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A KOHN-NIRENBERG EXAMPLE USING LOWER DEGREE

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ABSTRACT. We will construct polynomials of degree 6 in z and $\overline{z}$ on $\mathbb{C}^2$ which gives, via its coefficient β as a parameter, a family of pseudoconvex domains Ω_β in $\mathbb{C}^2$ with the origin being a boundary point, and show that the domains Ω_β has no peak functions of class c^1 at the origin and has no holomorphic support functions for $1 \leq \beta < \frac{9}{5}$.

1. Introduction

Let us consider a plurisubharmonic function on $\mathbb{C}^2$

$$\rho_1(z,w) = Re\,(w) + p_1(z), \tag{1}$$

where $p_1(z) = |z|^8 + \frac{15}{7}|z|^2 Re\, z^6$ for $(z,w) \in \mathbb{C}^2$. Then the pseudoconvex domain $\Omega \subset \mathbb{C}^2$ defined by

$$\Omega = \{\, (z,w) \in \mathbb{C}^2 \mid \rho_1(z,w) < 0 \,\}$$

gives the well-known Kohn–Nirenberg example on $\mathbb{C}^2$ [3]; the domain Ω is Levi pseudoconvex but does not have a holomorphic support function at the origin. In particular, the domain Ω has no peak function of class C^1 at the origin [1].

Recall that for a domain $\Omega \subset \mathbb{C}^n$ and $p \in \partial\Omega$, we say that p has a holomorphic support function for the domain Ω provided that there is

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a neighbourhood U_p of p and a holomorphic function $f_p : U_p \to \mathbb{C}$ such that

$$\{ z \in U_p \mid f_p(z) = 0 \} \cap \overline{\Omega} = \{ p \}.$$

Now let

$$k(z) = Re\, z + p_1(z). \tag{2}$$

Then $k(z)$ is obviously subharmonic, but not convex near 0 in $\mathbb{C}$ along the imagiary axis in $\mathbb{C}$. Moreover it is easy to see that the function f on $\mathbb{C}$ defined by

$$f(z) = Re\,(\alpha z + \beta z^n), \quad \alpha, \beta \in \mathbb{C}$$

is not convex near 0 for any $\alpha, \beta \neq 0$, and $n \geq 2$ along the line generated by the n-th root of $-\overline{\beta}/|\beta|$.

2. Main propositions

One property of the function k near 0 is as following

LEMMA 1. *There is no holomorphic change of coordinates g nea 0 satisfying $g(0) = 0$, $g'(0) = 1$, and $k \circ g(z)$ is convex at 0.*

PROOF. Suppose there is such a function g and consider the Talor expansion of $k \circ g$ at 0. Let

$$g(z) = \sum_{n=}^{\infty} c_n z^n, \quad {}_1 = 1.$$

Successive appliction of the last fact mentioned above shows

$$k \circ g(z) = Re\,(z + c_8 z^8) + |z|^8 - \frac{16}{7} Re\,(z^7 \bar{z}) + O(|z|^9).$$

When z is real, we get $Re\, c_8 \geq \frac{9}{7}$ by convexity at 0. To get a contradiction, let $z = re^{i\frac{3\pi}{8}}, r \geq 0$. Hence

$$k \circ g(z) = Re\, z + \{-Re\, c_8 + 1 - \frac{16}{7} Re\, e^{i\frac{\pi}{4}}\} r^8 + O(r^9).$$

So $k \circ g$ is not convex at 0 since the coefficient of r^8 is negative. This completes the proof.

Now let

$$(3) \qquad h(z) = Re\, z + |z|^4 - \alpha |z|^2 Re\, z^2, \quad \alpha \in \mathbb{C}.$$

We can choose $\alpha = \frac{4}{3}$ for h to be subharmonic. Then the Laplacian of h is identically zero along the real axis, and h can be convexfied by the holomorphic function $g(z) = z + \frac{1}{3}z^4$. Suppose

$$(4) \qquad r(z) = Re\, z + |z|^6 - \alpha(z^5\bar{z} + \bar{z}^5 z).$$

We can choose $\alpha = \frac{9}{10}$ to make r subharmonic, which implies that r is not convex at 0 along the real axis.

