

THE EXISTENCE OF SOLUTIONS OF A NONLINEAR WAVE EQUATION

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ABSTRACT. In this paper we investigate the existence of solutions of a nonlinear wave equation $u_{tt} - u_{xx} = p(x, t, u)$ in H_0 , where H_0 is the Hilbert space spanned by eigenfunctions. If p satisfy condition $(p_1) - (p_3)$, this nonlinear wave equation has at least one solution.

0. Introduction

In this paper we investigate the existence of weak solutions of a nonlinear wave equation under Dirichlet boundary condition on the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and periodic condition on the variable t

$$(0.1) \quad u_{tt} - u_{xx} = p(x, t, u) \quad \text{in} \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R},$$

$$(0.2) \quad u\left(-\frac{\pi}{2}, t\right) = u\left(\frac{\pi}{2}, t\right) = 0,$$

$$(0.3) \quad u \text{ is } \pi - \text{periodic in } t \text{ and even } x,$$

where we shall describe the condition on the function p .

In [3, 4], the authors investigate the existence of multiple solutions of a nonlinear wave equation (0.1) when the function p consisted of semi-linear terms and the multiple $s\phi_{00}$ ($s \in \mathbb{R}$) of the positive eigenfunction.

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The existence of multiple solutions of a nonlinear wave equation with semi linearities crossing multiple eigenvalues was shown by a variational reduction method in Choi and Jung [3,4].

Let L be the one-dimensional wave operator, $Lu = u_{tt} - u_{xx}$. Let H_0 be the Banach space spanned by eigenfunctions of the wave operator L , with L^2 -norm. Then equation (0.1) with (0.2) and (0.3) is equivalent to the equation

$$(0.4) \quad Lu = p(x, t, u) \quad \text{in} \quad H_0.$$

In this paper we assume that the function p satisfies the following.

(p_1) $p(x, t, u)$ is even in x and belongs to $C\left(\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times \mathbb{R}, \mathbb{R}\right)$.

(p_2) There are constants $a_1, a_2 \geq 0$ such that

$$|p(x, t, \xi)| \leq a_1 + a_2|\xi|^s \text{ for } 0 \leq s \leq 1,$$

where in case $s = 1$, we further assume on p that the limits

$$\lim_{\xi \rightarrow \infty} \frac{p(x, t, \xi)}{\xi} = -b, \quad \lim_{\xi \rightarrow -\infty} \frac{p(x, t, \xi)}{\xi} = -c$$

exist and $-1 < b, c < 7$ with $\frac{1}{\sqrt{b+1}} + \frac{1}{\sqrt{c+1}} \neq 1$.

(p_3) There is a constant k with $-3 < k < 1$ such that $p(x, t, \xi) - k = o(|\xi|)$ as $\xi \rightarrow 0$.

In Section 1, we investigate the property of the Hilbert space H spanned by eigenfunctions of the one-dimensional wave operator. We also investigate the property of the Hilbert space.

In Section 2, we first show that the functional corresponding to (0.4) is continuously Fréchet differentiable in a Hilbert space (which is a subspace of the Hilbert space H_0) and we calculate several estimates for the functional. By using several estimates and the critical point for a C^1 -map in a Banach space, we show that equation (0.4) satisfying (p_1)-(p_3) has at least one solution.

1. The Hilbert space spanned by eigenfunctions

In this section we shall describe the Hilbert space spanned by the eigenfunctions of the one-dimensional wave operator L and investigate its properties.

When u is even in x and periodic in t with period π , the eigenvalue problem for $u(x, t)$,

$$(1.1) \quad \begin{aligned} Lu = \lambda u \quad \text{in} \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R}, \\ u\left(\pm\frac{\pi}{2}, t\right) = 0, \end{aligned}$$

has infinitely many eigenvalues

$$\lambda_{mn} = (2n + 1)^2 - 4m^2 \quad (m, n = 0, 1, 2, \dots)$$

and corresponding normalized eigenfunctions ϕ_{mn}, ψ_{mn} ($m, n \leq 0$) given by

$$\begin{aligned} \phi_{0n} &= \frac{\sqrt{2}}{\pi} \cos(2n + 1)x \quad \text{for } n \geq 0, \\ \phi_{mn} &= \frac{2}{\pi} \cos 2mt \cdot \cos(2n + 1)x \quad \text{for } m > 0, n \geq 0, \\ \psi_{mn} &= \frac{2}{\pi} \sin 2mt \cdot \cos(2n + 1)x \quad \text{for } m > 0, n \geq 0. \end{aligned}$$

