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NOTE ON THE CONFORMAL CLASSES OF TORI WITH L^p- BOUNDED SECOND FUNDAMENTAL FORM

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ABSTRACT. If a torus has L^p-bounded second fundamental form then it is included in the lower part of the moduli space. That is, its conformal class is bounded.

Let M be a flat torus with lattice generated by $\{(a,\ b),\ (c,\ d)\} \subset R^2$. By a conformal map $x : M \to R^3$, we mean $|\frac{\partial x}{\partial u}| = |\frac{\partial x}{\partial v}|$ and $\frac{\partial x}{\partial u} \cdot \frac{\partial x}{\partial v} = 0$. In this case, we say that M and $x(M)$ have the same conformal structures, or M and $x(M)$ belong to the same conformal class.

It is well known that $T = \{a + ib : 0 \leq a \leq 1/2,\ \sqrt{1-a^2} \leq b\}$ is the set of all conformal classes of tori. That is, any torus is conformally equivalent to a flat torus with lattice generated by $\{(1,\ 0),\ (a,\ b)\}$, $0 \leq a \leq 1/2$ and $\sqrt{1-a^2} \leq b$. We call the set T the moduli space of tori.

We consider a flat torus M with lattice generated by $\{(1,\ 0),\ (a,\ b)\}$, where $0 \leq a \leq 1/2$ and $\sqrt{1-a^2} \leq b$, that is, $M = \{(u,\ v) : 0 \leq v \leq b,\ \frac{a}{b}v \leq u \leq \frac{a}{b}v + 1\}$. We denote by $L^{2,p}(M,\ R^3)$ the set of all maps of M into R^3 whose derivatives up to order 2 are p-integrable. Let $x \in L^{2,p}(M,\ R^3)$ be a conformal immersion and $N : x(M) \to S^2$ be the Gauss map. Here, the conformal class of both M and $x(M)$ is $a + ib$. For $p > 2$ we define

$$F_p(x) = \int_M |II|^p dV = \int_M |dN|^p dV = \int_M |k_1^2 + k_2^2|^{\frac{p}{2}} dV,$$

where II is the second fundamental form, and k_1 and k_2 are the principal curvatures, and dV is the induced area element. Note that $F_2(x) =$

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$\int_M |dN|^2 dV = 4\int_M H^2 dV - 2\int_M K dV = 4\int_M H^2 dV$, where H is the mean curvature and K is the Gaussian curvature. $F_2(x)$ turns out to be the Willmore functional. In [3], Willmore computed $\int H^2$ for all circular tori in R^3 and found that $\int H^2 \geq 2\pi^2$ with equality only for the Clifford torus, and he conjectured that $\int H^2 \geq 2\pi^2$ for all embedded surfaces in R^3 which has genus one. In [2], Li and Yau proved that $\int_M H^2 \geq \pi^2(b + \frac{1}{b}) \geq 2\pi^2$, where M is a two dimensional flat torus in R^3 with lattice generated by $\{(1,\ 0),\ (a,\ b)\}$, $0 \leq a \leq \frac{1}{2}$, $\sqrt{1-a^2} \leq b$. In [1], Langer proved the existence of infimum of F_p in each component of $\mathrm{Imm}_{2,p} \cap \mathrm{Area}^{-1}(1)$, where $\mathrm{Imm}_{2,p}$ denotes the set of all immersions in $L^{2,p}$.

In this paper, we will show that if $F_p(x)$ is bounded and $a+ib$ denotes the conformal class of x in the moduli space then b should be bounded. In particular, F_p takes its infimum in the lower part of the moduli space. In other words, the imaginary part of the conformal class of the infimum given by Langer cannot be too big. We denote $M_{s,t} = \{(u,\ v) : 0 \leq v \leq 1,\ sv \leq u \leq sv + t$, where $s = \frac{a}{b},\ t = \frac{1}{b}\}$. Then $M_{s,t}$ is conformally equivalent to the flat torus M which has been already defined. We also denote the u-parameter curves on $x(M_{s,t})$ by $\Gamma_v := \Gamma_v(u) = x(u,\ v)$ for each v, $0 \leq v \leq 1$, and the length of the image curves of Γ_v under the Gauss map N by $l(N\Gamma_v)$. We fix a number $0 < \lambda < \frac{\pi}{8}$ so small that if $l(N\Gamma_v) \leq \lambda$ then we may say Γ_v sits on a plane tangent to the surface $x(M_{s,t})$. Then we have the following theorem.

