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A FUNCTIONAL CENTRAL LIMIT THEOREM FOR POSITIVELY DEPENDENT RANDOM FIELDS

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ABSTRACT. In this note we prove a functional central limit theorem for linearly positive quadrant dependent(LPQD) random fields, satisfying some assumption on covariances and the moment condition $\sup_{\underline{n}\in \mathcal{Z}^d} E|S_{\underline{n}}|^{2+\rho} < \infty$ for some $\rho > 0$. We also apply this notion to random measures.

1. Introduction

A random field is a collection of nondegenerate random variables indexed by $\mathcal{Z}^d$ and is denoted by $\{X_{\underline{j}} : \underline{j} \in \mathcal{Z}^d\}$. The random field $\{X_{\underline{j}} : \underline{j} \in \mathcal{Z}^d\}$ may be interpolated and rescaled to form a random element of d-dimensional Skorohod space $\mathcal{D}[0,1]^d$ by setting

$$W_n(\underline{t}) = n^{-\frac{d}{2}} \sum_{j_1=1}^{[nt_1]} \cdots \sum_{j_d=1}^{[nt_d]} X_{\underline{j}}$$

where $\underline{t} = (t_1, \cdots, t_d) \in [0,1]^d$, $\underline{j} = (j_1, \cdots, j_d)$ and $[\cdot]$ is the greatest integer function. $\{X_{\underline{j}} : \underline{j} \in \mathcal{Z}^d\}$ fulfills the functional central limit throrem if $W_n(\underline{t})$ converges weakly in $\mathcal{D}[0,1]^d$ to a d-dimensional Wiener process. This is equivalent to the convergence of finite dimensional distributions together with tightness.

In the past recent years there have been growing interests in concepts of positive dependence for random fields. Such concepts are of considerable use in deriving inequalities in probability and statistics. Lehmann[5] introduced a simple and natural definition of positive dependence.

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A random field $\{X_{\underline{j}} : \underline{j} \in Z^d\}$ is said to be pairwise positive quadrant dependent(pairwise PQD) if for any real $r_{\underline{i}}, r_{\underline{j}}$ and $\underline{i} \neq \underline{j}$

$$P\{X_{\underline{i}} > r_{\underline{i}}, X_{\underline{j}} > r_{\underline{j}}\} \geq P\{X_{\underline{i}} > r_{\underline{i}}\}P\{X_{\underline{j}} > r_{\underline{j}}\}.$$

A much stronger concept than PQD was considered by Esary, Proschan and Walkup[4]; A random field $\{X_{\underline{j}} : \underline{j} \in Z^d\}$ is said to be associated if for any subset $A \subset Z^d$ and for any pair of coordinatewise increasing function f, g: $R^{|A|} \to R$,

$$\mathrm{Cov}[f(X_{\underline{j}} : \underline{j} \in A), g(X_{\underline{j}} : \underline{j} \in A)] \geq 0$$

whenever the covariance is defined where $|A|$ is a cardinality of a set A. Association implies, in particular, nonnegative correlation of the random variables $X_{\underline{j}}$. For details concerning association, see Esary, Proschan and Walkup[4]. We say that a random field $\{X_{\underline{j}} : \underline{j} \in Z^d\}$ is linearly positive quadrant dependence(LPQD) if for any disjoint subsets A,B of Z^d and positive $r_{\underline{j}}'s$

$$\sum_{\underline{i}\in A} r_{\underline{i}} X_{\underline{i}} \quad \text{and} \quad \sum_{\underline{j}\in A} r_{\underline{j}} X_{\underline{j}} \quad \text{are} \quad PQD.$$

Newman[6] showed first that for positively dependent random fields approximate uncorrelatedness implies independence, such that useful theorems can be obtained. In the following years several extentions and generalizations of these results were given. Among them there exists a central limit theorem[6] as well as a functional central limit[3] for associated random fields. The following theorem is due to Newman[6].

