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DECOMPOSABLE RIGHT HALF SMASH PRODUCT SPACES

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ABSTRACT. It is shown that for any space A, the cofibration $X \xrightarrow{i} X \rtimes \Sigma A \xrightarrow{q} \Sigma A \wedge X$ is decomposable when X is a co-T-space. It is also obtain necessary and sufficient conditions for the cofibration $X \xrightarrow{i} X \rtimes A \xrightarrow{q} A \wedge X$ is trivial, in the sense of cofibre homotopy type

1. Introduction

Let $B \xrightarrow{i} E \xrightarrow{q} F$ be a cofibration with cofiber F. We say that the cofibration $B \xrightarrow{i} E \xrightarrow{q} F$ is *decomposable* if E and $B \vee F$ have the same homotopy type. The cofibration $B \xrightarrow{i} E \xrightarrow{p} F$ is *cofibre homotopically trivial* if there exists a homotopy equivalence $h : E \to B \vee F$ such that $hi = i_1$, where $B \xrightarrow{i_1} B \vee F \xrightarrow{p_2} F$ is the trivial cofibration. Clearly a cofibration is decomposable if it is cofibre homotopically trivial. The purpose of this paper is to investigate decomposability of the cofibration $X \xrightarrow{i} X \rtimes A \xrightarrow{q} A \wedge X$. In the case $A = S^1$, the first author studied[7] some properties of decomposability of the above cofibration. In Section 2, we obtain necessary and sufficient conditions for the cofibration $X \xrightarrow{i} X \rtimes A \xrightarrow{q} A \wedge X$ is cofibre homotopically trivial. It is shown that for any space A, the cofibration $X \xrightarrow{i} X \rtimes \Sigma A \xrightarrow{q} \Sigma A \wedge X$ is decomposable when X is a co-T-space. In Section 3, we introduce the concept of a T'_A map between T'_A-spaces. Throughout this paper, space means a space of homotopy type of 1-connected locally finite CW complex. We assume

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also that spaces have non-degenerate base points. Let $L(A, X)$ denote the spaces of maps from A to X with the compact open topology and $L_0(A, X)$ denote the space of base point preserving maps in $L(A, X)$. All maps shall mean continuous functions. The base point as well as the constant map will be denoted by $*$. We also denote by $[X, Y]$ the set of homotopy classes of pointed maps $X \to Y$. The identity map of space will be denoted by 1 when it is clear from the context. The diagonal map $\Delta : X \to X \times X$ is given by $\Delta(x) = (x, x)$ for each $x \in X$, the folding map $\nabla : X \vee X \to X$ is given by $\nabla(x, *) = \nabla(*, x) = x$ for each $x \in X$.

2. Decomposable right half smash product spaces

We consider $X \rtimes A$, *the right half smash of* X *and* A. This is the space obtained from $X \times A$ by pinching $* \times A$ to $*$, that is, $X \rtimes A = X \times A / * \times A$. Then there is also the canonical cofibration

$$X \xrightarrow{\ i\ } X \rtimes A \xrightarrow{\ q\ } A \wedge X,$$

where $i(x) = [x, *]$ and $q([x, a]) = < a, x >$.

DEFINITION 2.1. We say that the right half smash $X \rtimes A$ of X and A is *decomposable* if the cofibration $X \xrightarrow{i} X \rtimes A \xrightarrow{q} A \wedge X$ is decomposable.

Observe that if there exists a homotopy equivalence $h : X \rtimes A \to X \vee (A \wedge X)$ such that the following diagram is homotopy commute;

$$\begin{array}{ccccc} X & \xrightarrow{\ i\ } & X \rtimes A & \xrightarrow{\ q\ } & A \wedge X \\ \| & & \downarrow h & & \\ X & \xrightarrow{\ i_1\ } & X \vee (A \wedge X) & \xrightarrow{\ p_2\ } & A \wedge X, \end{array}$$

then there is a homotopy equivalence $\hat{h} : X \rtimes A \to X \vee (A \wedge X)$ such that $\hat{h} i = i_1$ and $h \sim \hat{h}$. Thus the cofibration $X \xrightarrow{i} X \rtimes A \xrightarrow{q} A \wedge X$ is cofibre homotopically trivial. Clearly the right half smash $X \rtimes A$ of X and A is decomposable if the cofibration$X \xrightarrow{i} X \rtimes A \xrightarrow{q} A \wedge X$ is cofibre homotopically trivial.

