Comm. Korean Math. Soc. **11** (1996), No. 1, pp. 109–115

NOTES ON COMMON FIXED POINT THEOREMS IN METRIC SPACES

KEEHWAN KIM AND KOUNG HEE LEEM

ABSTRACT. A number of authors have generalized contraction mapping theorems in metric spaces. In this paper, we give some common fixed point theorems related with the diameter of the orbit on metric spaces. We generalize the results of M.Ohta and G.Nikaido[6], also Tasković[8].

1. Introduction

Let (X, d) be a metric space. Let S and T be mappings from X into itself. For any $x, y \in X$, we denote the followings:

(1) $O_{S,T}(x) = \{S^iT^jx : i, j = 0, 1, 2, \cdots\}$,
(2) $O_{S,T}(x, y) = O_{S,T}(x) \cup O_{S,T}(y)$,
(3) $O_T(x) = \{T^nx : n = 0, 1, 2, \cdots\}$,
(4) $O_T(x, y) = O_T(x) \cup O_T(y)$.

For any subset A of X, the diameter of A is denoted by $\delta[A]$, i.e.,

$$\delta[A] = \sup\{d(x, y) : x, y \in A\}.$$

We say that $x \in X$ is regular if $\delta[O_{S,T}(x)] < \infty$.

The following Theorem is essentially due to *Tasković* and generalizes a great number of known results.

THEOREM 1. (Tasković[8]) *Let (X, d) be a bounded complete metric space and let T be a mapping of X into itself with property: $\exists \alpha \in [0, 1)$ such that*

$$d(Tx, Ty) \leq \alpha\delta[O_T(x, y)]$$

Received May 20, 1995. Revised November 18, 1995.
1991 AMS Subject Classification: 47H10.
Key words and phrases: common fixed point, diameter of the orbit.

for any $x, y \in X$. *Then* T *has a unique fixed point* x^*. *Moreover, for any* $u \in X$, *we have*

$$\lim_{n\to\infty} T^n u = x^*.$$

In 1994, Theorem 1 was generalized by *M. Ohta* and *G. Nikaido* as follows:

THEOREM 2. (*M. Ohta and G. Nikaido*[6]) *Let* (X, d) *be a bounded complete metric space. Assume that* T *is a continuous mapping of* X *into itself with property* (1.1) : $\exists k \in N$, $\exists \alpha \in [0,1)$ *such that*

$$d(T^k x, T^k y) \le \alpha\delta[O_T(x,y)] \tag{1.1}$$

for any $x, y \in X$. *Then* T *has a unique fixed point* x^* *and for any* $u \in X$,

$$\lim_{n\to\infty} T^n u = x^*.$$

Moreover, for any $n \in N$,

$$d(T^n u, x^*) \le \alpha^{[\frac{n}{k}]}\delta[O_T(u)],$$

where $[\frac{n}{k}]$ *is the largest integer not exceeding* $\frac{n}{k}$.

In this paper, we shall generalize the above theorems by considering the following more general conditions instead of (1.1) :

$\exists k \in N$, $\exists \alpha \in [0,1)$ such that, for any $x, y \in X$,

$$d((ST)^k x, (ST)^k y) \le \alpha\delta[O_{S,T}(x,y)], \tag{1.2}$$

$$d(S^k x, T^k y) \le \alpha\delta[O_{S,T}(x,y)]. \tag{1.3}$$

2. Main Results

If we assume that each of S and T is a continuous mapping from X into itself, and $(ST)x = (TS)x$ for any $x \in X$, then we have following theorems.

THEOREM 3. *Let (X,d) be a metric space. Assume that T and S are continuous mappings from X into itself with property (1.2) and $(TS)x = (ST)x$ for any $x \in X$.*

If there exists a regular $u \in X$ such that $\{(ST)^n u\}$ has a cluster point x^*, then x^* is the unique common fixed point of T and S. Moreover, for any $n \in N$, we have

$$d((ST)^n u, x^*) \leq \alpha^{[\frac{n}{k}]}\delta[O_{S,T}(u)].$$

PROOF. For any positive integers n, i, j, p and q, it follows from (1.2) that

$$\begin{aligned} d(S^{n+k+i}T^{n+k+j}u, S^{n+k+p}T^{n+k+q}u) & \\ &\leq \alpha\delta[O_{S,T}(S^{n+i}T^{n+j}u, S^{n+p}T^{n+q}u)] \\ &\leq \alpha\delta[O_{S,T}(ST)^n u)]. \end{aligned}$$

