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WEAK CONVERGENCE THEOREMS FOR NON-LIPSCHITZIAN ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN UNIFORMLY CONVEX BANACH SPACES

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Abstract. In this paper, we give a necessary and sufficient condition for the weak convergence of trajectories of non-lipschitzian asymptotically nonexpansive mappings.

I. Introduction

A wider class of nonexpansive mappings known as asymptotically nonexpansive mappings [4] has been used by Gornicki [6] to give the necessary and sufficient condition for weak convergence of trajectories of such mappings to include the result of Bose [1] as special case.

It is well known that every asymptotically nonexpansive mapping is non-lipschitzian asymptotically nonexpansive but the converse is not true [7]. The purpose of this paper is to give a necessary and sufficient condition for weak convergence of a non-lipschitzian asymptotically nonexpansive mapping to show the result of Gornicki [6] as special case.

II. Preliminaries and Notations

First of all, we give some basic definitions. Let $(E, \|\cdot\|)$ be a Banach space and K be a subset of E. A mapping $T : K \to K$ is said to be nonexpansive if for any $x, y \in K$

$$\|Tx - Ty\| \le \|x - y\|.$$

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A mapping $T : K \to K$ is said to be asymptotically nonexpansive [4] if for each $x, y \in K$ and $i = 1, 2, 3, \cdots$,

$$\|T^i x - T^i y\| \leq k_i \|x - y\|,$$

where $\{k_i\}$ is a fixed sequence of positive reals such that $k_i \to 1$ as $i \to +\infty$. The class of asymptotically nonexpansive mappings is essentially wider than the class of nonexpansive mappings [4].

A mapping $T : K \to K$ is called non-lipschitzian asymptotically nonexpansive [7] if for each $x \in K$

$$\lim_{i \to \infty} \left\{ \sup_{y \in K} \left[\|T^i x - T^i y\| - \|x - y\| \right] \right\} \leq 0.$$

It is important to note that every asymptotically nonexpansive mapping is necessarily non-lipschitzian asymptotically nonexpansive but the converse is not true [7] as given below:

If a mapping $T : K \to K$ is asymptotically nonexpansive, then there exists a sequence $\{k_i\}$ of constants such that $k_i \to 1$ as $i \to \infty$ and for which

$$\|T^i x - T^i y\| \leq k_i \|x - y\|$$

for all $x, y \in K$ and $i = 1, 2, 3, \cdots$. Thus, we have

$$\begin{aligned} \|T^i x - T^i y\| - \|x - y\| &\leq (k_i - 1)\|x - y\| \\ &\leq |k_i - 1| \delta(K) \end{aligned}$$

and

$$\lim_{i \to \infty} \left\{ \sup_{y \in K} \left[\|T^i x - T^i y\| - \|x - y\| \right] \right\} \leq \lim_{i \to \infty} |k_i - 1| \delta(K) = 0,$$

where $\delta(K)$ denotes the diameter of K.

Let E be a uniformly convex Banach space, $\{x_n\}$ be a bounded sequence in E and K be a closed convex subset of E. Consider a functional $r : E \to [0, +\infty)$ defined by

$$r(x) = \overline{\lim_{n \to \infty}} \|x_n - x\|, \quad x \in E$$

The infimum of $r(\cdot)$ over K is called the radius of the sequence $\{x_n\}$ with respect to K and is denoted by $r(K, \{x_n\})$. A point z in K is called an asymptotic center of the sequence $\{x_n\}$ with respect to K if

$$r(z) = \min\{r(x) : x \in K\}.$$

The set of all asymptotic centers is denoted by $A(K, \{x_n\})$.

LEMMA 1. [5] *Every bounded sequence $\{x_n\}$ in a uniformly convex Banach space E has a unique asymptotic center with respect to any closed convex subset K of E, i.e., $A(K,\{x_n\}) = \{z\}$, and for all $x \in K$ with $x \neq z$*

$$\overline{\lim_{n\to\infty}} \|x_n - z\| < \overline{\lim_{n\to\infty}} \|x_n - x\|.$$

LEMMA 2. [2] *Let $\{x_n\}$ be a bounded sequence in a closed convex subset K of a uniformly convex Banach space E and $A(K,\{x_n\}) = \{z\}$. If $\{y_m\} \subset K$ and $r(y_m) \to r(K,\{x_n\})$ as $m \to +\infty$, then $y_m \to z$ as $m \to +\infty$.*

The weak convergence of a sequence $\{x_n\}$ in a Banach space E will be denoted by $x_n \overset{\omega}{\to} x$, while the strong convergence by $x_n \to x$. The set of fixed points of a mapping T will be denoted by $F(T)$.