LEMMA 2.2. *There exist a map* $h : X \rtimes A \to X \vee (A \wedge X)$ *and a homotopy equivalence* $h' : A \wedge X \to A \wedge X$ *such that* $hi \sim i_1$ *and* $h'q \sim p_2 h$ *if and only if the cofibration* $X \xrightarrow{i} X \rtimes A \xrightarrow{q} A \wedge X$ *is cofibre homotopically trivial.*

PROOF. Suppose there exist a map $h : X \rtimes A \to X \vee (A \wedge X)$ and a homotopy equivalence $h' : A \wedge X \to A \wedge X$ such that $hi \sim i_1$ and $h'q \sim p_2 h$. It is sufficient to show that the above map $h : X \rtimes A \to X \vee (A \wedge X)$ is a homotopy eqivalence. By applying the five lemma to the homology sequences arising from the diagram

$$\begin{array}{ccccc} X & \xrightarrow{i} & X \rtimes A & \xrightarrow{q} & A \wedge X \\ \| & & \Big\downarrow h & & \Big\downarrow h' \\ X & \xrightarrow{i_1} & X \vee (A \wedge X) & \xrightarrow{p_2} & A \wedge X, \end{array}$$

$h : X \rtimes A \to X \vee (A \wedge X)$ is a homology isomorphism. By the Van Kampen theorem, $\pi_1(X \vee (A \wedge X)) = 0$. Since $(X \times A, * \times A)$ has the AHEP, $X \times A \cup C(* \times A)$ and $X \rtimes A$ have the same homotopy type and $\pi_1(X \rtimes A) \cong \pi_1(X \times A \cup C(* \times A)) \cong \pi_1(X \times A \cup C(* \times A), C(* \times A)) \cong \pi_1(X \times A, * \times A) = 0$. Since all spaces are simply connected, the assertion follows. If the cofibration $X \xrightarrow{i} X \rtimes A \xrightarrow{q} A \wedge X$ is cofibre homotopically trivial, then there is a homotopy equivalence $h : X \rtimes A \to X \vee (A \wedge X)$ such that $hi = i_1$. Then the map $h : X \rtimes A \to X \vee (A \wedge X)$ induces a homotopy equivalence $\bar{h} : A \wedge X \to A \wedge X$ with $\bar{h}q = p_2 h$. Similary, it follows from the homology ladder of h and the five lemma, and the simply connectivity of space $A \wedge X$.

COROLLARY 2.3. *If* X *is a co-*H*-space, the cofibration* $X \xrightarrow{i} X \rtimes A \xrightarrow{q} A \wedge X$ *is cofibre homotopically trivial.*

PROOF. If $\mu : X \to X \vee X$ is a comultiplication on X, one can define a map $h : X \rtimes A \to X \vee (A \wedge X)$ by $h([x, a]) = (p_1\mu(x), < a, p_2\mu(x) >)$. Then hi is homotopic to i_1 and $p_2 h$ is homotopic to q. Thus we know, from Lemma 2.2, that the cofibration $X \xrightarrow{i} X \rtimes A \xrightarrow{q} A \wedge X$ is cofibre homotopically trivial.

In fact, it is well known[4] that $\Sigma X \rtimes A$ is homotopy equivalent to $\Sigma X \vee \Sigma(X \wedge A)$, and $\Sigma X \to \Sigma X \rtimes A \equiv \Sigma(X \rtimes A)$ is a co-H- group inclusion. It is also known[3] that if X is a co-H-group, then $X \rtimes A$ is a co-H-group.

