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ON REGULAR GROUPS OVER THEIR ENDOMORPHISM RINGS

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ABSTRACT. Let G be an abelian group of finite rink and E be the endomorphism ring of G. Then G is a left E-module by defining $f \cdot a = f(a)$ for $f \in E$ and $a \in G$. In this case a condition for an E-module G to be regular is given.

The concept of (von Neumman) regularity of rings has been extended to modules by D. Fieldhouse [1] and R. Ware [6]. Call an R-module M is regular if given any $m \in M$, there exsits $f \in \mathrm{Hom}_R(M, R)$ with $m = f(m)m$. R. Ware dealt with projective modules only and studied the endomorphism ring of a regular projective module. For example, every projective module over a regular ring is regular and the endomorphism ring of a finitely generated regular projective module is a regular ring [6]. But, as an immediate consequence of the definition, a submodule of a regular module is regular. This means that there are regular modules which are not projective. Also, an example of a regular projective module whose endomorphism ring is not regular is given in [7]. In this paper we study the relationship between regular groups and their endomorphism rings.

Throughout this paper G is an abelian group of finite rank n and E is the endomorphism ring of G. For the general notation, terminology and results we refer to [2], [3], [4] and [5].

The following theorem characterizes regular modules.

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THEOREM [7, THEOREM 2.2]. *For an R-module M, the following conditions are equivalent:*

(1) *M is regular.*
(2) *For every $m \in M$, Rm is projective and is a direct summand of M.*
(3) *For every $m_1, \dots, m_t \in M$, $\sum_{i=1}^{t} Rm_i$ is projective and is a direct summand of M.*

THEOREM [7, THEOREM 2.8]. *Let $\{M_\alpha\}$ be a family of left R-modules. Then $\oplus_\alpha M_\alpha$ is regular if and only if each M_α is regular.*

Let G be an abelian group of finite rank and E be the endomorphism ring of G. Then G is an E-module by defining $f \cdot a = f(a)$ for $f \in E$ and $a \in G$. Consider G as a $\mathbb{Z}$-module. Suppose G is regular over $\mathbb{Z}$ and $0 \neq x \in G$. If $\mathbb{Z}x$ is finite cyclic, then $\mathbb{Z}x$ is not projective as a $\mathbb{Z}$-module. Therefore $\mathbb{Z}x$ is infinite cyclic. Suppose $n\mathbb{Z}x = \mathbb{Z}nx = \mathbb{Z}x$ for all $n \in \mathbb{N}$. Then $\mathbb{Z}x$ is (torsion free) divisible and is isomorphic to $\mathbb{Q}$. But $\mathbb{Z}x$ is projective over $\mathbb{Z}$. Hence $\mathbb{Z}nx \subsetneq \mathbb{Z}x$ for some $n \in \mathbb{N}$ and we have $\mathbb{Z}x \oplus A = G = \mathbb{Z}nx \oplus B$ for some subgroups A and B of G. Here $x = knx + b$ for some $k \in \mathbb{Z}$ and $b \in B$. From $nx = kn^2x + nb$, we have $nx - kn^2x = nb \in \mathbb{Z}nx \cap B = (0)$. Since x is of infinite order and $n > 0$, n is a unit in $\mathbb{Z}$. This contradicts the fact that $\mathbb{Z}nx \subsetneq \mathbb{Z}x$. Therefore G is not regular over $\mathbb{Z}$. But there exist abelian groups which are regular over their endomorphism rings. For example, the rational numbers $\mathbb{Q}$ is regular over $\mathbb{Q}$. The next results characterize groups which are regular over their endomorphism rings.

LEMMA 1. *If G is simple over E, then E is regular.*

PROOF. Since G is simple over E, we have $G = Ex$ for $0 \neq x \in G$.
Case 1. The order of x is finite.