THEOREM A (NEWMAN, 1980). *Let $\{X_{\underline{j}} : \underline{j} \in Z^d\}$ be a strictly stationary associated random field with $EX_{\underline{j}} = 0$ and $EX_{\underline{j}}^2 < \infty$. Assume*

$$0 < \sum_{\underline{j}\in Z^d} Cov(X_{\underline{0}}, X_{\underline{j}}) = \sigma^2 < \infty. \tag{1.1}$$

Then the finite dimensional distribution of $W_n(\underline{t})$ converges in distribution to those of the Wiener process with diffusion σ^2.

It is well-known that if $d = 1, 2$ a strictly stationary associated random field $\{X_{\underline{j}} : \underline{j} \in \mathcal{Z}^d\}$ fulfills the functional central limit theorem[8,9]. However, it is still open whether the functional central limit theorem holds $d > 2$. Applying tightness criteria of Birkel and Wichura[1], Burton and Kim[3] proved the following functional central limit theorem for stationary associated random fields.

THEOREM B (BURTON, KIM, 1988). *Let $\{X_{\underline{j}} : \underline{j} \in \mathcal{Z}^d\}$ be a stationary associated random field with $EX_{\underline{j}} = 0, EX_{\underline{j}}^2 < \infty$. Assume that there exists a positive constant C such that for all $\underline{n}$ with positive components*

(1.2) $$E[|S_{\underline{n}}/|\underline{n}|^{\frac{1}{2}}|^{2+\delta} \le C \quad \text{for some} \quad 0 < \delta \le 1,$$

where $\underline{n} = (n_1, \cdots, n_d)$ and $|\underline{n}| = n_1 \times n_2 \times \cdots \times n_d$. Then $\{X_{\underline{j}} : \underline{j} \in \mathcal{Z}^d\}$ fulfills the functional central limit theorem.

In this note, we extend Theorems A and B to the case of LPQD random fields and apply these results to LPQD random mrasures.

2. The functional central limit theorem

In this section we will show that the weak convergence of the finite dimensional distributions of W_n to those of Wiener process W, given in Newman[6], still hold for an LPQD random field.

LEMMA 2.1 (NEWMAN, 1984). *Let $Y_1, \cdots, Y_m$ be LPQD random variables with $EY_{\underline{j}} = 0, EY_{\underline{j}}^2 < \infty$. Then for any real $r_1, \cdots, r_m$,*

(2.1)
$$\left| E\left[exp\Big(i\sum_{j=1}^{m} r_j Y_j\Big)\right] - \prod_{j=1}^{m}[E(exp(ir_jY_j))]\right| \le \sum_{j,k=1, j<k}^{m} |r_j||k_k| Cov(Y_j, Y_k).$$

PROOF. See the proof of Theorem 10 of [7].

For each $n \ge 1$ we consider the LPQD block random variables $\{X_{\underline{k}}^n : \underline{k} \in \mathcal{Z}^d\}$ defined by $X_{\underline{k}}^n = n^{-\frac{d}{2}}(S_{\underline{k}}^n - ES_{\underline{k}}^n)$, where $S_{\underline{k}}^n = \sum_{\underline{j} \in B_{\underline{k}}^n} X_{\underline{j}}$, and $B_{\underline{k}}^n$ is a block of side length n located near $n\underline{k}$, i.e. $B_{\underline{k}}^n = \{\underline{j} : nk_i \le j_i < n(k_i + 1) \text{ for } i = 1, \cdots, d\} = n\underline{k} + B_{\underline{0}}^n$(cf.[6]).