Let n be a fixed integer and define

$$(1.2) \quad \lambda_n^+ = \inf_m \{\lambda_{mn} : \lambda_{mn} > 0\} = 4n + 1,$$

$$(1.3) \quad \lambda_n^- = \sup_m \{\lambda_{mn} : \lambda_{mn} < 0\} = -4n - 3.$$

Letting $n \rightarrow \infty$, we obtain that $\lambda_n^+ \rightarrow +\infty$ and $\lambda_n^- \rightarrow -\infty$. Hence, it is easy to check that the only eigenvalues in the interval $(-15, 9)$ are given by

$$\lambda_{32} = -11 < \lambda_{21} = -7 < \lambda_{10} = -3 < \lambda_{00} = 1 < \lambda_{11} = 5.$$

Let Q be the square $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and H_0 the Hilbert space defined by

$$H_0 = \{u \in L^2(Q) : u \text{ is even in } x\}.$$

The set of functions $\{\phi_{mn}, \psi_{mn}\}$ is an orthonormal basis in H_0 . Let us denote an element u , in H_0 , by

$$u = \sum (h_{mn}\phi_{mn} + k_{mn}\psi_{mn}),$$

and we define a subspace H of H_0 as follows

$$H = \left\{ u \in H_0 : \sum |\lambda_{mn}|(h_{mn}^2 + k_{mn}^2) < \infty \right\}.$$

Then this space is a Banach space with norm

$$\| \|u\| \| = \left[\sum |\lambda_{mn}|(h_{mn}^2 + k_{mn}^2) \right]^{\frac{1}{2}}.$$

Since $|\lambda_{mn}| \geq 1$ for all m, n , we have that

- (i) $\| \|u\| \| \geq \|u\|$, where $\|u\|$ denotes the L^2 norm of u ,
- (ii) $\|u\| = 0$ if and only if $\| \|u\| \| = 0$,
- (iii) $Lu \in H$ implies $u \in H$.

We note that 1 belongs to H_0 , but does not belong to H . Hence we can see that the space H is a proper subspace of H_0 . The following lemma is very important in this paper.

LEMMA 1.1. *Suppose that c is not an eigenvalue of L and let $u \in H_0$. Then we have $(L - c)^{-1}u \in H$.*

PROOF. When n is fixed, λ_n^+ and λ_n^- were defined in (1.2) and (1.3):

$$\lambda_n^+ = 4n + 1,$$

$$\lambda_n^- = -4n - 3.$$

We see that $\lambda_n^+ \rightarrow +\infty$ and $\lambda_n^- \rightarrow -\infty$ as $n \rightarrow \infty$. Hence the number of elements in the set $\{\lambda_{mn} : |\lambda_{mn}| < |c|\}$ is finite, where λ_{mn} is an eigenvalue of L . Let

$$u = \sum (h_{mn}\phi_{mn} + k_{mn}\psi_{mn}).$$

Then

$$(L - c)^{-1}u = \sum \left(\frac{1}{\lambda_{mn} + c} h_{mn} \psi_{mn} + \frac{1}{\lambda_{mn} + c} k_{mn} \psi_{mn} \right).$$

Hence we have the inequality

$$\begin{aligned} |||(L - c)^{-1}u||| &= \sum |\lambda_{mn}| \frac{1}{(\lambda_{mn} + c)^2} (h_{mn}^2 + k_{mn}^2) \\ &\leq C \sum (h_{mn}^2 + k_{mn}^2) \end{aligned}$$

for some C , which means that

$$|||(L - c)^{-1}u||| \leq C_1 \|u\|, \quad C_1 = \sqrt{C}.$$

LEMMA 1.2. *If p satisfies (p_1) - (p_2) , the map $u(x, t) \rightarrow p(x, t, u(x, t))$ belongs to $C(H_0, H_0)$.*

PROOF. We note that the function $p(x, t, u(x, t))$ is even in x . If $u \in H_0$, by (p_2) , then

$$\begin{aligned} (1.4) \quad \int_Q |p(x, t, u)|^2 dx dt &\leq \int_Q (a_1 + a_2 |u|^s)^2 dx dt \\ &\leq a_3 \int_Q (1 + |u|^2) dx dt \end{aligned}$$

for $s \leq 0$, which shows that $p : H_0 \rightarrow H_0$.

