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SOME EXTENSIONS ON THE INJECTIVE COVER AND PRECOVER

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ABSTRACT. In this paper, we show relations between injective covers and direct sums, some commutative properties, and composition properties in the injective covers.

1. Introduction

Using the dual of a categorical definition of an injective envelope, Enochs in [3] defined an injective cover. The existence of an injective cover is not for all cases but every left$R$-module has an injective cover if and only if a ring R is left noetherian (Theorem 2.1. [3]). In this paper we show relations between injective covers and direct sums, some commutative properties, and composition properties in the injective covers. Some information about the structure of the injective cover of modules both general and in some special cases can be found in Enochs[3], Ashan,Enochs[1], Cheatham,Enochs,Jenda[2] and Park[4],[5].

DEFINITION 1.1. An injective cover of a left R-module M is a linear map $\phi : E \to M$ with E injective such that:

(1) Any diagram

$$\begin{array}{ccc} & & E' \\ & & \downarrow \\ E & \xrightarrow{\phi} & M \end{array}$$

with E' injective can be completed.

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(2) The diagram

$$\begin{array}{ccc} & & E \\ & & \downarrow \phi \\ E & \xrightarrow{\phi} & M \end{array}$$

can be completed only by automorphism of E.

Note that $\phi: E \to M$ is called an injective precover if ϕ satisfies (1) only.

2. Injective Cover and Precover

PROPOSITION 2.1. *If $\phi : E \to M$ is an injective cover and $E' \subset ker(\phi)$ is an injective submodule of $ker(\phi)$, then $E' = 0$.*

PROOF. Since $E' \subset M$ is an injective submodule E' is a direct summand of E. So let $E' \oplus S = E$ for some S. Consider the following diagram

$$\begin{array}{ccc} E' \oplus S & & \\ & \overset{\phi}{\searrow} & \\ E' \oplus S & \xrightarrow{\phi} & M \end{array}$$

Define $\sigma : E' \oplus S \to E' \oplus S$ by $\sigma(x,y) = (0,y)$. We claim σ makes the diagram commutative. We have $\phi(\sigma(x,y)) = \phi(0,y) = \phi_1(0) + \phi_2(y) = \phi_2(y)$ where $\phi_1 = \phi|_{E'}$ and $\phi_2 = \phi|_S$ and $\phi(x,y) = \phi_1(x)+\phi_2(y) = \phi_2(y)$ since $\phi_1(x) = \phi_1|_{E'}(x) = 0$. So we have the following commutative diagram

$$\begin{array}{ccc} E' \oplus S & & \\ \sigma \downarrow & \overset{\phi}{\searrow} & \\ E' \oplus S & \xrightarrow{\phi} & M \end{array}$$

But since ϕ is injective cover of M, σ is an automorphism. But $\sigma(x,y) = (0,y)$ is an automorphism only if $E' = 0$.

PROPOSITION 2.2. *Suppose $\sigma : E \to M$ and $\sigma' : E' \to M$ are both injective covers of M, then there is an isomorphism between E and E'.*

PROOF. Consider the following diagram

$$\begin{array}{ccc} E & & \\ & \overset{\sigma}{\searrow} & \\ E' & \overset{\sigma'}{\to} & M \end{array}$$

Since σ' is an injective cover of M, there is a linear map $f : E \to E'$ such that $\sigma = \sigma' \circ f$. Since σ is an injective cover of M, there is a linear map $g : E' \to E$ such that $\sigma' = \sigma \circ g$. So $\sigma = (\sigma \circ g) \circ f = \sigma \circ (g \circ f)$. Thus $g \circ f \in \mathrm{Aut}E$. And $\sigma' = (\sigma' \circ f) \circ g = \sigma' \circ (f \circ g)$ so $f \circ g \in \mathrm{Aut}E'$. Hence, $f : E \to E'$ is an isomorphism.

