Comm. Korean Math. Soc. **11** (1996), No. 2, pp. 359–366

TEICHMULLER EXTREMAL MAPPINGS ON THE UNIT DISK*

J. H. KEUM AND M. K. LEE

ABSTRACT. In this paper, we provide two Teichmüller extremal mappings of the unit disk, having different boundary values but the same dilatation.

1. Introduction

Let P_K be the parabolic region in the complex z-plane

$$P_K = \left\{ z = x + iy | 4\frac{1}{K}\left(x + \frac{1}{K}\right) \geq y^2, \quad x > -\frac{1}{K} \right\}.$$

In [2], E. Blum had shown that the horizontal stretch

$$\sigma(x + iy) = Kx + K - \frac{1}{K} + iy$$

parallel to the axis of the parabola of P_1 is a Teichmüller extremal mapping with dilatation K from the parabolic region P_1 onto P_K. (For the definition of Teichmüller extremality, see §2.)

It is well known that the map

$$\tau(x + iy) = \sqrt{K}x + \frac{i}{\sqrt{K}}y$$

is a Teichmüller extremal mapping with dilatation K from the strip $0 < y < 1$ onto the strip $0 < y < \frac{1}{\sqrt{K}}$. In [4], E. Reich found two surjective conformal mappings H_1 and H_2

Received October 16, 1995. Revised January 12, 1996.
1991 AMS Subject Classification: 30C75,30F60.
Key words and phrases: Teichmüller extremal mapping, conformal mapping, quasiconformal mapping.
* Partially supported by GARC.

$$H_1 : P_1 \to \text{the strip } 0 < y < 1$$
$$H_2 : P_K \to \text{the strip } 0 < y < \frac{1}{\sqrt{K}}$$

such that the mappings $H_2^{-1} \circ \tau \circ H_1$ and σ have the same values on the boundary ∂P_1, but have different complex dilatations in P_1. This means that, for a parabolic region, there exist at least two Teichmüller extremal mappings with the same boundary values and the same dilatation K.

In principle, all quasiconformal mappings of a parabolic region can be normalized to self-quasiconformal mappings of the unit disk (Riemann mapping theorem) and hence one can deduce that on the unit disk there exist at least two Teichmüller extremal mappings with the same boundary values and the same dilatation. However, such a normalization process is obscure and no conformal isomorphism between a parabolic region and the unit disk has ever been provided explicitly. This was also pointed out by E. Reich in [4].

In this paper we normalize the map $H_2^{-1} \circ \tau \circ H_1$ above to give a Teichmüller extremal mapping of the unit disk. (Proposition 3.1, Corollary 3.2). Unfortunately we have failed to normalize the other map σ. This is mainly due to the obscurity of the normalization process. Instead, we construct another Teichmüller extremal mapping of the unit disk having different boundary values but the same dilatation K. (Proposition 3.2).

2. Teichmüller spaces and Teichmüller extremal mappings

In this section, we recall some definitions about Teichmüller space and Teichmüller extremal mapping. For more details, we refer to [1], [3], [4].

Let p and n be nonnegative integers with $2p - 2 + n > 0$. In order to define Teichmüller space $T_{p,n}$, we choose a topological oriented surface $\Sigma = \Sigma_{p,n}$ obtained by removing n distinct points from a compact surface $\hat{\Sigma}$ of genus p. We call two orientation - preserving homeomorphisms of Riemann surfaces S, S' onto Σ

$$f : S \to \Sigma \quad \text{and} \quad f' : S' \to \Sigma$$

equivalent if there is a commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\ f\ } & \Sigma \\ \varphi\big\downarrow & & \big\downarrow\psi \\ S' & \xrightarrow{\ f'\ } & \Sigma \end{array}$$

where φ is a conformal isomorphism and ψ a homeomorphism homotopic to the identity. The equivalence classes

$$[f] = [f : S \to \Sigma]$$

of homeomorphisms of compact Riemann surfaces of genus p with n punctures onto Σ are called *marked Riemann surfaces* of type (p, n) or points of the *Teichmüller space* $T_{p,n}$. The topology in $T_{p,n}$ can be derived from the Teichmüller metric. The Teichmüller distance between two points $[f_1]$ and $[f_2]$ of $T_{p,n}$ is defined by

$$< [f_1], [f_2] >= \frac{1}{2} \log \inf K(f)$$

where f ranges over all quasiconformal mappings in the homotopy class of $f_1^{-1} \circ f_2$, and $K(f)$ is the (maximal) dilatation of f. And let

$$K^*_{f_1^{-1}\circ f_2} = \inf K(f) = e^{2<[f_1],[f_2]>}$$

denote the extremal dilatation corresponding to $f_1^{-1} \circ f_2$. If a quasiconformal map $f : S_1 \to S_2$ is homotopic to $f_2^{-1} \circ f_1$ and $K(f) = K^*_{f_1^{-1}\circ f_2}$, then f is called an *extremal map*.

