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ON THE H^s_ω-WAVE FRONT SETS

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ABSTRACT. In this paper we extend the concept of the Sobolev wave front set of a distribution to the one of the generalized Sobolev wave front set of a generalized distribution, and we investigate the relations among these concepts. Finally, we prove the local property of these sets.

I. Introduction

In this paper we will extend the concept of the H^s-wave front set of a distribution to the one of the H^s_ω-wave front set of a generalized distribution. The generalized Sobolev spaces H^s_ω were introduced in [7] and the concept of the ω-wave front set $\omega - WF(u)$, which we need later, was introduced in [5]. Throughout this paper Ω denotes an open subset of R^n, and ω denotes an element of the set $\mathcal{M}_c$, the set of all continuous real-valued functions ω on R^n which satisfy the following conditions :

$$(\alpha)\quad 0 = \omega(0) \le \omega(\xi+\eta) \le \omega(\xi) + \omega(\eta), \quad \xi, \eta \in R^n.$$

$$(\beta)\quad \int_{R^n} \frac{\omega(\xi)}{(1+|\xi|)^{n+1}} d\xi < \infty.$$

(γ) $\omega(\xi) \ge a + \log(1+|\xi|)$ for some constant a.

(δ) $\omega(\xi)$ is radial and increasing.

And the same notations as in [3] for the spaces of test functions and the spaces of the generalized distributions are used. In [7], the generalized Sobolev spaces H^s_ω was defined as the set of all distributions $u \in \mathcal{D}'_\omega(R^n)$ such that $(\|u\|^\omega_s)^2 = \int e^{2s\omega(\xi)} |\hat{u}(\xi)|^2 d\xi < \infty$. If $u \in \mathcal{D}'_\omega(\Omega)$, we define

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the H^s_ω-singular support of u, denoted by H^s_ω-sing supp u, to be the complement in Ω of the largest open subset U of Ω such that $u|_U \in H^s_{\omega loc}(U)$, the set of all $u \in \mathcal{D}'_\omega(\Omega)$ such that $\phi u \in H^s_\omega$ for all $\phi \in \mathcal{D}_\omega(\Omega)$. Note that if W is an open neighborhood of H^s_ω-sing supp u then there exist $u_1 \in H^s_{\omega loc}(\Omega)$ and $u_2 \in \mathcal{D}'_\omega(\Omega)$ such that $u = u_1 + u_2$ and supp $u_2 \subseteq W$, by means of a locally finite partition of unity. Recall that sing$_\omega$supp u of $u \in \mathcal{D}'_\omega(\Omega)$ is defined as the complement in Ω of the largest open subset U of Ω such that $u|_U \in \mathcal{E}_\omega(U)$.

LEMMA 1. *The sing$_\omega$supp u is the closure of the union over all s of H^s_ω-sing supp u.*

PROOF. It suffices to show that $\mathcal{E}_\omega(U) = \cap_s H^s_{\omega loc}(U)$ for any open subset U of R^n. Applying Paley-Wiener theorem, we know that $\mathcal{E}_\omega(U)$ is a subset of $H^s_{\omega loc}(U)$ for any open subset U and all s. Conversely, if u is in $H^s_{\omega loc}(U)$ for all s then ϕu is in H^s_ω for all ϕ in $\mathcal{D}_\omega(U)$. Hence $\int e^{2s\omega(\xi)}|\widehat{\phi u}(\xi)|^2 d\xi$ is finite for all s in R. Then, by applying Hölder's inequality, we know that $\int |\widehat{\phi u}(\xi)| e^{\lambda\omega(\xi)} d\xi$ is finite for all $\lambda \in R$, which shows that u is in $\mathcal{E}_\omega(U)$.

II. The H^s_ω-wave front sets

We now introduce the concept of the H^s_ω-wave front set. If $u \in \mathcal{D}'_\omega(\Omega)$ we define the H^s_ω-wave front set of u, denoted by $\omega - WF_s(u)$, to be the complement in $\Omega \times R^n - (0)$ of the set of point (x_0, ξ_0) such that there is an open neighborhood U of x_0 in Ω and a conic open neighborhood V of ξ_0 in $R^n - (0)$ such that for each $\phi \in \mathcal{D}_\omega(U)$ we have

$$\int_V |\widehat{\phi u}(\xi)|^2 e^{2s\omega(\xi)} d\xi < \infty.$$

If $(x_0, \xi_0) \notin \omega - WF_s(u)$ one says that u is microlocally in H^s_ω at (x_0, ξ_0), or simply that u is in H^s_ω at (x_0, ξ_0). We need a convenient description of the definition of $\omega - WF_s(u)$.

