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CHARACTERIZATIONS OF CONICAL LIMIT POINTS FOR KLEINIAN GROUPS

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ABSTRACT. For a nonelementary discrete group Γ of hyperbolic isometries acting on $B^m (m \geq 2)$, we give a topological characterization of conical limit points using admissible pairs.

1. Introduction

Let Γ be a nonelementary discrete group of hyperbolic isometries acting on the Poincarè disc B^{m+1}, $m \geq 1$. The discrete group Γ acts as a discontinuous group on B^{m+1}, and Γ acts on S^m, the boundary of B^{m+1}, as a group of conformal homeomorphisms, but need not be discontinuous there. The set of points of S^m at which Γ does not act discontinuously is the *limit set*. An alternative definition of limit set would be that a point p in S^m is a limit point for the discrete group Γ if for one, and hence every point x in B^{m+1}, the orbit $\Gamma(x)$ accumulates at p, and then the set of limit points is the limit set $\Lambda(\Gamma)$ or simply Λ. The *ordinary set* $\Omega(\Gamma)$ is its complement $S^m \setminus \Lambda(\Gamma)$. Then we see that $\Omega(\Gamma)$ is the maximal domain of discontinuity for Γ in S^m.

It is well known (see [**B**] and [**G**] for example) that hyperbolic isometries acting on the Poincarè disc B^{m+1} are Möbius transformations. One says that a discrete subgroup of Möbius group with a non-empty ordinary set is called a *Kleinian group*. A subgroup Γ of Möbius group is said to be *elementary* if and only if $\Lambda(\Gamma)$ consists of only 0, 1 or 2 points. Otherwise it is said to be nonelementary group. So we are more interested in the nonelementary Kleinian groups.

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Now let $p \in S^m$ be a limit point. By a neighborhood of p, we will always mean an open neighborhood of p in S^m. One says that an open set U can be concentrated at p, if for every neighborhood V of p, there exists an element $\gamma \in \Gamma$ so that $p \in \gamma(U)$ and $\gamma(U) \subseteq V$. Equivalently, U can be concentrated at p if and only if the set of translates of U contains a local basis for the topology of S^m at p. If an element γ can be selected so that $\gamma^{-1}(p)$ is arbitrarily close to p, then one says that U can be concentrated with control at p. An element γ can be selected in Γ so that $\gamma^{-1}(p)$ is arbitrarily close to p if and only if for every neighborhood V of p, there exists an element γ in Γ so that $p \in \gamma(V)$.

DEFINITION 1.1. The limit point p is called a *weak concentration point* for Γ if there exists an open set U that can be concentrated at p.

DEFINITION 1.2. The limit point p is called a *controlled concentration point* for Γ if there exists an open set U that can be concentrated with control at p.

We may see easily that every controlled concentration point is a weak concentration point from definitions.

REMARK. If an open set can be concentrated at p, then every smaller open set can be concentratated at p. Since S^m is locally connected, we may assume that the open set U in the above Definitions 1.1 and 1.2 is round open ball in S^m. And we can assume that the open set U is a neighborhood of p in S^m without loss generality. While the open set does not contain p, it is always possible to avoid it by replacing U by $\gamma(U)$ for some suitable γ in Γ. Let W be a round open ball in S^m whose boundary is spanned by $m-$ dimensional hyperbolic hyperplane H in B^{m+1}, and that the closure of H makes two components in $B^{m+1} \cup S^m$. In this note, we will say that U is a *half- ball* in $B^{m+1} \cup S^m$ if U is the component containing W. We are going to use such U as a neighborhood of p in $B^{m+1} \cup S^m$, if necessary.

DEFINITION 1.3. The limit point p is called a *conical limit point* for Γ if there is a sequence $\{\gamma_i\}_{i=1}^{\infty}$ of elements of Γ, a hyperbolic ray λ in B^{m+1} ending at p, and a constant $r > 0$ such that $\{\gamma_i(0)\}_{i=1}^{\infty}$ converges to p within the r -neighborhood $N(\lambda, r)$ of λ in B^{m+1}.

2. Characterizations of Conical Limit Points

In this section, we will examine an analytic and a geometric characterizations of conical limit points for nonelementary Kleinian groups and then we will give a topological characterization of conical limit points using the analytic and geometric characterizations.

Hereafter, we use $d(x, y)$ for the hyperbolic distance between x and y in B^{m+1}.