Locally the extremality can be defined as follows. Let S_1 and S_2 be two regions in the complex z-plane. Then a quasiconformal map $f : S_1 \to S_2$ with (maximal) dilatation K is called *extremal* if every quasiconformal mapping $\tilde{f}$ which agrees with f on the boundary of S_1 and is homotopic to f has a maximal dilatation $\tilde{K} \geq K$. An extremal quasiconformal map is called *uniquely extremal* if the strict inequality $\tilde{K} > K$ holds whenever $\tilde{f} \neq f$.

A quasiconformal map f is called a *Teichmüller mapping* if its complex dilatation has the form

$$\frac{f_{\bar z}}{f_z} = \chi(z) = k\frac{\overline{\phi(z)}}{|\phi(z)|}$$

where k is a constant with $0 < k < 1$, and ϕ is a holomorphic quadratic differential. We call f a *Teichmüller extremal mapping* if f is both Teichmüller and extremal.

3. Two Teichmüller extremal mappings of the unit disk having the samedilatation K

LEMMA 3.1. *Let $f : S_1 \to S_2$ be a Teichmüller mapping associated with a holomorphic quadratic differential ϕ. Suppose that $F : \tilde{S}_1 \to S_1$ and $G : S_2 \to \tilde{S}_2$ are conformal. Then $\tilde{f} = G \circ f \circ F : \tilde{S}_1 \to \tilde{S}_2$ is a Teichmüller mapping associated with $\tilde{\phi} = F_z(z)^2\phi(F(z))$.*

PROOF. Since $F_{\bar z} = 0$ and $G_{\bar z} = 0$, we have

$$\begin{aligned}
\tilde{f}_{\bar z} &= G_z(f \circ F) \cdot (f \circ F)_{\bar z} + G_{\bar z}(f \circ F) \cdot \overline{(f \circ F)}_{\bar z} \\
&= G_z(f \circ F) \cdot \left[f_z(F) \cdot F_{\bar z} + f_{\bar z}(F) \cdot \overline{F}_{\bar z}\right] \\
&= G_z(f \circ F) \cdot f_{\bar z}(F) \cdot \overline{(F_z)} \\
\tilde{f}_z &= G_z(f \circ F) \cdot \left[f_z(F) \cdot F_z + f_{\bar z}(F) \cdot \overline{F}_z\right] \\
&= G_z(f \circ F) \cdot \left[f_z(F) \cdot F_z + f_{\bar z}(F) \cdot \overline{(F_{\bar z})}\right] \\
&= G_z(f \circ F) \cdot f_z(F) \cdot F_z.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\tilde{f}_{\bar z}}{\tilde{f}_z} &= \frac{G_z(f \circ F) \cdot f_{\bar z}(F) \cdot \overline{(F_z)}}{G_z(f \circ F) \cdot f_z(F) \cdot F_z} \\
&= \frac{f_{\bar z}(F) \cdot \overline{(F_z)}}{f_z(F) \cdot F_z} \\
&= k\frac{\overline{\phi(F)}}{|\phi(F)|} \cdot \frac{\overline{(F_z^2)}}{F_z \cdot \overline{F_z}} \\
&= k\frac{\overline{F_z^2 \cdot \phi(F)}}{|F_z^2 \cdot \phi(F)|},
\end{aligned}$$

which proves that $\tilde{\phi}(z) = F_z(z)^2\phi(F(z))$.

COROLLARY 3.1. *In the above setting, the mapping f is Teichmüller extremal with dilatation K if and only if $\tilde{f}$ is Teichmüller extermal with dilatation K.*

PROOF. It is enough to show that the extremality is preserved under the composition with conformal mappings. From the proof of Lemma 3.1, we have

$$\frac{f_{\overline{z}}}{f_z} = k\frac{\overline{\phi(z)}}{|\phi(z)|} \quad \text{iff} \quad \frac{\tilde{f}_{\overline{z}}}{\tilde{f}_z} = k\frac{\overline{\tilde{\phi}(z)}}{|\tilde{\phi}(z)|}.$$

Hence

$$K(f) = K(\tilde{f}) = \frac{k+1}{k-1}.$$

So, the dilatation and hence, the extremality is preserved under the composition with conformal mappings.

PROPOSITION 3.1. *Let D denote the open unit disk and $D^+ = \{z \in D | Imz > 0\}$. Then the mapping $h : D^+ \to D^+$*

$$h(z) = \frac{i\left(e^{kln\left|\frac{1-iz}{1+iz}\right|+iarg\frac{1-iz}{1+iz}} - 1\right)}{1 + e^{Kln\left|\frac{1-iz}{1+iz}\right|+iarg\frac{1-iz}{1+iz}}}$$

is a Teichmüller extremal mapping with dilatation K.