LEMMA 2.2 (NEWMAN, 1980). *Let $\{X_{\underline{j}} : \underline{j} \in \mathcal{Z}^d\}$ be a stationary nonnegatively correlated random field with $EX_{\underline{j}} = 0, EX_{\underline{j}}^2 < \infty$. Assume that (1.1) holds. Then for any $\underline{k}, \underline{j} \in \mathcal{Z}^d$,*

$$\lim_{n\to\infty} E(X_{\underline{k}}^n)^2 = \sigma^2, \tag{2.2}$$

$$\lim_{n\to\infty} E(X_{\underline{0}}^{m_n} - X_{\underline{0}}^n)^2 = 0 \quad if \quad \lim_{n\to\infty} \frac{m_n}{n} = 1, \tag{2.3}$$

$$\lim_{n\to\infty} Cov(X_{\underline{k}}^n, X_{\underline{j}}^n) = 0 \quad if \quad \underline{k} \neq \underline{j}. \tag{2.4}$$

PROOF. See the proof Lemma 4 of [6].

THEOREM 2.3. *Let $\{X_{\underline{j}} : \underline{j} \in \mathcal{Z}^d\}$ be a stationary LPQD random field with $EX_{\underline{j}} = 0, EX_{\underline{j}}^2 < \infty$. Assume that (1.1) holds. Then the finite dimensional distributions of W_n converges in distribution to those of Wiener process W with diffusion σ^2.*

PROOF. As in the proof of Theorem 2 of Newman [6], we use Lemmas 2.1 and 2.2 and the simple fact that

$$X_{\underline{0}}^{ml} = l^{-\frac{d}{2}} \sum_{\underline{j}\in B_{\underline{0}}^l} X_{\underline{j}}^m, \tag{2.5}$$

to obtain

$$E\exp\left(i\sum_{\underline{k}\in\wedge} r_{\underline{k}} X_{\underline{k}}^n\right) \longrightarrow \exp\left(-\sum_{\underline{k}\in\wedge} \frac{\sigma^2 r_{\underline{k}}^2}{2}\right)$$

for any finite subset $\wedge \subset \mathcal{Z}^d$ and any choice of real $r_{\underline{k}}'s$. This completes the proof.

THEOREM 2.4. *Let $\{X_{\underline{j}} : \underline{j} \in \mathcal{Z}^d\}$ be a stationary LPQD random field with $EX_{\underline{j}} = 0, EX_{\underline{j}}^2 < \infty$. Assume that (1.2) holds. Then $\{X_{\underline{j}} : \underline{j} \in \mathcal{Z}^d\}$ fulfills the functional central limit theorem.*

PROOF. Since (1.2) implies (1.1) according to Theorem 2.3, it remains to prove the tightness of the sequence $\{W_n, n \geq 1\}$. From (1.2) there exists a constant $C < \infty$ such that

$$(2.6) \qquad E|W_n(\underline{t})|^{2+\rho} \leq C|\underline{t}|^{\frac{2+\rho}{2}} \text{ for all } \underline{0} \leq \underline{t} \leq \underline{1} \text{ and some } 0 < \rho \leq 1.$$

where $\underline{t} = (t_1, t_2, \cdots, t_d)$ and $|\underline{t}| = t_1 \times t_2 \times \cdots \times t_d$.
Due to (2.6) and the equation (1) and Theorem 1 of Birkel and Wichura[1], for each $\epsilon > 0, \eta > 0$, we can find $\delta > 0$ such that

$$P\{w(W_n, \delta) > \epsilon\} < \eta$$

for all sufficiently large n, where $w(W_n, \delta) = \sup\{|W_n(\underline{t}) - W_n(\underline{s})| : ||\underline{t} - \underline{s}|| < \delta\}$ and $||\underline{t} - \underline{s}|| = \max_{1 \leq j \leq d} |t_j - s_j|$. This and Theorem 15.5 of Billingsley[2] yield the tightness of the sequence $\{W_n, n \geq 1\}$, which completes the proof.