THEOREM 2. *Let $u \in \mathcal{D}'_\omega(\Omega)$ and let $(x_0, \xi_0) \in \Omega \times R^n - (0)$. Then $(x_0, \xi_0) \notin \omega - WF_s(u)$ if and only if there exist $u_1 \in H^s_{\omega loc}(\Omega)$ and*

$u_2 \in \mathcal{D}'_\omega(\Omega)$ such that $u = u_1 + u_2$ and $(x_0, \xi_0) \notin \omega - WF(u_2)$. Moreover, we may choose $u_1 \in H^s_{\omega c}(\Omega) = \{u \in H^s_\omega | supp\ u$ is a compact subset of $\Omega\}$.

PROOF. Choosing $\chi \in \mathcal{D}_\omega(\Omega)$ with $\chi = 1$ in a neighborhood of x_0 and replacing u_1 by χu_1 and u_2 by $(1-\chi)u_1 + u_2$, the last statement follows. Suppose we have u_1 and u_2 with the properties indicated. Since $(x_0, \xi_0) \notin \omega - WF(u_2)$ there is an open neighborhood U of x_0 and a conic open neighborhood V of ξ_0 such that for each $N \geq 0$ and each $\phi \in \mathcal{D}_\omega(U)$ we have , for each $\xi \in V$,

$$|\widehat{\phi u_2}(\xi)| \leq C_{N,\phi} e^{-N\omega(\xi)}.$$

Then

$$\int_V |\widehat{\phi u}(\xi)|^2 e^{2s\omega(\xi)} d\xi \leq 2\int_V |\widehat{\phi u_1}(\xi)|^2 e^{2s\omega(\xi)} d\xi + 2\int_V |\widehat{\phi u_2}(\xi)|^2 e^{2s\omega(\xi)} d\xi$$
$$\leq 2(\|\phi u_1\|^\omega_s)^2 + 2C^2_{N,\phi} \int_V e^{2(s-N)\omega(\xi)} d\xi$$

which is finite if N is large enough. Conversely, if $(x_0, \xi_0) \notin \omega - WF_s(u)$, choose an open neighborhood U of x_0 and a conic open neighborhood V of ξ_0 such that

$$\int_V |\widehat{\phi u}(\xi)|^2 e^{2s\omega(\xi)} d\xi < \infty$$

for each $\phi \in \mathcal{D}_\omega(U)$. Choose $\chi \in \mathcal{D}_\omega(U)$ such that $\chi = 1$ in an open neighborhood W of x_0. Define a function g by $g(\xi) = 1$ if $\xi \in V$ and $g(\xi) = 0$ if $\xi \notin V$. Then

$$\int_{R^n} |g(\xi)\widehat{\chi u}(\xi)|^2 e^{2s\omega(\xi)} d\xi = \int_V |\widehat{\chi u}(\xi)|^2 e^{2s\omega(\xi)} d\xi < \infty.$$

Hence we have

$$\int e^{-n\omega(\xi)} |g(\xi)\widehat{\chi u}(\xi)| e^{s\omega(\xi)} d\xi \leq \left(\int e^{-2n\omega(\xi)} d\xi\right)^{\frac{1}{2}}$$
$$\left(\int |g(\xi)\widehat{\chi u}(\xi)|^2 e^{2s\omega(\xi)} d\xi\right)^{\frac{1}{2}} < \infty.$$

Therefore, there is an element $u_1 \in \mathcal{F}_\omega$ such that $\hat{u_1}(\xi) = g(\xi)\widehat{\chi u}(\xi)$. Here $\mathcal{F}_\omega$ is the function space which introduced in [3]. Hence $u_1 \in H^s_\omega$ and $\hat{u_1}(\xi) = g(\xi)\widehat{\chi u}(\xi)$. Then $u = u_1 + u_2$ where $u_2 = (1-\chi)u + w$ and $w = \chi u - u_1$. In particular, for any $\phi \in \mathcal{D}_\omega(W)$ we have $\phi u_2 = \phi w$. Since $\chi u \in \mathcal{E}'_\omega(U)$, by applying the Paley-Wiener theorem, we have $w \in H^t_\omega$ for some t and $\hat{w}(\xi) = \widehat{\chi u}(\xi) - \hat{u_1}(\xi) = (1-g(\xi))\widehat{\chi u}(\xi)$. If $\phi \in \mathcal{D}_\omega(W)$ then, from Corollary 1.8.13 in [3], we have

$$\widehat{\phi u_2}(\xi) = \widehat{\phi w}(\xi) = (2\pi)^{-n}\hat{\phi} * \hat{w}(\xi) = (2\pi)^{-n}\int \hat{\phi}(\xi-\eta)(1-g(\eta))\widehat{\chi u}(\eta)d\eta.$$