THEOREM. *Given constants A, C, $p > 2$ and all s and t, where $s = a/b$, $t = 1/b$ for some $(a,\ b)$, $0 \leq a \leq 1/2$ and $\sqrt{1-a^2} \leq b$, let Ω be the set of all conformally immersed surfaces $x : M_{s,t} \to R^3$ satisfying $\mathrm{Area}(x) = \int_{M_{s,t}} dV < A$, $F_p(x) = \int_{M_{s,t}} |dN|^p dV < C$, and $\int_{M_{s,t}} x dV = 0$. Then there exists $\alpha > 0$, independent of s, t and x, such that $t > \frac{\lambda^2 \alpha}{2} C^{-\frac{2}{p}} A^{-\frac{p-2}{p}}$.*

Note that $\Omega \subset \mathrm{Imm}_{2,p}(M_{s,t},\ R^3) \subset C^1(M_{s,t},\ R^3)$. To prove the theorem we need several lemmas.

LEMMA 1. *For each $x \in \mathrm{Imm}_{2,p}(M_{s,t},\ R^3)$ the arc length function $l : [0,\ 1] \to R$ defined by $l(v) = l(N\Gamma_v)$ is continuous.*

PROOF. By definition, $N\Gamma_v = N \circ x(u, v)$ is a u-parameter curve on S^2 for $sv \leq u \leq sv + t$. Set $u - sv = c$ and we denote by $n(c, v) = N\Gamma_v$ for $0 \leq c \leq t$.

Let $v^j \to v$ in $[0, 1]$. Given $\varepsilon > 0$, choose a partition $P = \{c_0, c_1, \cdots, c_n\}$ of $[0, t]$ such that

$$l(N\Gamma_{v^j}) = l(n(c, v^j)) \leq \sum_{k=1}^{n} |n(c_k, v^j) - n(c_{k-1}, v^j)| + \frac{\varepsilon}{2}$$

for a fixed j satisfying $(*)$ below and

$$l(N\Gamma_v) = l(n(c, v)) \leq \sum_{k=1}^{n} |n(c_k, v) - n(c_{k-1}, v)| + \frac{\varepsilon}{2}.$$

Since $x \in L^{2,p} \subset C^1$, n is uniformly continuous on $M_{s,t}$. Choose j sufficiently large so that

$$(*) \qquad |n(c, v^j) - n(c, v)| \leq \frac{\varepsilon}{4n}$$

for all $c \in [0, t]$. Then we obtain

$$\begin{aligned}
&\sum_{k=1}^{n} |n(c_k, v^j) - n(c_{k-1}, v^j)| \\
&\quad \leq \sum_{k=1}^{n} (|n(c_k, v^j) - n(c_k, v)| + |n(c_k, v) - n(c_{k-1}, v)| \\
&\qquad + |n(c_{k-1}, v) - n(c_{k-1}, v^j)|) \\
&\quad \leq \sum_{k=1}^{n} |n(c_k, v) - n(c_{k-1}, v)| + \frac{\varepsilon}{2},
\end{aligned}$$

and so

$$\begin{aligned}
l(N\Gamma_{v^j}) &\leq \sum_{k=1}^{n} |n(c_k, v^j) - n(c_{k-1}, v^j)| + \frac{\varepsilon}{2} \\
&\leq \sum_{k=1}^{n} |n(c_k, v) - n(c_{k-1}, v)| + \varepsilon \\
&\leq l(N\Gamma_v) + \varepsilon.
\end{aligned}$$

Similarly, we obtain $l(N\Gamma_v) \leq l(N\Gamma_{v^j}) + \varepsilon$. Hence

$$|l(N\Gamma_{v^j}) - l(N\Gamma_v)| \leq \varepsilon.$$

Lemma 1 just says that the lengths of two close curves must also be close.