COROLLARY 2.4. *$S^n \rtimes S^r$ is homotopy equivalent to $S^n \vee S^{n+r}$ for $n > 1, r \geqq 1$.*

THEOREM 2.5. *The cofibration $X \xrightarrow{i} X \rtimes A \xrightarrow{q} A \wedge X$ is cofibre homotopically trivial if and only if there is a map $r : A \wedge X \to X \rtimes A$ such that $qr \sim 1_{A\wedge X}$.*

PROOF. Suppose that there are homotopy equivalences $h : X \rtimes A \to X \vee (A \wedge X)$ and $h' : A \wedge X \to A \wedge X$ such that $hi \sim i_1$ and $p_2 h \sim h'q$. Consider the map $r = h^{-1} i_2 h' : A \wedge X \to A \wedge X \to X \vee (A \wedge X) \to X \rtimes A$. Then $qr = qh^{-1} i_2 h' \sim h'^{-1} p_2 i_2 h' \sim 1_{A\wedge X}$. On the other hand, suppose there is a map $r : A \wedge X \to X \rtimes A$ such that $qr \sim 1_{A\wedge X}$. Let $k = \nabla(i \vee r) : X \vee (A \wedge X) \to X \rtimes A \vee (X \rtimes A) \to X \rtimes A$. Then $ki_1 = i$ and $qk \sim p_2$. For let $h_s : A \wedge X \to A \wedge X$ be the homotopy from qr to $1_{A\wedge X}$. Then $h_s p_2 : X \vee (A \wedge X) \to A \wedge X$ is a homotopy from qk to p_2. This proves the theorem.

DEFINITION 2.6. A *T'_A-structure* on a space X will be a map $r : A \wedge X \to X \rtimes A$ such that qr is homotopic to $1_{A\wedge X}$. We could include the T'_A-structure in the definition of T'_A-space by defining a *T'_A-space* as a couple (X, r).

A space X is called *T'-space*[7] if X is T_{S^1}-space. A based map $f : X \to A$ is called *cocyclic*[5] if there exists a map $\phi : X \to X \vee A$ such that $j\phi \sim (1 \times f)\Delta$, where $j : X \vee A \to X \times A$ is the inclusion and $\Delta : X \to X \times X$ is the diagonal map. The *dual Gottlieb set* denoted by $DG(X, A)$ the set of all homotopy classes of cocyclic maps from X to A.

PROPOSITION 2.7. *Let X be a $T'_{\Sigma A}$-space. Then there is an one-to-one correnspondence between the set of homotopy classes of $T'_{\Sigma A}$-structures on a space X and the set $DG(\Sigma(A \wedge X), X)$ of all homotopy classes of cocyclic maps from $\Sigma(A \wedge X)$ to X.*

PROOF. Since X is a $T'_{\Sigma A}$-space, there exist homotopy equivalences $h : X \rtimes \Sigma A \to X \vee \Sigma(A \wedge X)$ and $h' : \Sigma(A \wedge X) \to \Sigma(A \wedge X)$ such that $hi \sim i_1$ and $p_2 h \sim h'q$. Define a map $\phi : \{[r] \in [\Sigma(A \wedge X), X \rtimes \Sigma A] | qr \sim 1_{\Sigma(A \wedge X)}\} \to DG(\Sigma(A \wedge X), X)$ by $\phi([r]) = [p_1 hrh'^{-1}]$. Also, consider the map $\psi : DG(\Sigma(A \wedge X), X) \to \{[r] \in [\Sigma(A \wedge X), X \rtimes \Sigma A] | qr \sim 1_{\Sigma(A \wedge X)}\}$ given by $\psi([f]) = [h^{-1}(f \vee 1)\mu h']$, where $\mu : \Sigma(A \wedge X) \to \Sigma(A \wedge X) \vee \Sigma(A \wedge X)$ is the comultiplication on $\Sigma(A \wedge X)$. Clearly $\phi\psi = 1$. Now since $p_2 hrh'^{-1} \sim 1_{\Sigma(A \wedge X)}, (p_1 hrh'^{-1} \vee 1)\mu \sim (p_1 hrh'^{-1} \vee p_2 hrh'^{-1})\mu \sim hrh'^{-1}$. Thus $\psi\phi([r]) = [h^{-1}(p_1 hrh'^{-1} \vee 1)\mu h'] = [h^{-1}hrh'^{-1}h'] = [r]$. This proves the proposition.

