbb{R})} \int_{\mathcal{S}'(\mathbb{R})} |\phi(x)|^2 d\mu(x) d\mu(y) \right)^{\frac{1}{2}} \\ &\qquad \left(\int_{\mathcal{S}'(\mathbb{R})} \int_{\mathcal{S}'(\mathbb{R})} |f(x+(tK)^{\frac{1}{2}}y)|^2 d\mu(x) d\mu(y) \right)^{\frac{1}{2}} \\ &\leq \left\{ \int_{\mathcal{S}'(\mathbb{R})} |\phi(x)|^2 d\mu(x) \right\}^{\frac{1}{2}} \left\{ \int_{\mathcal{S}'(\mathbb{R})} |f((I+tK)^{\frac{1}{2}}x)|^2 d\mu(x) \right\}^{\frac{1}{2}} \\ &= \sqrt{M} \parallel \phi \parallel_2 \leq \sqrt{M} \parallel \phi \parallel_{2,p} \end{aligned}$$

for any $p \geq 0$, where the inequalities are justified by Hölder's inequality and Lemma 3.1.

PROPOSITION 3.3[8]. *Let K be a continuous linear operator from $\mathcal{S}'(\mathbb{R})$ into $\mathcal{S}(\mathbb{R})$ such that it is positive and self-adjoint on $L^2(\mathbb{R})$. Let f be a measurable function on $\mathcal{S}'(\mathbb{R})$ with $\int_{\mathcal{S}'(\mathbb{R})} |f((I+tK)^{\frac{1}{2}}x)|^2 d\mu(x) < \infty$ for all $t > 0$. Then the equation*

$$\frac{\partial u(t,x)}{\partial t} = \frac{1}{2}\tilde{\Xi}_{0,2}(\kappa)u(t,x), \quad u(0,\cdot) = f$$

has a unique weak solution. Moreover, the solution is given by

$$u(t,x) = \int f(x + (tK)^{\frac{1}{2}}y)d\mu(y).$$

REMARK. By Lemma 3.2 we note that $P_{t,K}f$ is in $(\&)^*$ for all $f \in \mathcal{A}$ and that we have an another expression [8] of (3.1);

(3–2)

$$\ll P_{t,K}f, \phi \gg = \int_{\mathcal{S}'(\mathbb{R})}\int_{\mathcal{S}'(\mathbb{R})} f(x)\phi(x - \sqrt{tK}y)w(t,x,y)d\mu(x)d\mu(y)$$

where $w(t,x,y) = \exp[\sqrt{t} < x, \sqrt{K}y > -\frac{1}{2}t < y, Ky >]$.

We define a functional $G_K f : (\mathcal{S}) \to \mathbb{R}$ by

$$\ll G_K f, \phi \gg = \int_0^\infty \mathrm{e}^{-\lambda t} \ll P_{t,K}f, \phi \gg dt.$$

LEMMA3.4. *Let $f \in \mathcal{A}$. Then $G_K f$ is a continuous linear functional on $(\mathcal{S})$, i.e. $G_K f \in (\mathcal{S})^*$.*

PROOF. We note from the proof of Lemma 3.2 that $| \ll P_{t,K}f, \phi \gg | \leq \sqrt{M} \parallel \phi \parallel_{2,p}$. Thus we easily see that for every $p \geq 0$

$$\begin{aligned}
| \ll G_K f, \phi \gg | &= | \int_0^\infty \mathrm{e}^{-\lambda t} \ll P_{t,K}f, \phi \gg dt| \\
&\leq \int_0^\infty \mathrm{e}^{-\lambda t} | \ll P_{t,K}f, \phi \gg |dt \\
&\leq \sqrt{M} \parallel \phi \parallel_{2,p} \int_0^\infty \mathrm{e}^{-\lambda t} dt \\
&\leq \sqrt{M} \parallel \phi \parallel_{2,p} .
\end{aligned}$$

Let $\vartheta_\kappa(x) \equiv < x, Kx > -\mathrm{Trace}_{L^2(\mathbb{R})}K$ and let ∇_κ be the operator $\nabla_\kappa\phi(x) = < Kx, \phi'(x) >$ on $(\mathcal{S})$. Define

$$\Lambda_\kappa = \Xi_{0,2}(\kappa) - 2\nabla_\kappa + \vartheta_\kappa.$$

It is easy to check [8] that Λ_κ is continuous from $(\mathcal{S})$ into itself and $\tilde{\Xi}_{0,2}(\kappa) = \Lambda_\kappa^*$.