To prove the continuity of this map, observe that it is continuous at u if and only if

$$f(x, t, z(x, t)) = p(x, t, z(x, t) + u(x, t)) - p(x, t, u(x, t))$$

is continuous at $z = 0$.

Therefore we can assume that $u = 0$ and $p(x, t, 0) = 0$. Let $\epsilon > 0$. We claim there is a $\delta > 0$ such that $\|u\| \leq \delta$ implies $\|P(\cdot, u)\| \leq \epsilon$. By (p_1) and $p(x, t, 0) = 0$, given any $\hat{\epsilon} > 0$, there is a $\hat{\delta} > 0$ such that

$$|p(x, t, \xi)| \leq \hat{\epsilon} \quad \text{if } (x, t) \in Q \quad \text{and} \quad |\xi| \leq \hat{\delta}.$$

Let $u \in H_0$ with $\|u\| \leq 0$, δ being free for now, and set

$$Q_1 = \{(x, t) \in Q : |\delta| \leq \beta\}.$$

Then we have

$$(1.5) \quad \int_{Q_1} |p(x, t, u(x, t))|^2 dx dt \leq \hat{\epsilon}^2 |Q_1| \leq \hat{\epsilon}^2 \pi^2,$$

where $|Q_1|$ denotes the measure of Q_1 .

Choose $\hat{\epsilon}$ so that $\hat{\epsilon} \pi \leq \frac{\epsilon}{2}$. Let $Q_2 = Q - Q_1$. Then as in (1.4)

$$(1.6) \quad \int_{Q_2} |p(x, t, u(x, t))|^2 dx dt \leq a_3(|Q_3| + \delta^2).$$

Moreover

$$(1.7) \quad \delta^2 \geq \int^{Q_2} |u|^2 dx dt \geq \beta^2 |Q_2|$$

or $|Q_2| \leq (\delta\beta^{-1})^2$. Combining (1.6)-(1.7) we have

$$(1.8) \quad \int_{Q_2} |p(x, t, u)|^2 dx dt \leq a_3(1 + \beta^{-2})\delta^2.$$

Choose δ so that $a_3(1 + \beta^{-2})\delta^2 \leq (\frac{\epsilon}{2})^2$. Then (1.5) and (1.8) imply $\|p(\cdot, u)\| \leq 0$ if $\|u\| \leq \delta$. This completes the proof.

With the above lemmas 1.1 and 1.2, we can obtain the following

PROPOSITION 1.1. *Assume that the function p satisfies the conditions (p_1) - (p_2) and that there exists a solution, in H_0 , of the equation*

$$(1.9) \quad Lu = p(x, t, u) \quad \text{in} \quad H_0.$$

Then it belongs to H .

PROOF. Assume that u belongs to the Hilbert space H_0 . Since p satisfies (p_1) and (p_2) , it follows from Lemma 1.2 that $p(x, t, u)$ belongs to H_0 . Equation (1.9) is equivalent to

$$u = L^{-1}[p(x, t, u)].$$

Hence it follows from Lemma 1.1 that u belongs to H .

With the aid of Proposition 1.1, the investigation of the existence of solutions in H_0 of (1.9) reduces to the investigation of the existence of solutions in H of (1.9).

2. The existence of a solution

In this section we investigate the existence of a nontrivial solution of the one-dimensional wave equation (1.9) described in Section 1. By Proposition 1.1, the problem (1.9) in H_0 is reduced to one in the subspace H of the Hilbert space H_0 . Hence we consider the problem

$$(2.1) \quad Lu = p(x, t, u(x, t)) \quad \text{in} \quad H,$$

where the function p satisfies $(p_1) - (p_3)$ described in Section 0.