LEMMA 2.1 [B-M]. *Let p be a conical limit point of a discrete group Γ (that is, there is a sequence $\{\gamma_i\}_{i=1}^{\infty}$ of elements in Γ, a hyperbolic ray λ in B^{m+1} ending at p and a constant $r > 0$ such that $\{\gamma_i(0)\}_{i=1}^{\infty}$ converges to p within $N(\lambda, r)$). Then*

(1) *For each x in B^{m+1}, there is a constant $s > 0$ such that $\{\gamma_i(x)\}_{i=1}^{\infty}$ converges to p within $N(\lambda, s)$.*
(2) *For each hyperbolic ray μ in B^{m+1} ending at p, there is a constant $t > 0$ such that $\gamma_i(0)$ converges to p within $N(\mu, t)$.*

From the above Lemma 2.1, we can use $\lambda = [0, p)$ without loss of generality.

First of all, we give a chracterization of Beadon and Maskit.

THEOREM 2.2. *The following are equivalent.*

(1) *For some (hence every) hyperbolic ray λ in B^{m+1} ending at p, there is a constant $r > 0$ such that $\gamma_i(0)$ converges to p within $N(\lambda, r)$ (that is, p is a conical limit point).*
(2) *For some (hence every) hyperbolic ray λ ending at p, there is a compact subset K of B^{m+1} such that, for all i , $K \cap \gamma_i^{-1}(\lambda) \neq \emptyset$.*

Now we give a geometric characterization of conical limit points whose proof is different from Beardon-Maskit's.

THEOREM 2.3. *A limit point p is a conical limit point for Γ if and only if there is a sequence γ_i of elements of Γ such that $\gamma_i^{-1}(p)$ converges to q and $\gamma_i^{-1}(0)$ converges to r where $r \neq q$.*

PROOF. Let λ be the half-geodesic in B^{m+1} that runs from 0 to p. Let B be some closed ball centered at 0. Since p is a conical limit point, there exist a sequence of elements γ_i in Γ such that, for all i, $B \cap \gamma_i^{-1}(\lambda) \neq \emptyset$

(See Theorem 2.2 (2)). Then we can choose $p_i \in \gamma_i(B) \cap \lambda$, for each i. Therefore there exists a sequence of points p_i on λ so that $\lim p_i = p$ and $\gamma_i^{-1}(p_i) \in B$. Let v_i denote the unit (for the hyperbolic metric) tangent vector to λ to p_i, pointing in the direction of p. Since the space $T_1(B)$ of unit tangent vectors to points of B is compact, $T(\gamma_i^{-1})(v_i) \longrightarrow \omega_0$ for some $\omega_0 \in T_1(B)$. Since B is compact, $\gamma_i^{-1}(p_i) \rightarrow w$ for some $w \in B$. Then w is the starting point of ω_0. Let μ be the geodesic which has ω_0 as the tangent vector at w. Let q, r be two ending points of μ. Since $B \cap \mu \neq \emptyset$, $q \neq r$. Furthermore $\gamma_i^{-1}(p) \rightarrow q$ and $\gamma_i^{-1}(0) \rightarrow r$.

Conversely, let μ be the geodesic from r to q. Since $\gamma_i^{-1}(0) \rightarrow r$ and $\gamma_i^{-1}(p) \rightarrow q$, $\gamma_i^{-1}([0,p)) \rightarrow \mu$. Let $d(0,\mu) = s < \infty$. Then for sufficiently large i, $d(0, \gamma_i^{-1}[0,p)) < s + 1 = r$. Thus $d(\gamma_i(0), [0,p)) < r$. Therefore p is a conical limit point.

Here are two definitions which are needed for the topological approach of conical limit points.

DEFINITION 2.4. (U_1, U_2) is called an *admissible pair* at p if U_1, U_2 are open neighborhoods of p with $\overline{U_2} \subset U_1, U_2 \cap \Lambda \neq \emptyset$ and $\Lambda \not\subset \overline{U_1}$.

DEFINITION 2.5. An admissible pair (U_1, U_2) can be *concentrated* at p if for every neighborhood V of p, there exists an element $\gamma \in \Gamma$ so that $\gamma(U_1) \subseteq V$ and $p \in \gamma(U_2)$.

We now state the main theorem in this paper.

THEOREM 2.6. *A limit point p is a conical limit point if and only if there exists an admissible pair (U_1, U_2) concentrated at p.*

PROOF. Suppose that p is a conical limit point Then by Theorem 2.3, there exist a sequence of element γ_i in Γ such that $\gamma_i^{-1}(p)$ converges to q and $\gamma_i^{-1}(0)$ converges to r where $r \neq q$.