3. The LPQD random measure

For a random measure X we define the λ-renormalization of X to be the signed random measure X_λ, where $X_\lambda(B) = \lambda^{-\frac{d}{2}}[X(\lambda B) - EX(\lambda B)]$ for bounded Borel set $B \subset R^d$, and consider X_λ as a random element of $\mathcal{D}[0,1]^d$ by setting $X_\lambda(\underline{t}) = X_\lambda([\underline{0}, \underline{t}])$, where $[\underline{0}, \underline{t}]$ is the rectangle $[0, t_1] \times \cdots \times [0, t_d]$. A random measure X fulfills the functional central limit theorem with parameter γ^2 if, as $\lambda \to \infty$, X_λ converges in distribution to the d-dimensional Wiener measure on $\mathcal{D}[0,1]^d$ with the constant γ^2, where γ^2 is defined in (3.1)(cf.[3]).

DEFINITION 3.1. A random measure X is said to be LPQD if the family of random variables $\{X(B) : B$ a bounded Borel$\}$ is LPQD.

A simple argument using Chebyshev's inequality allows us to extend the functional central limit for LPQD random fields to random measures.

THEOREM 3.2. *Let X be a stationary LPQD random measure with $EX^2(B) < \infty$ for a bounded set B. Assume*

$$(3.1) \qquad 0 < \sum_{\underline{k} \in \mathcal{Z}^d} Cov(X(I), X(I + \underline{k})) = \gamma^2 < \infty,$$

where I is the unit cube in R^d. Then X satisfies the central limit theorem.

PROOF. Let X denote a random interval function subject to the conditions of theorem. Consider the distribution of

$$\frac{X(\lambda I) - EX(\lambda I)}{\lambda^{\frac{d}{2}}}$$

as $\lambda \to \infty$. Let $Y_{\underline{j}} = X(I + \underline{j}), \underline{j} \in \mathcal{Z}^d$. Then $\{Y_{\underline{j}} \ \underline{j} \in \mathcal{Z}^d\}$ is a family of stationary LPQD random variables for which

$$\gamma^2 = \sum_{\underline{j} \in \mathcal{Z}^d} \mathrm{Cov}(Y_{\underline{0}}, Y_{\underline{j}}) < \infty.$$

From Theorem 2.3 it follows that

$$\left\{ \frac{S_{\underline{n}}^k - ES_{\underline{n}}^k}{k^{\frac{k}{2}}} : \underline{n} \in \mathcal{Z}^d \right\} \to \{W_{\underline{n}} : \underline{n} \in \mathcal{Z}^d\} \text{ as } k \to \infty,$$

in the sense of convergence of finite dimensional distributions, where $\{W_{\underline{n}} : \underline{n} \in \mathcal{Z}^d\}$ are i.i.d., Gaussian random variables with mean zero and variance η, and $S_{\underline{n}}^k = \sum_{\underline{j} \in [k(I+\underline{n})]} Y_{\underline{j}}$, where $[B] = \{\underline{n} = (n_1, \cdots, n_d) \in \mathcal{Z}^d : \underline{n} \in B\} = \mathcal{Z}^d \cap B$. Let $\mathcal{D}_\lambda = \{\underline{j} \in \mathcal{Z}^d : I + \underline{j}\} \subset [\lambda]I$, where $[\lambda]$ denotes the greatest integer in λ. Also let $I_\lambda^0 = \lambda I / [\lambda] I$. Then $X(\lambda I) = X(I_\lambda^0) + \sum_{\underline{j} \in \mathcal{D}_\lambda} Y_{\underline{j}}$.

Moreover, $X(\lambda I) - EX(\lambda I) = [X(I_\lambda^0) - EX(I_\lambda^0)] + \sum_{\underline{j} \in \mathcal{D}_\lambda} (Y_{\underline{j}} - EY_{\underline{j}})$.