PROOF. The mappings

$$g_1 : D^+ \to \{z = x + iy | x > 0, \quad x^2 + y^2 > 1\}$$
$$g_2 : \{z = x + iy | x > 0, \quad x^2 + y^2 > 1\} \to \{z = x + iy | x > 0, \; 0 < y < 1\}$$

given by

$$g_1(z) = \frac{1 - iz}{1 + iz}$$
$$g_2(z) = \frac{1}{\pi} ln|z| + i\left(\frac{argz}{\pi} + \frac{1}{2}\right) \left(-\frac{\pi}{2} \leq argz \leq \frac{\pi}{2}\right)$$

are both surjective and conformal.

Consider the extremal mapping g_3 associated with $\phi(z) \equiv 1$,

$$g_3 : \{z = x + iy | x > 0, \quad 0 < y < 1\} \to \{z = x + iy | x > 0, \quad 0 < y < 1\}$$

given by

$$g_3(x + iy) = Kx + iy.$$

Then we see that

$$h(z) = (g_2 g_1)^{-1} \circ g_3 \circ (g_2 g_1)(z), \quad z \in D^+.$$

By Corollary 3.1, h is a Teichmüller extremal mapping with dilatation K.

COROLLARY 3.2. *The mapping $g : D \to D$ defined by*

$$g(z) = \begin{cases} h(z), & Imz \geq 0 \\ \overline{h(\overline{z})}, & Imz < 0 \end{cases}$$

is a Teichmüller extremal mapping with dilatation K.

PROPOSITION 3.2. *Let D denote the unit disk. Then the mapping*

$$f : D \to D$$

$$f(z) = \begin{cases} \frac{(x^2+x+y^2)+iKy}{(x+1)-iKy}, & z = x + iy \neq -1 \\ -1, & z = -1 \end{cases}$$

is a Teichmüller extremal mapping with dilatation K.

PROOF. It is clear that the restriction of f to the boundary ∂D

$$f|_{\partial D}(z) = \begin{cases} \frac{(x+1)+iKy}{(x+1)-iKy}, & z \neq -1 \\ -1, & z = -1 \end{cases}$$

is an homeomorphism of ∂D.

Since

$$\frac{\partial f}{\partial x} = \frac{\{(x+1)^2 - y^2\} - i2K(x+1)y}{\{(x+1) - iKy\}^2}$$

$$\frac{\partial f}{\partial y} = \frac{2(x+1)y + iK\{(x+1)^2 - y^2\}}{\{(x+1) - iKy\}^2}$$

we have

$$\begin{aligned}
\frac{f_{\bar{z}}}{f_z} &= \frac{\frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)}{\frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)} \\
&= \frac{(1-K)\{(x+1)^2 - y^2 + i2(x+1)y\}}{(1+K)\{(x+1)^2 - y^2 - i2(x+1)y\}} \\
&= \frac{1-K}{1+K}\frac{\{(x+1)+iy\}^2}{\{(x+1)-iy\}^2} \\
&= k\frac{\overline{\phi_0(z)}}{|\phi_0(z)|}
\end{aligned}$$

where

$$\phi_0(z) = \frac{4}{(z+1)^4} \quad \text{and} \quad k = \frac{1-K}{1+K}.$$

Hence, f is a Teichmüller mapping with dilatation K.

To prove the extremality we choose the region S_K

$$S_K = \{z = x + iy | x < K\}.$$

Consider the following functions on $D \setminus \partial D$

$$F(z) = \frac{2z}{z+1} : D \setminus \partial D \to S_1$$

$$G(z) = \frac{2Kz}{z+1} : D \setminus \partial D \to S_K.$$

Then F and G are conformal. Let

$$g_4(z) = Kx + iy : S_1 \to S_K$$

be the extremal mapping associated with $\phi(z) \equiv 1$. Then we see that

$$f(z) = G^{-1} \circ g_4 \circ F(z), \quad z \in D \setminus \partial D$$

and hence, by Corollary 3.1, that f is a Teichmüller extremal mapping with dilatation K.

REMARK. ne may compute the complex dilatation ϕ_0 of f in the following way using Lemma 3.1

$$\begin{aligned}\phi_0(z) &= F_z(z)^2\phi(F(z)) \\ &= \left\{\frac{d}{dz}\left(\frac{2z}{z+1}\right)\right\}^2 \\ &= \frac{4}{(z+1)^4}.\end{aligned}$$

REMARK. Unfortunately, the map g of Corollary 3.2 and the map f of Proposition 3.2 do not have the same boundary values.

References

1. L. Bers. *Finite dimensional Teichmüller spaces and generalizations*, Bull. Amer. Math. :. **5** (1981), 131-172.
2. E. Blum, *Die Extremalität gewisser Teichmüllerscher Abbildungen des Einheitskreises*, Comment. Math. Helv. **44** (1969), 319-340.
3. E. Reich, *A criterion for unique extremality of Teichmüller mappings*, Indiana Univ. Math. J. **30** (1981), 441-447.
4. ———, *Nonuniqueness of Teichmüller extremal mappings*, Proc. Amer. Math. Soc. **8P** 183), 13-516.

Dej of Mathematics
Ko ersity
Se , Korea