Hence we have, for any $n \geq 0$,

$$\delta[O_{S,T}((ST)^{n+k}u)] \leq \alpha\delta[O_{S,T}((ST)^n u)]. \tag{2.1}$$

We claim that

$$d((ST)^n u, (ST)^{n+p}u) \leq \alpha^{[\frac{n}{k}]}\delta[O_{S,T}(u)] \tag{2.2}$$

holds for any $n, p \in N$. We can write $n = mk + l$ uniquely for some non-negative integers m, l with $0 \leq l \leq k-1$. Then, by (1.2) and (2.1), we have

(2.3)

$$\begin{aligned} d((ST)^n u, (ST)^{n+p}u) &= d((ST)^{mk+l}u, (ST)^{mk+l+p}u) \\ &\leq \alpha\delta[O_{S,T}((ST)^{(m-1)k+l}u, (ST)^{(m-1)k+l+p}u)] \\ &\leq \alpha\delta[O_{S,T}(ST)^{(m-1)k}u] \\ &\leq \alpha^2\delta[O_{S,T}(ST)^{(m-2)k}u] \\ &\leq \cdots \\ &\leq \alpha^m\delta[O_{S,T}(u)], \end{aligned}$$

which proves (2.2). This implies that $\{(ST)^n u\}$ is a Cauchy sequence and since it has a cluster point x^*, we have

$$x^* = \lim_{n\to\infty} (ST)^n u.$$

Since T and S are continuous,

$$x^* = \lim_{n\to\infty} (ST)^n u = \lim_{n\to\infty} (ST)^{n+1} u = \lim_{n\to\infty} (ST)(ST)^n u = (ST)x^*.$$

By (1.2), for any $i, j = 0, 1, \cdots$,

$$d((ST)^k S^i x^*, (ST)^k S^j x^*) \leq \alpha\delta[O_{S,T}(x^*)],$$
$$d((ST)^k T^i x^*, (ST)^k T^j x^*) \leq \alpha\delta[O_{S,T}(x^*)]$$

and

$$\begin{aligned} d((ST)^k S^i x^*, (ST)^k T^j x^*) &\leq \alpha\delta[O_{S,T}(S^i x^*, T^j x^*)] \\ &\leq \alpha\delta[O_{S,T}(x^*)]. \end{aligned}$$

This means that $\delta[O_{S,T}(x^*)] \leq \alpha\delta[O_{S,T}(x^*)]$. Since $\alpha \in [0,1)$, $\delta[O_{S,T}(x^*)] = 0$. Therefore $Tx^* = x^* = Sx^*$. Finally, let w be such that $Tw = w = Sw$. Then we have

$$d(w, x^*) = d((ST)^k w, (ST)^k x^*) \leq \alpha\delta[O_{S,T}(w, x^*)] = \alpha d(w, x^*).$$

Hence we have $w = x^*$. Taking the limit in (2.3) as p tends to infinity, we obtain

$$d((ST)^n u, x^*) \leq \alpha^{[\frac{n}{k}]}\delta[O_{S,T}(u)].$$

THEOREM 4. *Let (X, d) be a metric space. Assume that T and S are continuous mappings from X into itself with property (1.3) and $(TS)x = (ST)x$ for any $x \in X$.*

If there exists a regular $u \in X$ such that $\{(ST)^n u\}$ has a cluster point x^*, then x^* is the unique common fixed point of T and S. Moreover, for any $n \in N$, we have

$$d((ST)^n u, x^*) \leq \alpha^{[\frac{n}{k}]}\delta[O_{S,T}(u)].$$

PROOF. For any positive integers n, i, j, p and q, it follows from (1.3) that

$$\begin{aligned} d(&S^{n+k+i}T^{n+k+j}u, S^{n+k+p}T^{n+k+q}u) \\ &\leq \alpha\delta[O_{S,T}(S^{n+i}T^{n+k+j}u, S^{n+k+p}T^{n+q}u)] \\ &\leq \alpha\delta[O_{S,T}(ST)^n u)]. \end{aligned}$$

Hence we have, for any $n \geq 0$,

$$\delta[O_{S,T}((ST)^{n+k}u)] \leq \alpha\delta[O_{S,T}((ST)^n u)]. \tag{2.4}$$

We claim that

$$d((ST)^n u, (ST)^{n+p}u) \leq \alpha^{[\frac{n}{k}]}\delta[O_{S,T}(u)] \tag{2.5}$$

holds for any $n, p \in N$. We can write $n = mk + l$ uniquely for some non-negative integers m, l with $0 \leq l \leq k-1$. Then, by (1.3) and (2.3), we have