Due to the fact $I_\lambda^0 \subset ([\lambda] + 1)I/[\lambda]I$ and the nonnegative correlatedness, $VarX(I_\lambda^0) = o(\lambda^{d-1})$ as $\lambda \to \infty$. In particular, it follows from Chebyshev's inequality that $[X(I_\lambda^0) - EX(I_\lambda^0)]/\lambda^{\frac{d}{2}}$ converges in probability to zero as $\lambda \to \infty$. Since $[\lambda]^d \sim \lambda^d$ as $\lambda \to \infty$ the result follows from Theorem 2.3.

THEOREM 3.3. *Let X be a stationary LPQD random measure. Assume that there is a positive constant K depending only on $\delta(0<\delta\le 1)$ such that for all rectangular boxes $B\supseteq I$,*

$$E[|X(B)-EX(B)|^{2+\delta}]\le K|B|^{1+\frac{\delta}{2}}, \tag{3.2}$$

where $|B|$ is a Lebesgue measure of B. Then X fulfills the functional central limit theorem.

PROOF. By (3.2),(3.1) holds. Set

$$\begin{aligned} X_\lambda(\underline{t}) &= \lambda^{-\frac{d}{2}}[X(\lambda[\underline{0},\underline{t}])-EX(\lambda[\underline{0},\underline{t}])] \\ &= \lambda^{-\frac{d}{2}}\sum_{\underline{0}\le\underline{k}\le[[\lambda]\underline{t}]}[X(I+\underline{k})-EX(I+\underline{k})] \\ &\quad+\lambda^{-\frac{d}{2}}[X(I_\lambda)-EX(I_\lambda)], \end{aligned} \tag{3.3}$$

where $I_\lambda=[\underline{0},[\lambda\underline{t}]]/[\underline{0},[[\lambda]\underline{t}]]$. Since $[\lambda]^d\sim\lambda^d$ as $\lambda\to\infty$, the first term of the right hand side of (3.3) fulfills the functional central limit theorem according to Theorem 2.4. Since $I_\lambda\subset[\underline{0},[[\lambda]\underline{t}]+1]/[\underline{0},[\lambda\underline{t}]]$ $X(B)$'s are nonnegatively correlated with $Var(I_\lambda)=o(\lambda^{d-1})$ as $n\to\infty$, and thus the second term of the right-hand side of (3.3), $\lambda^{-\frac{d}{2}}[X(I_\lambda)-EX(I_\lambda)]$ converges in probability to zero as $\lambda\to\infty$ by Chebyshev's inequality. This completes the proof.

4. An Example

Finally, we give an example. Poisson center random measures have been used as models of infinite divisibility and self-similarity as well as models of natural phenomena as in storm systems and galaxies (see [3] and the reference therein). These are constructed as follows. Let U be a stationary Poisson point random field with parameter ρ. Let $V=\{V_{\underline{x}}:\underline{x}\in R^d\}$ be a collection of i.i.d. random measures with $E[V_{\underline{x}}(R^d)]<\infty$. Then we say that X is a cluster process with centers U and members V if

$$X(B)=\sum_{\underline{x}\in U(\underline{x})>0}V_{\underline{x}}(B-\underline{x})$$

for each bounded Borel set B. We denote X by $[U,V]$.

THEOREM 4.1 (BURTON, KIM, 1988). *Let $X = [U, V]$ as above. Let B be a rectangular box in R^d and $0 < \delta \leq 2$; then there is a constant K depending only on δ and $|B|$ so that*

$$E[|X(B)|^{2+\delta}] \leq KE[(V_{\underline{x}}(R^d))^{2+\delta}].$$

Moreover, if $E[(V_{\underline{x}}(R^d))^{2+\delta}] < \infty$ then X satisfies (3.2).

THEOREM 4.2. *Let $X = [U, V]$ as above. Then X is LPQD. Moreover, if $E[(V_{\underline{x}}(R^d))^{2+\delta}] < \infty$ then X fulfills a functional central limit theorem.*

PROOF. Note that association implies LPQD. Since X is associated (see [3]), X is also LPQD, and thus X fulfills a functional central limit theorem according to Theorem 3.3 and Theorem 4.1.

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