LEMMA 2. *For each $x \in Imm_{2,p}(M_{s,t},\ R^3)$ we define a set $E_x = \{v \in [0,\ 1] : l(N\Gamma_v) > \lambda\}$. Then the Lebesgue measure of E_x, $m(E_x) > 0$.*

PROOF. Suppose that $m(E_x) = 0$. Then $l(N\Gamma_v) \leq \lambda$ for all $v \in [0,\ 1]$, since by the continuity of the arc length (Lemma 1), $l(N\Gamma_{v_0}) > \lambda$ for some v_0 implies $l(N\Gamma_v) > \lambda$ for all $v \in (v_0 - \delta,\ v_0 + \delta)$ for some $\delta > 0$, which means that $m(E_x) \geq m((v_0 - \delta,\ v_0 + \delta)) = 2\delta > 0$. By the definition of λ, for each $v \in [0,\ 1]$ $N\Gamma_v$ sits roughly at a point on the unit sphere S^2. Since the Gauss map is onto, we have two points $v_0 \neq v_1 \in [0,\ 1]$ such that $N\Gamma_{v_0}$ and $N\Gamma_{v_1}$ are sufficiently close to the south pole SP and the north pole NP of S^2, respectively. In other words, $N\Gamma_{v_0}$ and $N\Gamma_{v_1}$ pass through SP and NP, respectively. For any fixed c, with $0 \leq c \leq t$, $n(sv + c,\ v) = N \circ x(sv + c,\ v)$ is a curve C_1 from a point P_1 near SP to a point P_2 near NP for $v \in [v_0,\ v_1]$ and $n(sv + c,\ v)$ is a curve C_2 from P_2 to P_1 for $v \in [0,\ 1] - [v_0,\ v_1]$. Let $T_i = \{w \in S^2 : |w - z| < 2\lambda$ for some $z \in C_i\}$, a tubular neighborhood of C_i in S^2. Since $N\Gamma_v$ is almost a point in S^2, for each $v \in [0,\ 1]$ $N\Gamma_v$ must be close to a point in C_1 or C_2, that is, $N\Gamma_v \subset T_1 \cup T_2$ for all $v \in [0,\ 1]$. So we have $S^2 = N \circ x(M_{s,t}) = \cup_{v\in[0,\ 1]} N\Gamma_v \subset T_1 \cup T_2$. But $\lambda < \frac{\pi}{8}$ was chosen so small that $T_1 \cup T_2$ is certainly not all of S^2. This gives a contradiction.

LEMMA 3. *There exists $\alpha > 0$ such that $m(E_x) \geq \alpha$ holds for all $x \in \Omega$ and all s and t.*

PROOF. Suppose that this is not the case. By the compactness theorem of J. Langer [1], we may choose $x^j : M_{s,t} \to R^3$ in Ω such that for a sequence of surface diffeomorphisms $\{\phi^j\}$, $x^j \circ \phi^j$ converges in the C^1 topology to an immersion $y \in L^{2,p}(M_{s,t},\ R^3)$ and $m(E_{x^j \circ \phi^j}) < 2^{-j}$. We claim that $E_y = \emptyset$ or $m(E_y) = 0$, which contradicts to Lemma 2.

For the brevity of the notation, we denote $y^j = x^j \circ \phi^j$. Suppose that $v_0 \in E_y$. Then we have $l(N\Gamma_{v_0}) > \lambda$, where Γ_{v_0} is a u-parameter curve on $y(M_{s,t})$. Let $\beta = l(N\Gamma_{v_0})$ and $\nu = \beta - \lambda > 0$. We denote by $\Gamma_v^j = y^j(u,\ v)$ the u-parameter curve on $y^j(M_{s,t})$ for $sv \leq u \leq sv + t$. As in Lemma 1, we set $sv - u = c$ and we denote by $n^j(c,\ v) = N\Gamma_v^j$ and $n(c,\ v) = N\Gamma_v$ for $0 \leq c \leq t$.

Choose a partition $P = \{c_0,\ c_1,\ \cdots,\ c_n\}$ of $[0,\ t]$ such that

$$l(N\Gamma_v^j) \leq \sum_{k=1}^{n} |n^j(c_k,\ v) - n^j(c_{k-1},\ v)| + \frac{\nu}{4}$$

for a fixed j satisfying (**) below and

$$l(N\Gamma_v) \leq \sum_{k=1}^{n} |n(c_k,\ v) - n(c_{k-1},\ v)| + \frac{\nu}{4}.$$