(2.6)

$$\begin{aligned} &d((ST)^n u, (ST)^{n+p}u) \\ =&d((ST)^{mk+l}u, (ST)^{mk+l+p}u) \\ \leq&\alpha\delta[O_{S,T}(S^{(m-1)k+l}T^{mk+l}u, S^{mk+l+p}T^{(m-1)k+l+p}u)] \\ \leq&\alpha\delta[O_{S,T}(ST)^{(m-1)k}u] \\ \leq&\alpha^2\delta[O_{S,T}(ST)^{(m-2)k}u] \\ \leq&\cdots \\ \leq&\alpha^m\delta[O_{S,T}(u)], \end{aligned}$$

which proves (2.5). This implies that $\{(ST)^n u\}$ is a Cauchy sequence and since it has a cluster point x^*, we have

$$x^* = \lim_{n\to\infty}(ST)^n u.$$

Since T and S are continuous,

$$x^* = \lim_{n\to\infty}(ST)^n u = \lim_{n\to\infty}(ST)^{n+1}u = \lim_{n\to\infty}(ST)(ST)^n u = (ST)x^*.$$

Moreover, for any $i, j = 0, 1, \cdots$,

$$\begin{aligned} d(S^k T^i x^*, T^k S^j x^*) &\leq \alpha\delta[O_{S,T}(T^i x^*, S^j x^*)] \\ &\leq \alpha\delta[O_{S,T}(x^*)] \end{aligned}$$

This means that $\delta[O_{S,T}(x^*)] \leq \alpha\delta[O_{S,T}(x^*)]$. Since $\alpha \in [0,1)$, $\delta[O_{S,T}(x^*)] = 0$. Therefore $Tx^* = x^* = Sx^*$. Finally, let w be such that $Tw = w = Sw$. Then we have

$$d(w, x^*) = d((ST)^k w, (ST)^k x^*) \leq \alpha\delta[O_{S,T}(w, x^*)] = \alpha d(w, x^*).$$

Hence we have $w = x^*$. Taking the limit in (2.6) as p tends to infinity, we obtain

$$d((ST)^n u, x^*) \leq \alpha^{[\frac{n}{k}]}\delta[O_{S,T}(u)].$$

COROLLARY 5. *Let (X, d) be a bounded complete metric space and let T and S be continuous mappings with property either (1.3) or (1.4), $(ST)x = (TS)x$ for any $x \in X$. Then S and T have a unique common fixed point x^*, and $\lim_{n\to\infty}(ST)^n u = x^*$ for any $u \in X$. Moreover, for any $n \in N$,*

$$d((ST)^n u, x^*) \leq \alpha^{[\frac{n}{k}]}\delta[O_{S,T}(u)].$$

PROOF. In accordance to the proof of theorem 3 or 4, we have that the sequence $\{(ST)^n u\}$ converges to x^* for any $u \in X$. By the proof of theorem 3 or 4, we obtain that x^* is a unique fixed point and $\lim_{n\to\infty}(ST)^n u = x^*$ for any $u \in X$.

COROLLARY 6. *Let (X, d) be a bounded complete metric space and assume that T is a continuous mapping of X into itself with property (1.1) : Then T has a unique fixed point x^* and for any $u \in X$,*

$$\lim_{n\to\infty} T^n u = x^*.$$

PROOF. It suffices to assume that $S = I$ or $S = T$ in Corollary 5.

References

1. R. Bianchini, *Su un problema di S. Reich riguardante la teoria dei punti fissi*, Boll. Un. Math. Ital. **5** (1972), 103-108.
2. J.Daneš, *Two fixed point theorems in topological and metric spaces*, Bull. Austr. Math. Soc. **14** (1961), 259-265.
3. J. Dugundji, *Positive definite functions and coincidences*, Fundamenta Math. **40** (1976), 131-142.
4. B. Fisher, *Results on fixed points*, Bull. Acad. Sci. Poloae **25** (1977), 1253-1256.
5. M. Hegedüš and T. Szilagyi, *Equivalent conditions and a new fixed point theorem in the theory of contractive mapping*, Math. Japonica **25** (1980), 147-157.
6. M. Ohta and G. Nikaido, *Remarks on fixed point theorems in complete metric spaces*, Math. Japonica **38** (1994), 287-290.
7. M. Shih and C. Yeh, *On fixed point theorems of contractive type*, Proc. Amer. Soc. **85** (1982), 465-468.
8. M. R. Tasković, *Some results in the fixed point theory II*, Publ. Inst. Math **27** (1980), 249-258.
9. ———, *Some new principles in fixed point theory*, Math. Japonica **35** (1990), 645-666.

Department of Mathematics
Yeungnam University
Kyongsan 712-749, Korea
E-mail: khkim@ynucc.yeungnam.ac.kr