We now state the main theorem of this section.

THEOREM 2.1. *Assume that p satisfies $(p_1) - (p_3)$. Then (2.1) possesses at least one solution.*

We now consider the functional associated with (2.1)

$$(2.2) \quad I(u) = \int_Q \left[\frac{1}{2}(-|u_t|^2 + |u_x|^2) - P(x, t, u) \right] dx dt,$$

where $P(x, t, \xi) = \int_0^\xi p(x, t, \eta) d\eta$.

Since every solution in H_0 of (2.1) belongs to H , it suffices to investigate the solutions in H of (2.1). On the other hand, the weak solutions of (2.1) coincide with the critical points of $I(u)$. Hence we consider the functional $I(u)$ in H .

In order to apply critical point theory to the functional $I(u)$ given by (2.2), we have to know whether $I \in C^1(H, \mathbb{R})$ and $I(u)$ satisfies the Palais-Smale condition.

First we prove that I is continuous and Fréchet differentiable in H .

LEMMA 2.1. *If p satisfies (p_1) - (p_3) , then the functional $I(u)$ is continuous and Fréchet differentiable in H , and $I'(u)$ is continuous in H with*

$$(2.3) \quad I'(u)\phi = \int_Q [Lu\phi - p(x, t, u)\phi] dx dt$$

for all $\phi \in H$. Moreover

$$J(u) = \int_Q P(x, t, u) dx dt$$

is weakly continuous and $J'(u)$ is compact.

PROOF. Let u be in H . We now prove that $I(u)$ is continuous at u . We first consider

$$\begin{aligned} I(u+v) - I(u) &= \int_Q [u(v_{tt} - v_{xx}) + \frac{1}{2}v(v_{tt} - v_{xx})] dx dt \\ &\quad - \int_Q [P(x, t, u+v) - P(x, t, u)] dx dt. \end{aligned}$$

Let $u = \sum h_{mn}\phi_{mn} + k_{mn}\psi_{mn}$, $v = \sum \tilde{h}_{mn}\phi_{mn} - \tilde{k}_{mn}\psi_{mn}$. Then, by using Schwartz inequality, we have

$$\left| \int_Q u(v_{tt} - v_{xx}) dx dt \right| \leq \|u\| \cdot \|v\|,$$

$$\left| \int_Q \frac{1}{2}v(v_{tt} - v_{xx}) dx dt \right| \leq \|v\|^2.$$

On the other hand, by the Mean Value Theorem, we have

$$P(x, t, \xi + \eta) - P(x, t, \xi) = P(x, t, \xi + \theta\eta)\eta \quad \text{for some } \theta \in (0, 1).$$

Therefore by (p_2) , we have, for $s \leq 1$,

$$\begin{aligned}
 (2.4) \quad & \int_Q |P(x, t, u + v) - P(x, t, u)| dx dt \\
 &= \int_Q |p(x, t, u + \theta v)| |v| dx dt \\
 &\leq \int_Q [a_1 + a_2(|u| + |v|)^s] |v| dx dt \\
 &\leq a_1 \|v\| + a_2 \|u\| \|v\| + a_2 \|v\|^2 \\
 &\leq a_1 \|v\| + a_2 \|u\| \cdot \|v\| + a_3 \|v\|^2 \\
 &= (a_1 + a_2 \|u\| + a_3 \|v\|) \|v\|.
 \end{aligned}$$

With the above results, we can see that $I(u)$ is continuous at u .

Now let us prove that $I(u)$ is Fréchet differentiable at $u \in H$ with equation (2.3). To prove equation (2.3), we compute the following

$$\begin{aligned}
 & |I(u + v) - I(u) - I'(u)v| \\
 &= \left| \int_Q [u(v_{tt} - v_{xx}) + \frac{1}{2}v(v_{tt} - v_{xx}) - (P(x, t, u + v) - P(x, t, u)) \right. \\
 &\quad \left. - (u_{tt} - u_{xx})v + p(x, t, u)v] dx dt \right| \\
 &= \left| \int_Q \frac{1}{2}vLv dx dt - \int_Q [P(x, t, u + v)P(x, t, u) - p(x, t, u)v] dx dt \right| \\
 &\leq \frac{1}{2} \|v\|^2 + \int_Q |p(x, t, u + \theta v) - p(x, t, u)| |v| dx dt.
 \end{aligned}$$