Let H_1 denote the m-dimensional hyperbolic hyperplane in B^{m+1} that passes through p_1 and is perpendicular to $[0,p)$ where p_1 is a point on $\gamma_1(B) \cap [0,p)$ (See proof of Theorem 2.3). Let U_1 be the component of $\partial B^{m+1} - \partial H_1$ containing p. By convergence, we may pass to a further subsequence (getting a new H_1 and a new U_1) so that $\gamma_i \gamma_1^{-1}(H_1)$ meets $[0,p)$, with its normal vector at the intersection points making an angle bounded below $\pi/2$ with the tangent vector to $[0,p)$. Then, for all i,

$\gamma_i\gamma_1^{-1}(U_1)$ contain p and diameters limit to 0. So $\gamma_i\gamma_1^{-1}(U_1)$ contains a local basis for p. Let H_2 denote the m – dimensional hyperbolic hyperplane that passes through p_2 and is perpendicular to $[0,p)$ where p_2 is a point on $\gamma_2(B)\cap[0,p)$. Let U_2 be the component of $\partial B^{m+1}-\partial H_2$ containing p with $\overline{U_2}\subset U_1$. We can get a new H_2 and U_2 satisfing $\overline{U_2}\subset U_1$. On the other hand, $q\in\gamma_1^{-1}(U_2)$. Since $\gamma_i^{-1}(p)\longrightarrow q$, $\gamma_i^{-1}(p)\in\gamma_1^{-1}(U_2)$. Therefore $p\in\gamma_i\gamma_1^{-1}(U_2)$. Then $\{\gamma_i\gamma_1^{-1}(U_1)\}$ contains a local basis for p and hence for each neighborhood V of p there is a $\gamma_n\in\Gamma$ such that $p\in\gamma_n\gamma_1^{-1}(U_2)\subset\gamma_n\gamma_1^{-1}(U_1)\subset V$. Therefore $(U\ ,U_2)$ is an admissible pair at p that can be concentrated at p.

Conversely, suppose a pair (U_1,U_2) can be concentrated at p. Let $\{V_i\}$ be a sequence of neighborhoods of p such that $\operatorname{diam}V_i\longrightarrow 0$. For each i, there exists $\gamma_i\in\Gamma$ such that $\gamma_i(U_1)\subseteq V_i$, $p\in\gamma_i(U_2)$. Then $\gamma_i^{-1}(p)\in U_2\subseteq\overline{U_2}$. Since $\overline{U_2}$ is compact in S^m, $\gamma_i^{-1}(p)\longrightarrow q$, for some $q\in\overline{U_2}$. Since Γ act discontinuously on B^{m+1}, $\gamma_i^{-1}(0)\longrightarrow r$ for some $r\in S^m$. If we show that $r\neq q$, then this proof is complete from Theorem 2.3.

Now we may assume that V_i and U_1 are half ball in $B^{m+1}\cup S^m$ (See Remark in the introduction). Suppose $q=r$. Since for all $x\in B^{m+1}$, $\gamma_i^{-1}(x)$ converges to r, there exists x_0 in $B^{m+1}\setminus U_1$ so that for all sufficiently large i, $\gamma_i^{-1}(x_0)\in U_1$. Then

$$\gamma_i^{-1}(B^{m+1}\setminus U_1)\subseteq\gamma_i^{-1}(B^{m+1}\setminus V_i)=B^{m+1}\setminus\gamma_i^{-1}(V_i)\subseteq B^{m+1}\setminus U_1$$

for all sufficiently large i. Therefore $\gamma_i^{-1}(x_0)\in\gamma_i^{-1}(B^{m+1}\setminus U_1)\subseteq B^{m+1}\setminus U_1$. So $\gamma_i^{-1}(x_0)\notin U_1$. Thus we have a contradiction. Therefore r is not equal to q.

From Theorem 2.6, one can easily deduce the following corollaries.

COROLLARY 2.7. *Every conical limit point is a weak concentration point.*

PROOF. U_1 in Theorem 2.6 is an open set that can be concentrated at p.

COROLLARY 2.8. *Every controlled concentration point is a conical limit point.*

PROOF. Let U be an open set that can be concentrated with control at p. Then we can see that for any open set $V \subset U$ with $\bar{V} \subset U$, (U, V) is a concentrated pair at p.

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