Since $\chi u \in \mathcal{E}'_\omega(R^n)$, by Paley-Wiener theorem, we have

$$|\widehat{\chi u}(\eta)| \leq Ce^{M\omega(\eta)}$$

for some constant M. Thus

$$|\widehat{\phi u_2}(\xi)| \leq C_{N,\phi}\int_{R^n - V} e^{-N\omega(\xi-\eta)}e^{M\omega(\eta)}d\eta$$

for each $N \geq 0$. Choose an open cone V' with $\xi_0 \in V'$ and $V' \cap S^{n-1}$ relatively compact in V. Let m be an integer such that $0 < \frac{1}{m} < \text{dist}(V' \cap S^{n-1}, \partial V)$. Then $|\xi - \eta| \geq \frac{1}{m}|\xi|$ for $\xi \in V'$ and $\eta \in R^n - V$. From the condition (α) on ω we have $\omega(\frac{1}{m}\xi) \geq \frac{1}{m}\omega(\xi)$. And $e^{-(M+k)\omega(\xi-\eta)} \leq e^{(M+k)\omega(\xi)}e^{-(M+k)\omega(\eta)}$ for all ξ and η if $M + k \geq 0$. Hence by the condition (δ) on ω, we have for $\xi \in V'$

$$|\widehat{\phi u_2}(\xi)| \leq C_{mN+M+k,\phi}e^{-N\omega(\xi)}e^{(M+k)\omega(\xi)}\int e^{-k\omega(\eta)}d\eta.$$

Taking $k > n$, we see that for each integer $N \geq 0$, there is a constant $C'_{N,\phi}$ such that $|\widehat{\phi u_2}(\xi)| \leq C'_{N,\phi}e^{-N\omega(\xi)}$ for all $\phi \in \mathcal{D}_\omega(W)$ and all $\xi \in V'$. Thus $(x_0, \xi_0) \notin \omega - WF(u_2)$.

We need

LEMMA 3. *Let $u \in \mathcal{D}'_\omega(\Omega)$. If K is a compact subset of Ω, Γ is a closed cone in $R^n - (0)$ and $K \times \Gamma$ is disjoint from $\omega - WF_s(u)$ then $|\widehat{\phi u}(\xi)| \leq C_\phi e^{-s\omega(\xi)}$ for each $\phi \in \mathcal{D}_\omega(K)$ and $\xi \in \Gamma$*

PROOF. Let $x_0 \in K$ and $\xi_0 \in \Gamma$. Since $(x_0, \xi_0) \notin \omega - WF_s(u)$, we have $u = u_1 + u_2$ where $u_1 \in H^s_{\omega loc}(\Omega)$ and $(x_0, \xi_0) \notin \omega - WF(u_2)$. Thus there is an open neighborhood U of x_0 and a conic open neighborhood V of ξ_0 such that $|\widehat{\phi u_2}(\xi)| \le C_{N,\phi} e^{-N\omega(\xi)}$ for each $\phi \in \mathcal{D}_\omega(U), \xi \in V$ and $N \ge 0$. If ψ is a local unit for $\operatorname{supp}(\phi u_1)$ in $\mathcal{D}_\omega(U)$ then

$$\begin{aligned}
|\widehat{\phi u_1}(\xi)|e^{s\omega(\xi)} &= |\widehat{\psi(\phi u_1)}(\xi)|e^{s\omega(\xi)} = (2\pi)^{-n}|\hat{\psi} * \widehat{\phi u_1}(\xi)|e^{s\omega(\xi)} \\
&= (2\pi)^{-n}\left|\int \hat{\psi}(\xi - \eta)\widehat{\phi u_1}(\eta)e^{s\omega(\xi)}d\eta\right| \\
&\le \int |\hat{\psi}(\xi - \eta)|e^{|s|\omega(\xi-\eta)}|\widehat{\phi u_1}(\eta)|e^{s\omega(\eta)}d\eta \\
&\le \|\psi\|^\omega_{|s|}\|\phi u_1\|^\omega_s.
\end{aligned}$$