THEOREM 3. *Let K be a closed convex (but not necessarily bounded) subset of a uniformly convex Banach space E. If $T : K \to K$ be a non-lipschitzian asymptotically nonexpansive mapping, then the following statements are equivalent:*

(a) *T has a fixed point in K;*
(b) *There is a point $x_0 \in K$ such that the sequence $\{T^n x_0\}$ of iterates is bounded.*

PROOF. (a)$\Rightarrow$(b) follows easily.

Conversely, to prove (b)$\Rightarrow$(a), assume that $x_0 \in K$ is such that the sequence $\{x_n\}$ defined by $x_n = T^n x_0$ is bounded. By Lemma 1, let $A(K,\{x_n\}) = \{z\}$, and let $\{y_m\}$ be a sequence in K defined by $y_m = T^m z$ for $m = 1, 2, \cdots$. We shall show that

$$r(y_m) = \overline{\lim_{n\to\infty}} \|x_n - y_m\| \to r(K,\{x_n\}) \quad \text{as } m \to +\infty.$$

By Lemma 2, this would imply that $y_m \to z$ as $m \to +\infty$, and because T is continuous, we have

$$Tz = T\big(\lim_{n\to\infty} T^m z\big) = \lim_{n\to\infty} T^{m+1} z = z.$$

For two integers $n > m \geq 1$, we have

$$\begin{aligned}\|x_n - y_m\| &= \|T^n x_0 - T^m z\| = \|T^m(T^{m-n}x_0) - T^m z\| \\ &= \|T^m x_{n-m} - T^m z\| \\ &= [\|T^m x_{n-m} - T^m z\| - \|x_{n-m} - z\|] + \|x_{n-m} - z\|\end{aligned}$$

and

$$r(y_m) \leq \sup_{y \in K} [\|T^m y - T^m z\| - \|y - z\|] + r(z).$$

This shows that $r(y_m) \to r(K, \{x_n\})$ as $m \to +\infty$. This comp letes the proof.

We say that a Banach space E satisfies the Opial's condition [8] if for each sequence $\{x_n\} \subset E$ weakly convergent to a point z and for all $y \in E$ with $y \neq z$,

$$\underline{\lim}_{n \to \infty} \|x_n - z\| < \underline{\lim}_{n \to \infty} \|x_n - y\|$$

It is easy to observe that the above inequality can be given an equivalent form in terms of the limit superior.

Examples of Banach spaces which satisfy the Opial's condition are Hilbert spaces and all the space ℓ^p $(1 < p < +\infty)$. On the other hand, $L^p[0, 2\pi]$ with $1 < p \neq 2$ fails to satisfy the Opial's condition [8].

LEMMA 4. [1] *Let K be a closed convex subset of a uniformly convex Banach space E satisfying the Opial's condition. If a sequence $\{x_n\} \subset K$ converges weakly to a point x, then x is the asymptotic center of $\{x_n\}$ in K.*

LEMMA 5. *Let K be a closed convex subset of a uniformly convex Banach space E satisfying the Opial's condition and $T : K \to K$ be a non-li pschitzian asymptotically nonexpansive mapping. Suppose that x_o is the asymptotic center of the bounded sequence $\{T^n x\}$ for some $x \in K$. If the weak limit z of a subsequence $\{T^{n_i} x\} \subset \{T^n x\}$ is a fixed point of T, then x_0 coincides with z.*

PROOF. Clearly, $r(K,\{T^n x\}) \geq r(K,\{T^{n_i} x\})$. Since $T^{n_i}x \xrightarrow{\omega} z$, by Lemma 4, we have $A(K,\{T^{n_i}x\}) = \{z\}$ and so, for any $\epsilon > 0$, we can choosean integer i_0 such that

$$\|z - T^{n_{i_0}}x\| \leq r(K,\{T^{n_i}x\}) + \epsilon/2.$$

Since z is a fixed point of T and T is non-lipschitzian asymptotically nonexpansive, we can choose an integer J such that for all $j \geq J$

$$\begin{aligned} &\|z - T^{n_{i_0}+j}x\| \\ \leq\ &\sup_{y\in K}\left[\|T^j z - T^j y\| - \|z - y\|\right] + r(K,\{T^{n_i}x\}) + \epsilon/2 \\ \leq\ &r(K,\{T^{n_i}x\}) + \epsilon \leq r(K,\{T^n x\}) + \epsilon. \end{aligned}$$

It follows therefore that

$$\overline{\lim_{n\to\infty}}\, \|z - T^n x\| = r(K,\{T^n x\})$$

and hence x_0 being the unique point with this property, we have $z = x_0$. This completes the proof.