Consider the map $e'_A : X \to L_0(A, A \wedge X)$ given by $e'_A(x)(a) = \langle a, x \rangle$. Then e'_A is well defined and continuous.

DEFINITION 2.8. A space X is called a *co-T_A-space* if $e'_A : X \to L_0(A, A \wedge X)$ is cocyclic.

It is easily shown that any co-H-space is a co-T_A-space. A space X is called *a co-T-space*[6] if $e' : X \to \Omega\Sigma X$ is cocyclic. It is known[6] that X is a co-T-space if and only if $DG(X, \Omega A) = [X, \Omega A]$ for any space A, where $DG(X, \Omega A) = \{[f] \in [X, \Omega A] | f : X \to \Omega A \text{ is cocyclic}\}$.

LEMMA 2.9[5].

(1) *If $f : X \to A$ is cocyclic and $g : A \to B$ is any map, then $gf : X \to B$ is cocyclic.*

(2) *If $h : Y \to X$ is a map which has a left homotopy inverse and $f : X \to A$ is cocyclic, then $fh : Y \to A$ is cocyclic.*

We show that a co-T_A-space may be characterized by the dual Gottlieb set as follows;

THEOREM 2.10. *X is a co-T_A-space if and only if $DG(X, L_0(A, B)) = [X, L_0(A, B)]$ for any space B.*

PROOF. Suppose X is a co-T_A-space. Let $f : X \to L_0(A, B)$ be a map. Then there is a map $\bar{f} : A \times X \to B$ given by $\bar{f}(a, x) = f(x)(a)$. Consider the map $\hat{f} : A \wedge X \to B$ with $\hat{f}q = \bar{f}$, where $q : A \times X \to A \wedge X$ is the quotient map. From the fact $f = \hat{f}_* e'_A : X \to L_0(A, B)$ and e'_A is cocyclic, we know, from Lemma 2.9 (1), that $f : X \to L_0(A, B)$ is a cocyclic. Conversely, take $B = A \wedge X$. Then $e'_A: X \to L_0(A, A \wedge X)$ is cocyclic.

COROLLARY 2.11. *If X is a co-T-space, then for any space A, X is a co-$T_{\Sigma A}$-space.*

PROOF. We know that there is a homeomorphism $\theta : L_0(\Sigma A, B) \to L_0(S^1, L_0(A, B))$ given by $\theta(f)(s)(a) = f < a, s >$. We show that $DG(X, L_0(\Sigma A, B)) = [X, L_0(\Sigma A, B)]$ for any space B. Let $f : X \to L_0(\Sigma A, B)$ be a map. Since X is a co-T-space, we know that $DG(X, L_0(S^1, L_0(A, B))) = [X, L_0(S^1, L_0(A, B))]$. Thus we know, from Lemma 2.9, that $\theta f : X \to L_0(S^1, L_0(A, B))$is cocyclic and $f \sim \theta^{-1}(\theta f) : X \to L_0(\Sigma A, B)$ is cocyclic.