Hence $|\widehat{\phi u_1}(\xi)| \le C'_\phi e^{-s\omega(\xi)}\|\phi u_1\|^\omega_s$ for all $\xi \in R^n$. Therefore

$$(*) \qquad |\widehat{\phi u}(\xi)| \le C_\phi e^{-s\omega(\xi)}$$

for each $\phi \in \mathcal{D}_\omega(U)$ and $\xi \in V$. Since $\Gamma \cap S^{n-1}$ is compact, we can find open conic neighborhoods V_1, ..., V_m covering Γ and the corresponding open neighborhoods U_1, ..., U_m of x_0 such that $(*)$ holds for $\phi \in \mathcal{D}_\omega(U_j)$ and $\xi \in V_j, j = 1, ..., m$. Setting $U = \cap_{j=1}^m U_j$ we now have $(*)$ for $\phi \in \mathcal{D}_\omega(U)$ and $\xi \in \Gamma$. A finite number of such sets U will cover K and therefore, by a (finite) partition of unity argument, the lemma follows.

We have the following inclusion.

THEOREM 4. *Let $u \in \mathcal{D}'_\omega(\Omega)$. If $t < s$ then $\omega - WF_t(u) \subseteq \omega - WF_s(u) \subseteq \omega - WF(u)$. Moreover, $\omega - WF(u)$ is the closure of the union over all s of $\omega - WF_s(u)$.*

PROOF. The first inclusion is clear by definition. If $(x_0, \xi_0) \notin \omega - WF(u)$ then $u = 0 + u$ is a decomposition of the type occuring in Theorem 2. Thus $(x_0, \xi_0) \notin \omega - WF_s(u)$. Finally if (x_0, ξ_0) is in the complement of the closure of the union of the conic sets $\omega - WF_s(u)$ then there is a conic neighborhood W of (x_0, ξ_0) disjoint from $\omega - WF_s(u)$ for each s. Choose an open neighborhood U of x_0 and a conic neighborhood

V of ξ_0 such that $U \times (V \cap S^{n-1})$ is relatively compact in W. Then by Lemma 3 we have $|\widehat{\phi u}(\xi)| \leq C_{N,\phi} e^{-N\omega(\xi)}$ for each $\phi \in \mathcal{D}_\omega(U)$, $\xi \in V$ and $N \geq 0$. Thus $(x_0, \xi_0) \notin \omega - WF(u)$.

We also have

THEOREM 5. *Let $\pi : \Omega \times R^n - (0) \to \Omega$ be the projection map. If $u \in \mathcal{D}'_\omega(\Omega)$ then $\pi(\omega - WF_s(u)) = H^s_\omega$-sing supp u. Hence $u \in H^s_{\omega loc}(\Omega)$ if and only if $\omega - WF_s(u) = \Phi$.*

PROOF. If $x_0 \notin H^s_\omega$-sing supp u there is an open neighborhood U of x_0 such that $\phi \in \mathcal{D}_\omega(U)$ implies $\phi u \in H^s_\omega$. By definition of $\omega - WF_s(u)$ we then have $(x_0, \xi) \notin \omega - WF_s(u)$ for each ξ and therefore $x_0 \notin \pi(\omega - WF_s(u))$. Conversely, suppose that $x_0 \notin \pi(\omega - WF_s(u))$. Then $(x_0, \xi) \notin \omega - WF_s(u)$ for each ξ. By compactness of S^{n-1} we can find open cones $V_1, \dots, V_m$ covering $R^n - (0)$ and open neighborhoods $U_1, \dots, U_m$ of x_0 such that $\int_{V_j} |\widehat{\phi u}(\xi)|^2 e^{2s\omega(\xi)} d\xi < \infty$ for $\phi \in \mathcal{D}_\omega(U_j)$, $j = 1, \dots, m$. Thus if $U = \cap_{j=1}^m U_j$ then $\phi u \in H^s_\omega$ for each $\phi \in \mathcal{D}_\omega(U)$. Hence $x_0 \notin H^s_\omega$-sing supp u. Now the last statement is obvious.

We need the following lemma.