THEOREM 2.3. *If $\phi : E \to M$ is an injective cover and E' is a left injective module then $E \oplus E' \to M \oplus E'$ is an injective cover.*

PROOF. Since $\phi : E \to M$ is an injective cover any diagram

$$\begin{array}{ccc} E'' & & \\ & \searrow & \\ E & \overset{\phi}{\to} & M \end{array}$$

with E'' injective can be completed. So define $\sigma : E \oplus E' \to M \oplus E'$ by $\sigma(a, b) = (\phi(a), b)$. Let E'' be a left injective module and let $\tau_1 : E'' \to M$ and $\tau_2 : E'' \to E'$ be linear maps. So let $\tau : E'' \to M \oplus E'$ be a linear map by $\tau(x) = (\tau_1(x), \tau_2(x))$, for $x \in E''$. Consider the following diagram

$$\begin{array}{ccc} E'' & & \\ & \overset{\tau}{\searrow} & \\ E \oplus E' & \overset{\sigma}{\to} & M \oplus E' \end{array}$$

Let $\tau_1 : E'' \to E$ completes the following diagram

$$\begin{array}{ccc} E'' & & \\ \tau_1 \downarrow & \overset{\tau_2}{\searrow} & \\ E & \overset{\phi}{\to} & M \end{array}$$

Now define $r : E'' \to E \oplus E'$ by $r(x) = (r_1(x), \tau_2(x)), x \in E''$. Then $\tau(x) = (\tau_1(x), \tau_2(x))$ and $\sigma \circ \tau(x) = \sigma(\tau_1(x), \tau_2(x)) = (\phi(r_1(x)), \tau_2(x))$. But $\phi(r_1(x)) = \tau_1(x)$ so $(\tau_1(x), \tau_2(x)) = (\phi(r_1(x)), \tau_2(x))$. Thus we have the following commutative diagram

$$\begin{array}{ccc} E'' & & \\ \tau\downarrow & \overset{\tau}{\searrow} & \\ E \oplus E' & \overset{\sigma}{\to} & M \oplus E' \end{array}$$

Now let $h : E \oplus E' \to E \oplus E'$ be a linear map such that the following diagram commutes

$$\begin{array}{ccc} E \oplus E' & & \\ h\downarrow & \overset{\sigma}{\searrow} & \\ E \oplus E' & \overset{\sigma}{\to} & M \oplus E' \end{array}$$

Let $h(a,b) = (f(a,b), g(a,b)) = (f_1(a) + f_2(b),\ g_1(a) + g_2(b))$ where $f : E \oplus E' \to E$ and $g : E \oplus E' \to E'$

$$f_1 : E \to E, \quad f_2 : E' \to E$$

$$g_1 : E \to E', \quad g_2 : E' \to E'$$

are linear maps giving f and g respectively. Then $\sigma(h(a,b)) = \sigma(a,b)$. But $\sigma(h(a,b)) = \sigma(f_1(a)+f_2(b), g_1(a)+g_2(b)) = (\phi(f_1(a)+f_2(b), g_1(a)+g_2(b))$ and $\sigma(a,b) = (\phi(a), b)$. So $(\phi(f_1(a) + f_2(b)), g_1(a) + g_2(b)) = (\phi(a), b)$. Thus take $b = 0$ then $g_1(a) + g_2(b) = 0$, so $g_1(a) = 0$, so $g_2(b) = b$, and $\phi(f_1(a)) = \phi(a)$. Therefore $f_1 : E \to E$ completes the following diagram

$$\begin{array}{ccc} E & & \\ f_1\downarrow & \overset{\phi}{\searrow} & \\ E & \overset{\phi}{\to} & M \end{array}$$

Thus f_1 is an automorphism. So $h(a,b) = (f(a,b), g(a,b)) = (f_1(a,b) + f_2(a,b), b)$. Now we claim h is an automorphism. Suppose $h(a,b) = 0$, then $(f_1(a) + f_2(b), b) = (0,0)$. So $b = 0$ implies $f_2(b) = 0$ implies

$f_1(a) = 0$ implies $a = 0$. Thus h is 1-1. Let $(x, y) \in E \oplus E'$. Since $h(a,b) = (f_1(a) + f_2(b), b)$ let $b = y$ then $f_1(a) + f_2(b) = x$. So let $a = f_1^{-1}(x - f_2(b))$ since f_1 is an automorphism so h is onto. Therefore h is an automorphism. Hence, we conclude that $\sigma : E \oplus E' \to M \oplus E'$ is an injective cover.