THEOREM 2.12. *Any co-T_A-space is a T'_A-space.*

PROOF. Let X be a co-T_A-space. Then we have a map $\theta : X \to X \vee L_0(A, A \wedge X)$ such that $j\theta$ is homotopic to $(1 \times e'_A)\Delta$, where $j : X \vee L_0(A, A \wedge X) \to X \times L_0(A, A \wedge X)$ is the inclusion. Let $h = k\theta : X \to X \vee L_0(A, A \wedge X) \to L(A, X \vee (A \wedge X))$, where $k(x, *) =$ constant map at $(x, *)$, $k(*, f) = i_2 f$. Then h gives rise to a map $\bar{h} : X \times A \to X \vee (A \wedge X)$ with $\bar{h}(x, a) = h(x)(a)$. Since $\bar{h}(*, a) = k\theta(*)(a) = (*, *)$, $\bar{h}$ induces a map $\hat{h} : X \rtimes A \to X \vee (A \wedge X)$. Let $p : L(A, X \vee (A \wedge X)) \to X \vee (A \wedge X)$ be the evaluation map, that is, $p(f) = f(*)$. Then $pk = (1 \vee *) : X \vee L_0(A, A \wedge X) \to X \vee (A \wedge X)$ and $(1 \vee *)\theta \sim i_1$. Thus $\hat{h}i = pk\theta = (1 \vee *)\theta \sim i_1$. Moreover, for a homotopy $f_t : X \to L_0(A, A \wedge X)$ from $p_2 j\theta$ to e'_A, let $g_t = \hat{f}_t q : X \rtimes A \to A \wedge X$, where $\hat{f}_t < a, x >= f_t(x)(a)$. Since $p_2 \hat{h} = \widehat{p_{2*}k\theta} = \widehat{p_2 j}\theta$, g_t is a homotopy from $p_2\hat{h}$ to q. Thus X is a T'_A-space. This proves the theorem.

From Corollary 2.11 and Theorem 2.12, we have the following corollary.

COROLLARY 2.13. *If X is a co-T-space, then for any space A, the cofibration $X \to X \rtimes \Sigma A \to \Sigma A \wedge X$ is decomposable.*

3. T'_A-maps

A T'_A-structure on a space X will be a map $r : A \wedge X \to X \rtimes A$ such that qr is homotopic to $1_{A\wedge X}$. We could include the T'_A-structure in the definition of T'_A-space by defining a T'_A-space as a couple (X, r).

DEFINITION 3.1. Let (X, r_X) and (Y, r_Y) be T'_A-spaces. A map $f : X \to Y$ is called a *T'_A-map with respect to r_X and r_Y* if the following diagram is homotopy commute;

$$\begin{array}{ccc} A \wedge X & \xrightarrow{1\wedge f} & A \wedge Y \\ \downarrow{\scriptstyle r_X} & & \downarrow{\scriptstyle r_Y} \\ X \rtimes A & \xrightarrow{f\rtimes 1} & Y \rtimes A. \end{array}$$

PROPOSITION 3.2. *Let X, Y be co-H-spaces and $f : X \to Y$ co-H-map. Then $f : X \to Y$ is a T'_A-map with respect to T'_A-structures which come from co-H-structures X and Y respectively.*

PROOF. Let $r_X : A \wedge X \to X \rtimes A$ and $r_Y : A \wedge Y \to Y \rtimes A$ be the T'_A-structures obtained from the co-H-structures μ_X and μ_Y of X and Y respectively, that is, $r_X = h_X^{-1} i_2 h'_X : A \wedge X \to X \rtimes A$ and $r_Y = h_Y^{-1} i_2 h'_Y : A \wedge Y \to Y \rtimes A$, where $h_X[x, a] = (p_1\mu_X(x), < a, p_2\mu_X(x) >)$ and $h_Y[y, a] = (p_1\mu_Y(y), < a, p_2\mu_Y(y) >)$. Now one can easily show that $(f \vee (1 \wedge f))h_X$ is homotopic to $h_Y(f \rtimes 1)$ from the fact $f : X \to Y$ is a co-H-map. Thus $(f \rtimes 1)h_X^{-1} \sim h_Y^{-1} h_Y (f \rtimes 1) h_X^{-1} \sim h_Y^{-1}(f \vee (1 \wedge f)) h_X h_X^{-1} \sim h_Y^{-1}(f \vee (1 \wedge f))$. Since $i_2 h'_Y(1 \wedge f) = (f \vee (1 \wedge f)) i_2$, the assertion follows.