By (p_1) , $p(x, t, \xi)$ is a continuous function of ξ and hence for given $\epsilon_1 > 0$, there exists $\delta > 0$ such that

$$|p(x, t, u + \theta v) - p(x, t, u)| < \epsilon_1$$

holds almost every when $\|v\| < \delta$. Therefore we have

$$\begin{aligned}
 |I(u + v) - I(u) - I'(u)v| &< \frac{1}{2} \|v\|^2 + \epsilon_1 \int_Q |v| dx dt \\
 &\leq \frac{1}{2} \|v\| + \epsilon_1 \|v\| \\
 &\leq \left(\frac{1}{2} \|v\| + \epsilon_1 \right) \|v\|,
 \end{aligned}$$

which proves equation (2.3).

It is clear that the first term in I' is continuous. Hence, to prove the continuity of $I'(u)$, it suffices to show that $J'(u)$ is continuous. Let $u_m \rightarrow u$ in H . Then $u_m \rightarrow u$ in H_0 and we have

(2.5)

$$\begin{aligned} \|J'(u_m) - J'(u)\|_{op} &= \sup_{\|\phi\| \leq 1} \left| \int_Q (p(x, t, u_m) - p(x, t, u)) \phi dx dt \right| \\ &\leq \sup_{\|\phi\| \leq 1} \int_Q |p(x, t, u_m) - p(x, t, u)| |\phi| dx dt \\ &\leq \sup_{\|\phi\| \leq 1} \|p(x, t, u_m) - p(x, t, u)\| \|\phi\| \\ &\leq \sup_{\|\phi\| \leq 1} \|p(x, t, u_m) - p(x, t, u)\| \cdot \|\phi\| \\ &\leq \|p(x, t, u_m) - p(x, t, u)\|, \end{aligned}$$

where $\|\cdot\|_{op}$ is the operator norm. Since the map $u(x, t) \rightarrow p(x, t, u(x, t))$ belongs to $C(H_0, H_0)$, the last term in the above inequalities tends to 0 as $m \rightarrow \infty$ and J' is continuous.

To prove that J is weakly continuous, let u_m converges to u in H . Then u_m converges to u in H_0 since $\|u\| \leq \|u\|$. Consequently, Lemma 1.2 implies $J(u_m) \rightarrow J(u)$.

Finally, to prove that J' is compact, let (u_m) be bounded in H . Then along a subsequence, u_m converges weakly to some $u \in H$ and $u_m \rightarrow u$ in H_0 . The proof is completed by (2.3).

LEMMA 2.2. *If p satisfies (p_1) - (p_4) , then the functional $I(u)$ satisfies the Palais-Smale condition. That is, any sequence (u_m) in H for which $I(u_m)$ is bounded and $I'(u_m) \rightarrow 0$ in H as $m \rightarrow \infty$ possesses a convergent subsequence.*

The verification for (PS) is simplified with the aid of the following result.

LEMMA 2.3. *Let p satisfy (p_1) - (p_2) and let (u_m) be a bounded sequence in H such that $I(u_m)$ is bounded and $I'(u_m) \rightarrow 0$ in H as $m \rightarrow \infty$. Then (u_m) has a convergent subsequence.*

PROOF. Suppose that $I(u_m)$ is bounded and $I'(u_m) \rightarrow 0$ in H as $m \rightarrow \infty$ for any bounded sequence (u_m) in H . Let $D : H \rightarrow H^*$ denote the duality map between H and its dual defined by

$$(Du)\phi = \int_Q Lu \cdot \phi dxdt \quad \text{for } u, \phi \in H.$$

Thus

$$D^{-1}I'(u) = u - D^{-1}J'(u).$$

By the continuity of D^{-1} and (2.10), we have

$$u_m = D^{-1}I'(u_m) + D^{-1}J'(u_m) \rightarrow D^{-1}J'(u_m),$$

where the limit being taken along the convergent subsequence of $J'(u_m)$.