THEOREM 2.4. *Suppose M_1, M_2 are left R-modules having injective covers $\phi_1 : E_1 \to M_1$, $\phi_2 : E_2 \to M_2$. Then for any linear map $f : M_1 \to M_2$ there is a linear map $g : E_1 \to E_2$ such that*

$$\begin{array}{ccc} E_1 & \overset{\phi_1}{\to} & M_1 \\ g \downarrow & & \downarrow f \\ E_2 & \overset{\phi_2}{\to} & M_2 \end{array}$$

is commutative. If f is an isomorphism then so is g.

PROOF. Consider the following diagram

$$\begin{array}{ccc} E_1 & \overset{\phi_1}{\to} & M_1 \\ & & \downarrow f \\ E_2 & \overset{\phi_2}{\to} & M_2 \end{array}$$

Let $\tau = f \circ \phi_1$ then $\tau : E_1 \to M_2$ is a linear map. So there is a linear map $g : E_1 \to E_2$ such that $\tau = \phi_2 \circ g$ since ϕ_2 is an injective cover of M_2. Now we have the following commutative diagram

$$\begin{array}{ccc} E_1 & \overset{\phi_1}{\to} & M_1 \\ g \downarrow & & \downarrow f \\ E_2 & \overset{\phi_2}{\to} & M_2 \end{array}$$

Suppose f is an automorphism. Then we have the following commutative diagram

$$\begin{array}{ccc} E_1 & \stackrel{\phi_1}{\rightarrow} & M_1 \\ g\downarrow & & \downarrow f \\ E_2 & \stackrel{\phi_2}{\rightarrow} & M_2 \\ h\downarrow & & \downarrow f^{-1} \\ E_1 & \stackrel{\phi_1}{\rightarrow} & M_1 \end{array}$$

Claim $h \circ g \in \mathrm{Aut} E_1$. Consider the following commutative diagram

$$\begin{array}{ccc} E_1 & \stackrel{\phi_1}{\rightarrow} & M_1 \\ h\circ g\downarrow & \stackrel{\phi_1}{\searrow} & \downarrow id \\ E_1 & \stackrel{\phi_1}{\rightarrow} & M_1 \end{array}$$

Since ϕ_1 is an injective cover of M_1, the lower triangle can be completed only by an automorphism. So we have $h \circ g \in \mathrm{Aut} E_1$. Similiarly we have the following commutative diagram

$$\begin{array}{ccc} E_2 & \stackrel{\phi_2}{\rightarrow} & M_2 \\ h\downarrow & & \downarrow f^{-1} \\ E_1 & \stackrel{\phi_1}{\rightarrow} & M_1 \\ g\downarrow & & \downarrow f \\ E_2 & \stackrel{\phi_2}{\rightarrow} & M_2 \end{array}$$

Claim $g \circ h \in \mathrm{Aut} E_2$. Consider the following commutative diagram

$$\begin{array}{ccc} E_2 & \stackrel{\phi_2}{\rightarrow} & M_2 \\ g\circ h\downarrow & \stackrel{\phi_2}{\searrow} & \downarrow id \\ E_2 & \stackrel{\phi_2}{\rightarrow} & M_2 \end{array}$$

Since ϕ_2 is an injective cover of M_2, the lower triangle can be completed only by an automorphism. So we have $g \circ h \in \mathrm{Aut} E_2$. Hence, g is an automorphism.