THEOREM 3.3. *Let $f : X \to Y$ be a $T'_{\Sigma A}$-map with respect to r_X and r_Y and C_f the mapping cone of f. If C_f is 1-connected, then C_f admits a $T'_{\Sigma A}$-structure such that the inclusion $i : Y \to C_f$ is a $T'_{\Sigma A}$-map with respect to r_Y and r_{C_f}.*

PROOF. We have a commutative diagram

$$\begin{array}{ccccccc} X \rtimes \Sigma A & \xrightarrow{f\rtimes 1} & Y \rtimes \Sigma A & \longrightarrow & C_{(f\rtimes 1)} & \xrightarrow{\equiv} & C_f \rtimes \Sigma A \\ \downarrow{\scriptstyle q_X} & & \downarrow{\scriptstyle q_Y} & & \downarrow{\scriptstyle q_C} & & \downarrow \\ \Sigma A \wedge X & \xrightarrow{(1\wedge f)} & \Sigma A \wedge Y & \longrightarrow & C_{(1\wedge f)} & \xrightarrow{\equiv} & \Sigma A \wedge C_f. \end{array}$$

Since X and Y are $T'_{\Sigma A}$-spaces, there are maps $r_X : \Sigma A \wedge X \to X \rtimes \Sigma A, r_Y : \Sigma A \wedge Y \to Y \rtimes \Sigma A$ such that $q_X r_X = 1_{\Sigma A \wedge X}, q_Y r_Y = 1_{\Sigma A \wedge Y}$. We want to construct a map $R : C_{(1\wedge f)} \to C_{(f \rtimes 1)}$ such that $q_C R$ is homotopic to $1_{C_{(1\wedge f)}}$. Since f is a $T'_{\Sigma A}$-map, there is a homotopy $G : (\Sigma A \wedge X) \times I \to Y \rtimes \Sigma A$ such that $G(,0) = r_Y(1 \wedge f), G(,1) = (f \rtimes 1) r_X$. Let us replaces q_Y by a fibration

$$\begin{array}{c} E_q \overset{\iota}{\underset{\nu}{\leftrightarrows}} Y \rtimes \Sigma A \\ \pi \searrow \swarrow q_Y \\ \Sigma A \wedge Y. \end{array}$$

Then $\pi\iota = q_Y, \nu\iota = 1$ and $\iota\nu \sim 1$. Since q_Y has a right homotopy inverse r_Y, the fibration $F_q \xrightarrow{i} E_q \xrightarrow{\pi} \Sigma A \wedge Y$ is satisfied with i inducing a monomorphism on generalized homotopy groups. Consider the homotopy $H : (\Sigma A \wedge X) \times I \to \Sigma A \wedge Y$ given by $H(< a, t, x >, s) = (1 \wedge f)(< a, t, x >)$. Then $\iota r_Y(1 \wedge f), \iota(f \rtimes 1) r_X : \Sigma A \wedge X \to E_q$ are homotopic and $H : (\Sigma A \wedge X) \times I \to \Sigma A \wedge Y$ is a homotopy such that $H(,0) = (1 \wedge f) = q_Y r_Y (1 \wedge f) = \pi(\iota r_Y(1 \wedge f)), H(,1) = (1 \wedge f) = q_Y(f \rtimes 1) r_X = \pi(\iota(f \rtimes 1) r_X)$. We can now use a result of Berstein and Harper(Lemma 1.8 [2]) and assume, without loss of generality, that $q_Y G \sim H$. We define $R : C_{(1\wedge f)} \to C_{(f\rtimes 1)}$ by

$$R([< a, t, x >, s]) = \begin{cases} G(< a, t, x >, 2s), 0 \leq s \leq 1/2 \\ [r_X < a, t, x >, 2s - 1], 1/2 \leq s \leq 1 \end{cases}$$

$$R(< a, t, y >) = r_Y(< a, t, y >).$$

One sees easily that $R : C_{(1\wedge f)} \to C_{(f\rtimes 1)}$ is well defined. We have also $q_C R \sim 1_{C_{(1\wedge f)}}$. Thus $R : C_{(1\wedge f)} \to C_{(f\rtimes 1)}$ gives a $T'_{\Sigma A}$-structure on C_f and $i : Y \to C_f$ is a $T'_{\Sigma A}$-map.

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