But, since (u_m) is bounded in H and J' is compact (cf. Lemma 2.1), $J'(u_m)$ has a convergent subsequence. This completes the lemma.

We now state the theorem (cf. [3]), which will be useful in the proof of Lemma 2.2.

THEOREM A. Let $-1 < a, b < 7$ and $\frac{1}{\sqrt{a+1}} + \frac{1}{\sqrt{b+1}} \neq 1$. Then the equation

$$Lu + au^+ - bu^- = 0 \quad \text{in } H_0$$

has only the trivial solution.

PROOF OF LEMMA 2.2. By Lemma 2.3, to verify (PS), we only need to show that if $|I(u_m)| \leq M$ and $I'(u_m) \rightarrow 0$ as $m \rightarrow \infty$ then (u_m) is a bounded sequence.

For m large, we have

$$Lu_n + p(x, t, u_n) = DI(u_n) \quad \text{in } H.$$

Assume that (PS) condition does not hold, that is, $\|u_n\| \rightarrow \infty$. Dividing by $\|u_n\|$ and taking $w_n = \|u_n\|^{-1}u_n$, we have

$$(2.9) \quad Lw_n + \frac{1}{\|u_n\|} p(x, t, u_n) = \frac{1}{\|u_n\|} DI(u_n).$$

Since $DI(u_n) \rightarrow 0$ as $n \rightarrow \infty$ and $\|u_n\| \rightarrow \infty$, the right hand side of (2.9) converges to 0 in H as $n \rightarrow \infty$. Moreover (2.9) shows that $\|u_n\|$ is bounded. Since L^{-1} is a compact operator, passing to a subsequence we obtain that $w_n \rightarrow w_0$ in H . Since $\|w_n\| = 1$ for all $n = 1, 2, \dots$, it follows that $\|w_0\| = 1$. Taking the limit of (2.9), we find

$$Lw_0 + bw_0^+ - cw_0^- = 0 \text{ (in case } s = 1) \quad \text{or} \quad Lw_0 = 0 \text{ (in case } 0 \leq s < 1)$$

with $\|w_0\| \neq 0$. This contradicts the fact (from Theorem A) that the above equation has only the trivial solution.

We state a critical point theory, which is very useful to show the existence of critical points of a C^1 -map in a Banach space.

LEMMA 2.5. *Let E be a real Banach space and $I \in C^1(E, \mathbb{R})$ satisfy (PS). Then the local minimum (or maximum) of I is a critical point of I .*

LEMMA 2.6. *Suppose that I satisfy $(p_1) - (p_3)$. Then, as $u \rightarrow 0$, we have*

$$(2.10) \quad I(u) = \frac{1}{2} \int_Q Lu \cdot u dxdt + \int_Q k u dxdt + o(\|u\|^2).$$

PROOF. By (p_3) , given any $\epsilon > 0$, there is a $\delta > 0$ such that $|\xi| \leq \delta$ implies

$$|P(x, t, \xi) - k\xi| \leq \frac{1}{2}\epsilon|\xi|^2$$

for all $(x, t) \in Q$. By (p_2) , there is a constant $A = A(\delta) > 0$ such that $|\xi| > \delta$ implies

$$|P(x, t, \xi) - k\xi| \leq A|\xi|^{s+1}$$

for all $(x, t) \in Q$. Combining these two estimates, for all $\xi \in \mathbb{R}$ and $(x, t) \in Q$, we have

$$|P(x, t, \xi) - k\xi| \leq \frac{\epsilon}{2}|\xi|^2 + A|\xi|^{s+1}.$$

On the other hand, there is a δ_1 such that $\|u\| < \delta_1$ implies

$$\int_{|u(x,t)| > \delta} |P(x, t, u) - ku| dxdt \leq \frac{\epsilon}{2} \|u\|.$$