The concept of asymptotic regularity is due to Browder and Petryshyn [3]. A mapping $T : K \to K$ is said to be (weakly) asymptotically regular at $x \in K$ if $T^{n+1}x - T^n x \to 0$ (weakly) as $n \to +\infty$.

Now we give our main result:

THEOREM 6. *Let E be a uniformly convex Banach space satisfying the Opial's condition and K be a closed convex (but not necessarily bounded) subset of E. If $T : K \to K$ is a non-lipschitzian asymptotically nonexpansive mapping and $x \in K$, then $\{T^n x\}$ converges weakly to a fixed point of T if and only if T is weakly asymptotically regular at x.*

PROOF. Let us assume that $T^n x \xrightarrow{\omega} p$ as $n \to +\infty$. We can show that $p \in F(T)$. By Lemma 4, $A(K,\{T^n x\}) = \{p\}$ and analogously as in Theorem 3, we have $p \in F(T)$. From $T^n x \xrightarrow{\omega} p$ as $n \to +\infty$, it follows that

$$T^{n+1}x - T^n x \xrightarrow{\omega} 0 \quad \text{as } n \to +\infty.$$

Conversely, now we are going to show the implication in the opposite way. From the assumption $T^{n+1}x - T^n x \xrightarrow{\omega} 0$ as $n \to +\infty$, we have

$$T^{n_i+m}x \xrightarrow{\omega} y \quad \text{as } i \to +\infty$$

for $m = 0, 1, \cdots$. By Lemma 4, we have

$$A(K, \{T^{n_i+m}x\}) = \{y\}$$

for $m = 0, 1, 2, \cdots$. Let $\{y_s\}$ be a sequence in K defined by $y_s = T^s y$ for $s = 1, 2, \cdots$. For integers $m > s \geq 1$, we have

$$\begin{aligned}\|y_s - T^{n_i+m}x\| &= \|T^s y - T^s(T^{n_i+m-s}x)\| \\ &= [\|T^s y - T^s(T^{n_i+m-s}x)\| - \|y - T^{n_i+m-s}x\|] \\ &\quad + \|y - T^{n_i+m-s}x\|,\end{aligned}$$

which implies that

$$r(y_s) \leq \sup_{u \in K}[\|T^s y - T^s u\| - \|y - u\|] + r(y).$$

By Lemma 2, $T^s y \to y$ as $s \to +\infty$ and, by the continuity of T, $Ty = y$. Let $w_\omega(x)$ denote the set of weak limits of subsequences of a sequence $\{T^n x\}$. From this part of the proof, we have $w_\omega(x) \subset F(T)$. By Lemma 5, $y \in w_\omega(x)$. This proves that $w_\omega(x) = \{z\}$ and so $T^n x \xrightarrow{\omega} z$ as $n \to +\infty$. This completes the proof.

From Theorem 6, we have the following corollaries:

COROLLARY 7. [6] *Let E be a uniformly covex Banach space satisfying the Opial's condition and K be a closed convex (but not necessarily boun ded) subset of E. If $T : K \to K$ is an asymptotically nonexpansive mapping and $x \in K$, then $\{T^n x\}$ converges weakly to a fixed point of T if and only if T is weakly asymptotically regular at x.*

PROOF. Since every asymptotically nonexpansive mapping is nonlipschitzian asymptotically nonexpansive mapping, the result follows from Theorem 6.

COROLLARY 8. [1] *Let K be a closed convex subset of a uniformly convex Banach space E satisfying the Opial's condition. Assume that a mapping $T : K \to K$ is asymptotically nonexpansive and weakly asymptotically regular and $F(T) \neq \emptyset$. Then for any $x \in K$, the sequence $\{T^n x\}$ of iterates is weakly convergent to a fixed point of T.*

PROOF. The result follows from Corollary 7.

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