THEOREM 3.5. *Let K be a continuous linear operator from $\mathcal{S}'(\mathbb{R})$ into $\mathcal{S}(\mathbb{R})$ such that it is positive and self-adjoint on $L^2(\mathbb{R})$. Let $f \in \mathcal{A}$. Then $u = G_K f$ is a weak solution of the equation*

$$(\lambda I - \frac{1}{2}\tilde{\Xi}_{0,2}(\kappa))u = f.$$

PROOF. First we need to show that $P_{t,K}f$ converges weakly to f as $t \to 0$, i.e.

$$\text{(3–3)} \qquad \lim_{t\to 0} \ll P_{t,K}f, \phi \gg = \ll f, \phi \gg, \quad \text{for all } \phi \in (\mathcal{S}).$$

In fact, this follows from (3.2), the Lebesgue Dominated Convergence Theorem and the inequality in the proof of Lemma 3.2. Now using the equality $\tilde{\Xi}_{0,2}(\kappa) = \Lambda_\kappa^*$, Proposition 3.3, the integral by parts formula and (3.3), we have for all $\phi \in (\mathcal{S})$

$$\begin{aligned}
\ll \frac{1}{2}\tilde{\Xi}_{0,2}(\kappa)G_K f, \phi \gg &= \ll G_K f, \frac{1}{2}\Lambda_\kappa\phi \gg \\
&= \int_0^\infty e^{-\lambda t} \ll P_{t,K}f, \frac{1}{2}\Lambda_\kappa\phi \gg dt \\
&= \int_0^\infty e^{-\lambda t} \ll \frac{1}{2}\tilde{\Xi}_{0,2}P_{t,K}f, \phi \gg dt \\
&= \int_0^\infty e^{-\lambda t} \ll \frac{\partial}{\partial t}P_{t,K}f, \phi \gg dt
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{\epsilon\to 0}\lim_{R\to\infty}\int_\epsilon^R e^{-\lambda t}\frac{\partial}{\partial t} \ll P_{t,K}f,\phi \gg dt \\
&= \lim_{\epsilon\to 0}\lim_{R\to\infty}[e^{-\lambda t} \ll P_{t,K}f,\phi \gg]_\epsilon^R \\
&\qquad + \lambda \lim_{\epsilon\to 0}\lim_{R\to\infty}\int_\epsilon^R e^{-\lambda t} \ll P_{t,K}f,\phi \gg dt \\
&= - \ll f,\phi \gg + \lambda \ll G_K f,\phi \gg \\
&= \ll -f + \lambda G_K f,\phi \gg .
\end{aligned}
$$

Hence the desired result is obtained.

References

1. D. M. Chung and U. C. Ji, *Groups of operators on white noise functionals and applications to Cauchy problems in white noise analysis*, Preprint.
2. L. Gross, *Potential theory on Hilbert space*, J.Functional Analysis **1** (1967), 123–181.
3. T. Hida, H. H. Kuo, J. Potthoff and L. Streit, *"White noise: An Infinite Dimensional Calculus"*, Monograph in preparation.
4. S. J. Kang, *Heat and Poission equations associated with Number operator in White Noise Analysis*, Soochow J. of Math. **20** (1994), 45-55.
5. H. H. Kn, *Gaussian measures in Banach spaces*, Lecture Notes in Math. Springer-Verlag **463** (1975).
6. ______, *Potential theory associated with Uhlenbeck-Ornstein process*, J. Functional analysis **21** (1976), 63–75.
7. ______, *Lecture on White noise analysis*, Soochow J. Math. **18** (1992), 229–300.
8. ______, *Fourier transform and heat equation in white noise analysis*, Preprint.
9. Yuh-Jia Lee, *Fundamental solutions for differential equations associated with the number operator*, Trans. Amer. Math. Soc. **268** (1981), 467-476.
10. ______, *A correction and some additions to" Fundamental solutions for differential equations associated with the number operator"*, Trans. Amer. Math. Soc. **276** (1983), 621-624.
11. N. Obata, *White noise calculus and Fock space, in "Lecture notes in Math."*, Springer-verlag, NewYork **1577** (1994).
12. M. A. Piech, *Parabolic equations associated with the number operator*, Trans. Amer. Math. Soc. **194** (1974), 213-222.

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