Therefore if $\|u\| < \delta_1$, we have

$$\begin{aligned} \left| J(u) - k \int_Q u dxdt \right| &\leq \frac{\epsilon}{2} \int_{|u(x,t)| \leq \delta} |u|^2 dxdt + A \int_{|u(x,t)| > \delta} |u|^{s+1} dxdt \\ &\leq \frac{\epsilon}{2} \|u\|^2 + \frac{\epsilon}{2} \|u\| \\ &\leq \epsilon \|u\|^2 \end{aligned}$$

for $s \leq 1$. Hence

$$\left| J(u) - k \int_Q u dxdt \right| \leq \epsilon \|u\|^2.$$

Since ϵ was arbitrary, $J(u) - k \int_Q u dxdt = o(\|u\|^2)$ as $u \rightarrow 0$. Therefore we have

$$I(u) = \frac{1}{2} \int_Q Lu \cdot u dxdt - J(u) = \frac{1}{2} \int_Q Lu \cdot u dxdt + \int_Q k u dxdt + o(\|u\|^2).$$

Let V be the subspace of H , spanned by the eigenfunctions of $\lambda_{mn} > 0$ and W be the orthogonal complement of V in H . Let $P : H \rightarrow V$ denote the orthogonal projection of H onto V and let $I - P : H \rightarrow W$ denote that of H onto W . Then every element u of H is expressed by $u = v + w$, where $v = Pu$, $w = (I - P)u$. Hence equation (2.1) is equivalent to a system

$$(2.11.a) \quad Lv = P(p(\cdot, \cdot, v + w)),$$

$$(2.11.b) \quad Lw = (I - P)(p(\cdot, \cdot, v + w)).$$

Let

$$\begin{aligned} I_1(v) &= \frac{1}{2} \int_Q Lv \cdot v dxdt - \int_Q P(x, t, v + w) dxdt, \\ I_2(w) &= \frac{1}{2} \int_Q Lw \cdot w dxdt - \int_Q P(x, t, v + w) dxdt. \end{aligned}$$

Then we have the following lemma.

LEMMA 2.7. *There is a neighborhood B_1 of 0 in V such that for any $v \in B_1$ there exists a solution $z \in W$ of equation (2.11.b) in W , where z is a local maximum $I_2(w)$. If we put $z = \theta(v)$ in B_1 , then θ is continuous in B_1 and we have*

$$DI(v + \theta(v))(w) = 0 \quad \text{for all } w \in W.$$

PROOF. Equation (2.11.b) is equivalent to

$$(2.12) \quad z = L^{-1}(I - P)(p(\cdot, \cdot, v + w))$$

We note that $L^{-1}(I - P)$ is a self-adjoint, compact, linear map from W into itself and the eigenvalues $L^{-1}(I - P)$ in W are λ_{mn}^{-1} with $\lambda_{mn} < 0$. Hence, by Lemma 2.6,

$$I_2(w) = -\frac{1}{2} \|w\|^2 + \int_Q k(v + w) dx dt + o(\|v\|^2) o(\|w\|^2).$$

Since $-3 < k < 1$, $I_2(w)$ has a local maximum z , which is a solution of (2.11.b). Let $z = \theta(v)$. By Lemma 1.2 and Lemma 2.2 of [3], θ is continuous in B_1 .

Let $v \in V$ and set $z = \theta(v)$. If $w \in W$, then from (2.2) we see that

$$\int_Q (-z_t w_t + z_x w_x + p(x, t, v + w)w) dx dt = 0.$$

Since $\int_Q v_t w_t = 0$ and $\int_Q v_x w_x = 0$, we have

$$DI(v + \theta(v))(w) = 0 \quad \text{for } w \in W.$$

PROOF OF THEOREM 2.1. It follows from Lemma 2.2 of [3] that if $\tilde{I} : V \rightarrow \mathbb{R}$ is defined by $\tilde{I}(v) = I(v + \theta(v))$ in B_1 , then \tilde{I} has a continuous Fréchet derivative $D\tilde{I}$ with respect to V and

$$D\tilde{I}(v)(h) = DI(v + \theta(v))(h) \quad \text{for all } h \in V.$$

If v_0 is a critical point of \tilde{I} , then $v_0 + \theta(v_0)$ is a solution of (2.1). Since V is the subspace of H of eigenfunctions of $\lambda_{mn} > 0$, \tilde{I} has a local minimum in B_1 , which is a critical value of \tilde{I} . This completes the proof of Theorem 2.1.

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