Let $n \in \mathbb{N}$ be the order of x. Suppose n is not prime. Then $n = st$ for some $s > 1$, $t > 1$. Since $sx \neq 0$ and G is simple over E, we have $Ex = Esx$. Therefore $x = fsx$ for some $f \in E$ and $tx = tfsx = fnx = 0$. This contradicts the fact that n is the order of x. Hence n is prime i.e., $px = 0$ for some prime p and $pG = pEx = Epx = 0$. Therefore G is an elementary p-group and is isomorphic to $\oplus^n \mathbb{Z}_p$ by [2, Theorem 8.5]. Hence E is isomorphic to $M_{n\times n}(\mathbb{Z}_p)$ which is a regular ring.

Case 2. The order of x is infinite.

Since $G = Ex$ is simple over E, G is torsion free. Let $n \in \mathbb{N}$ be arbitrary. Then $nG = nEx = Enx = G$. Therefore G is torsion free divisible and is isomorphic to $\oplus^n \mathbb{Q}$. Hence E is isomorphic to $M_{n\times n}(\mathbb{Q})$ which is a regular ring.

Note that a ring R is regular if and only if $M_{n\times n}(R)$ is regular [6, Corollary 3.7].

PROPOSITION 2. *If G is regular over E, then E is regular.*

PROOF. Let G be of rank n. Then, G contains no infinite direct sums of subgroups. By [7, Theorem 1.8], $G = Ex_1 \oplus \cdots \oplus Ex_k$ for some $k \leq n$ and each Ex_i is simple over E. Let E_i be the endomorphism ring of Ex_i as an abelian group i.e., $E_i = \mathrm{Hom}_{\mathbb{Z}}(Ex_i, Ex_i)$. Since each Ex_i is fully invariant in G, E is isomorphic to $E_1 \oplus \cdots \oplus E_k$ and $Ex_i = E_i x_i$ for each i. Let A be a nonzero submodule of $E_i x_i$ as an E_i-module. Then EA is a nonzero submodule of Ex_i as an E-module. Since Ex_i is simple over E and since $Ex_i = E_i x_i$, we have $A = E_i A = EA = Ex_i = E_i x_i$. Therefore $E_i x_i$ is simple over E_i and E_i is regular for all i by Lemma 1. Thus $E_1 \oplus \cdots \oplus E_k$ is regular and so is E.

Note that above E is semisimple.

COROLLARY 3. *If G is regular over E, then G is projective, injective and discrete over E.*

LEMMA 4. *If G is isomorphic to a finite direct sum of copies of the prime field F, then G is regular over E.*

PROOF. Let $G \simeq \oplus^n F$. Then E is isomorphic to $M_{n\times n}(F)$ (and E is regular). Thus E is semisimple over E and G is isomorphic to a left ideal of E which is a direct summand of E, hence projective over E. Moreover G is simple over E. Therefore G is regular over E.

PROPOSITION 5. *If E is regular, then G is regular over E.*

PROOF. If E is regular, then G is a direct sum of a torsion free divisible group and an elementary group by [3, Proposition 112.7]. Thus we can write $G = A_1 \oplus_p A_p$ where A_1 is isomorphic to $\oplus\mathbb{Q}$, p is prime and A_p is isomorphic to $\oplus\mathbb{Z}_p$. Let E_i be the endomorphism ring of the abelian group A_i. Then E_1 is isomorphic to $M_{k_1\times k_1}(\mathbb{Q})$ and E_p is isomorphic to $M_{k_p\times k_p}(\mathbb{Z}_p)$ for some k_i. From some facts on homomorphisms of groups it is easy to show that each A_i is fully invariant in G and E is isomorphic to $E_1 \oplus_p E_p$. Hence A_i is a submodule of G over E and $EA = E_iA_i$. Consider A_i as an E_i-module for each i. Then A_i is regular over E_i by Lemma 4. Hence for each $x_i \in A_i$, $Ex_i = E_ix_i$ is a direct summand of A_i and is projective over E_i. But E_i is isomorphic to a direct summand of E. Thus A_i is regular over E for each i and $G = A_1 \oplus_p A_p$ is regular over E by [7, Theorem 2.8].

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