THEOREM 2.5. *Suppose $\sigma : F \to M$ be an injective cover of M then for an injective module F' and an isomorphism $k : F' \to F, \sigma \circ k$ is an injective cover of M.*

PROOF. Consider the following diagram

$$\begin{array}{ccccc} F'' & & & & \\ & & & \overset{t}{\searrow} & \\ F' & \overset{k}{\rightarrow} & F \rightarrow & & M \end{array}$$

where $t : F'' \to M$ is a linear map. Since σ is an injective cover of M, there is a linear map $s : F'' \to F$ such that $\sigma \circ s = t$ so let $\tau : F'' \to F'$ be linear map such that $\tau = k^{-1} \circ s$, then we have the following commutative diagram

$$\begin{array}{ccccccc} F'' & & & & & & \\ \tau \downarrow & & \overset{s}{\searrow} & & \overset{h}{\searrow} & & \\ F' & \underset{k^{-1}}{\overset{k}{\rightleftarrows}} & F & & \rightarrow & & M \end{array}$$

So we have $h = (\sigma \circ k) \circ \tau$. Now consider the following diagram

$$\begin{array}{ccc} F'' & & \\ \tau \downarrow & \overset{\sigma \circ k}{\searrow} & \\ F' & \overset{\sigma \circ k}{\rightarrow} & M \end{array}$$

with τ completing the above diagram. We claim τ is an automorphism. Consider the following diagram

$$\begin{array}{ccccc} F' & \overset{k}{\rightarrow} & F & & \\ \tau \downarrow & & & \overset{\sigma}{\searrow} & \\ F' & \overset{k}{\rightarrow} & F & \overset{\sigma}{\rightarrow} & M \end{array}$$

Let $\phi: F \to F$ be a linear map such that $\phi = k \circ \tau \circ k^{-1}$, then we have the following commutative diagram

$$\begin{array}{ccc} F' & \xrightarrow{k} & F \\ \tau \downarrow & & \downarrow \phi \\ F' & \xrightarrow{k} & F \end{array}$$

Since

$$\begin{array}{ccccc} F' & \xrightarrow{k} & F & & \\ & & & \overset{\sigma}{\searrow} & \\ \tau \downarrow & & & & \\ F' & \xrightarrow{k} & F & \xrightarrow{\tau} & M \end{array}$$

Thus $\sigma(\phi(b)) = \sigma(k(\tau(k^{-1}(b))) = \sigma(k(k^{-1}(b)) = \sigma(b)$ for $b \in F$. Thus $\sigma(b) = \sigma \circ \phi(b)$. So we have the following commutative diagram

$$\begin{array}{ccc} F & & \\ \phi \downarrow & \searrow & \\ F & \xrightarrow{\sigma} & M \end{array}$$

and we also have the following commutative diagram

$$\begin{array}{ccccc} F' & \xrightarrow{k} & F & & \\ & & & \sigma & \\ \tau \downarrow & & \downarrow \phi & \searrow & \\ F' & \xrightarrow{k} & F & \xrightarrow{\sigma} & M \end{array}$$

But since σ is an injective cover of M, ϕ is an automorphism. But k is an isomorphism, so τ is an automorphism. Hence, $\sigma \circ k$ is an injective cover of M.

EXAMPLE 2.6. Let D be a divisible group and G be an abelian group and $\phi : D \to G$ be a linear map. Then $\phi(D)$ is a divisible subgroup of

G. But G has a largest divisible subgroup D'. So $\phi(D) \subset D' \subset G$. Thus consider the following diagram

$$\begin{array}{ccc} D & & \\ & \overset{\phi}{\searrow} & \\ D' & \overset{i}{\hookrightarrow} & G \end{array}$$

Then there is a linear map ${}_{D'}|\phi : D \to D'$ that completes the above diagram. And the following diagram

$$\begin{array}{ccc} D' & & \\ & \overset{i}{\searrow} & \\ D' & \overset{i}{\hookrightarrow} & G \end{array}$$

can be completed only by identity map of D'. Hence, $i : D' \to G$ is an injective cover of G.

EXAMPLE 2.7. Let $R = Z$ and $G = Z$, then the only divisible (injective) subgroup of Z is 0. So the zero map is an injective cover of Z.

EXAMPLE 2.8. Let $R = Z$ and $G = Q$, the a largest divisible subgroup of Q is Q itself. So identity map of Q is an injective cover of Q. In general given an abelian group G and a largest divisible subgroup of G, the inclusion map is an injective cover of G. For R a PID we have the same argument.

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