Since $y^j \to y$ in C^1, we have $n^j \to n$ in C^0. So we obtain

$$(**) \qquad ||n^j - n||_{C^0} < \nu/8n$$

for sufficiently large j, and we have

$$\begin{aligned}
&\sum_{k=1}^{n} |n^j(c_k,\ v) - n^j(c_{k-1},\ v)| \\
&\quad\leq \sum_{k=1}^{n} (|n^j(c_k,\ v) - n(c_k,\ v)| + |n(c_k,\ v) - n(c_{k-1},\ v)| \\
&\quad\quad + |n(c_{k-1},\ v) - n^j(c_{k-1},\ v)|) \\
&\quad< \sum_{k=1}^{n} |n(c_k,\ v) - n(c_{k-1},\ v)| + \frac{\nu}{4}.
\end{aligned}$$

So we obtain $l(N\Gamma_v^j) < l(N\Gamma_v) + \nu/2$. Similarly we have $l(N\Gamma_v) < l(N\Gamma_v^j) + \nu/2$, and hence $|l(N\Gamma_v^j) - l(N\Gamma_v)| < \nu/2$.

Now by lemma 1, there exists $\mu > 2^{-j-1}$ such that $|v - v_0| < \mu$ implies

$$|l(N\Gamma_v) - l(N\Gamma_{v_0})| < \frac{\nu}{2}.$$

In facts, we may assume $\mu > 2^{-j-1}$ by choosing j large enough. So for $v \in (v_0 - \mu,\ v_0 + \mu)$ we have

$$|l(N\Gamma_v^j) - l(N\Gamma_{v_0})| \\ \leq |l(N\Gamma_v^j) - l(N\Gamma_v)| + |l(N\Gamma_v) - l(\Gamma_{v_0})| < \nu,$$

and so $l(N\Gamma_v^j) > l(N\Gamma_{v_0}) - \nu = \beta - \nu = \lambda$. This implies $(v_0 - \mu,\ v_0 + \mu) \subset E_{y^j}$ and so $2\mu \leq m(E_{y^j}) < 2^{-j}$, which is a contradiction.

PROOF OF THE THEOREM. Note that the conformality of x implies $|\frac{\partial x}{\partial u}| = |\frac{\partial x}{\partial v}|$ and $\frac{\partial x}{\partial u} \cdot \frac{\partial x}{\partial v} = 0$, so $dV = |\frac{\partial x}{\partial u} \times \frac{\partial x}{\partial v}| dudv = \frac{1}{2}|dx|^2 dudv$. Using the lemmas and applying the Hölder's inequality, we obtain the following estimates.

$$\begin{aligned}
&C > F_p(x) \\
&= \int_{M_{s,t}} |dN|^p dV \geq A^{-\frac{p-2}{2}} \Big(\int_{M_{s,t}} |dN|^2 dV \Big)^{\frac{p}{2}} \\
&= A^{-\frac{p-2}{2}} \Big(\frac{1}{2} \int_{M_{s,t}} |dN|^2 |dx|^2 dudv \Big)^{\frac{p}{2}} \geq A^{-\frac{p-2}{2}} \Big(\frac{1}{2} \int_{M_{s,t}} |dn|^2 dudv \Big)^{\frac{p}{2}} \\
&\geq A^{-\frac{p-2}{2}} \Big(\frac{1}{2} \int_{E_x} \Big(\int_{sv}^{sv+t} |\frac{\partial n}{\partial u}|^2 du \Big) dv \Big)^{\frac{p}{2}} \\
&\geq A^{-\frac{p-2}{2}} \Big(\frac{1}{2} \int_{E_x} \frac{1}{t} \Big(\int_{sv}^{sv+t} |\frac{\partial n}{\partial u}| du \Big)^2 dv \Big)^{\frac{p}{2}} \\
&= A^{-\frac{p-2}{p}} \Big(\frac{1}{2} \int_{E_x} \frac{1}{t} l(N\Gamma_v)^2 dv \Big)^{\frac{p}{2}} > A^{-\frac{p-2}{2}} \Big(\frac{\lambda^2 m(E_x)}{2t} \Big)^{\frac{p}{2}} \\
&\geq A^{-\frac{p-2}{2}} \Big(\frac{\lambda^2 \alpha}{2t} \Big)^{\frac{p}{2}},
\end{aligned}$$

where $n(u,\ v) = N \circ x(u,\ v)$. Hence we have $\frac{1}{b} = \tau > \frac{\lambda^2 \alpha}{2} C^{-\frac{2}{p}} A^{-\frac{p-2}{p}}$, where $a + ib$ is the conformal class of x.

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