LEMMA 6. *Let $u \in \mathcal{D}'_\omega(\Omega)$. We have*

(a) *$\omega - WF_s(u)$ is a closed conic set in $\Omega \times R^n - (0)$.*

(b) *If W is an open subset of Ω, then $\omega - WF_s(u|_W) = \omega - WF_s(u) \cap \pi^{-1}(W)$ where $\pi : \Omega \times R^n - (0) \to \Omega$ is the projection map.*

(c) *If $\psi \in \mathcal{E}_\omega(\Omega)$ then $\omega - WF_s(\psi u) \subseteq \omega - WF_s(u)$.*

(d) *If $\psi \in \mathcal{E}_\omega(\Omega)$ and $1/\psi \in \mathcal{E}_\omega(W_\psi)$, where $W_\psi = \{x \in \Omega | \psi(x) \neq 0\}$, then $\omega - WF_s(\psi u) \cap \pi^{-1}(W_\psi) = \omega - WF_s(u) \cap \pi^{-1}(W_\psi)$.*

PROOF. (a) The complement of $\omega - WF_s(u)$ is the union of the open conic sets $U \times V$ provided by the definition. (b) Obvious. (c) If $(x_0, \xi_0) \notin \omega - WF_s(u)$ then there is a neighborhood U of x_0 and a conic neighborhood V of ξ_0 such that $\int_V |\widehat{\phi u}(\xi)|^2 e^{2s\omega(\xi)} d\xi < \infty$ for all $\phi \in \mathcal{D}_\omega(U)$. Since $\psi\phi \in \mathcal{D}_\omega(U)$ for all $\phi \in \mathcal{D}_\omega(U)$, we have $\int_V |\widehat{\phi(\psi u)}(\xi)|^2 e^{2s\omega(\xi)} d\xi < \infty$ for all $\phi \in \mathcal{D}_\omega(U)$. Hence $(x_0, \xi_0) \notin \omega - WF_s(\psi u)$. (d) By (b) it suffices to show that $\omega - WF_s(\psi u|_{W_\psi}) = \omega - WF_s(u|_{W_\psi})$, and therefore we may assume that $\psi(x) \neq 0$ for each $x \in \Omega$. Then (d) follows from (c) since $u = \frac{1}{\psi}(\psi u)$ and $\frac{1}{\psi} \in \mathcal{E}_\omega(W_\psi)$.

We have

THEOREM 7. (Strong Local Property) *If* $P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$, $a_\alpha \in \mathcal{E}_\omega(\Omega)$ *for* $|\alpha| \leq m$, *then* $\omega - WF_s(Pu) \subseteq \omega - WF_{s+m}(u)$ *for each* $u \in \mathcal{D}'_\omega(\Omega)$.

PROOF. In view of Lemma 6(c), it suffices to show that $\omega - WF_s(D_j u) \subseteq \omega - WF_{s+1}(u)$. If $\phi \in \mathcal{D}_\omega(\Omega)$ then $\phi D_j u = D_j(\phi u) - (D_j \phi) u$. Suppose $(x_0, \xi_0) \notin \omega - WF_{s+1}(u)$. Then there is an open neighborhood U of x_0 and a conic neighborhood V of ξ_0 such that $\int_V |\widehat{\phi u}(\xi)|^2 e^{2(s+1)\omega(\xi)} d\xi < \infty$ for all $\phi \in \mathcal{D}_\omega(U)$. Since $|\widehat{D_j(\phi u)}(\xi)| = |\xi_j| |\widehat{\phi u}(\xi)|$, we have

$$\int_V |\widehat{\phi(D_j u)}(\xi)|^2 e^{2s\omega(\xi)} d\xi \leq 2 \int_V |\xi|^2 |\widehat{\phi u}(\xi)|^2 e^{2s\omega(\xi)} d\xi + 2 \int_V |\widehat{(D_j \phi) u}(\xi)|^2 e^{2s\omega(\xi)} d\xi$$

for all $\phi \in \mathcal{D}_\omega(U)$. Since the condition (γ) on ω implies that $|\xi|^2 \leq e^a e^{2\omega(\xi)}$ for each $\xi \in R^n$, the first integral is finite. And the second one is also finite since $D_j \phi \in \mathcal{D}_\omega(U)$.

REMARK. Suppose that there is a constant a such that $a + \omega_1(\xi) \leq \omega_2(\xi)$ for all $\xi \in R^n$. Then $\omega_1 - WF_s(u) \subseteq \omega_2 - WF_s(u)$ for each $u \in \mathcal{D}'_{\omega_1}(\Omega)$ for $s \geq 0$. And the reverse inclusion holds for $s < 0$. If $\omega(\xi) = \log(1 + |\xi|)$ then $\omega - WF_s(u) = WF_s(u)$ for $u \in \mathcal{D}'(\Omega)$.

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