LEMMA 2. *There is no holomorphic change of coordinates g such that $g(0) = 0$, $g'(0) = 1$, and $r \circ g(z)$ is convex at 0.*

PROOF. Assume the existence of such a holomorphic change g of coordinate and consider the Taylor expansion of $r \circ g$ near 0. Let

$$g(z) = \sum_{n=1}^{\infty} c_n z^n, \quad c_1 = 1.$$

Then

$$r \circ g(z) = Re\,(z + c_6 z^6) + |z|^6 - \frac{5}{|}z|^2 Re\, z^4 + O(|z|^7).$$

Hence the coefficient of c_6 of z^6 in the Taylor expansion of $r(g(z))$ is such that $Re\, c_6 \geq \frac{4}{5}$ along the real axis and $Re\, c_6 \leq > -\frac{4}{5}$ along the imaginary axis by the convexity of $r \circ g$. Thus r is not convexifiable at 0.

Let

$$(5) \quad \rho_2(z, w) = Re(w) + p_2(z), \quad p_2(z) = \beta|z|^6 - \frac{9}{5}|z|^2 Re\, z^4, \quad \beta \geq 1$$

and a domain Ω_β in $\mathbb{C}^2$ given by

$$\Omega_\beta = \{ (z, w) \in \mathbb{C}^2 \mid \rho_2 < 0 \}.$$

Clearly ρ_2 is C^2–plurisubharmonic in Ω_β as a sum of two C^2–subharmonic functions on $\mathbb{C}$. Hence Ω_β is pseudoconvex. It is strictly pseudoconvex at every boundary point except for the line : $z = 0$, $Re\, w = 0$.

REMARK. (1) By adding $|zw|^2$ ($|z|^2+|w|^2$, resp.) to the plurisubharmonic function $\rho_2(z,w)$, we obtain a strictly pseudoconvex domain except at the origin where it is pseudoconvex (a strictly pseudoconvex domain, resp.)

(2) The domain Ω_β has no peak functions at 0 that extends to be holomorphic in a neighborhood of 0 (for if f were a peak function, then $f(z)-1$ would be a holomorphic support function at 0.)

The following Proposition shows that the pseudoconvex domains Ω_β has no C^1 peak functions at the origin.

PROPOSITION 3. *For $1\le\beta<\frac{9}{5}$, let U_β be a neighborhood of $0\in\partial\Omega_\beta$. Then there exists no function $f\in H(\Omega_\beta\cap U_\beta)\cap c^1(\overline{\Omega}_\beta\cap U_\beta)$ so that $f(0)=1$ and $|f|<1$ on $\overline{\Omega}_\beta\cap U_\beta\setminus\{0\}$.*

PROOF. Assume there exist such functions f and let $h(z,w)=Re\, f(z,w)-1$. Then $\alpha:=\frac{\partial h}{\partial n}(0)>0$ by the Hopf lemma. If $\epsilon>0$, there is a neighborhood $U_\epsilon(0)$ of 0 such that

$$\alpha-\epsilon<\frac{\partial h}{\partial n}(z,w)<\alpha+\epsilon \quad \text{whenever} \quad (z,w)\in U_\epsilon(0).$$

For $u<0$, let

$$\Delta_u=\{\,(z,u)\mid |z|<\sqrt[6]{5|u|/(5\beta+9)}\,\}.$$

Then $\Delta_u\subset\Omega$.

Notice that if $(z,u)\in\Delta_u$ and $z^4>0$ we have $(z,t)\in\overline{\Omega}_\beta$ for $u\le t\le(\frac{9}{5}-\beta)|z|^6$. Now follow the idea of the proof in [4, p. 123] to establish the following harmonic function h on D satisfying

$$\text{(i)} \qquad h(0)\ge-\alpha$$

$$\text{(ii)} \qquad h(\tilde z)\le-\alpha(1-|\tilde z|^6)$$

$$\text{(iii)} \qquad h(\tilde z)\le-\alpha(1-\frac{5\beta-9}{5\beta+9}|\tilde z|^6) \quad \text{whenever} \quad |\tilde z|^4>0$$

where $\tilde z:=z/\sqrt[6]{5|u|/(5\beta+9)}$.

Let

$$g(\tilde{z}) = (5\beta + 9)\frac{h(\tilde{z}) + \alpha}{\alpha}.$$

then g is harmonic on the unit disc D in $\mathbb{C}$, $g(0) = 0$, $g(\tilde{z}) \leq (5\beta+9)|\tilde{z}|^6$, and $g(\tilde{z}) \leq (5\beta - 9)|\tilde{z}|^6$ if $\tilde{z}^4 > 0$. By the harmonicity of g, there exists a holomorphic function F such that $Re\, F = g$ and $F(0) = 0$. Then

$$F(\tilde{z}) = c\tilde{z}^l + O(|\tilde{z}|^{l+1}).$$

Since $g(\tilde{z}) \leq (5\beta+9)|\tilde{z}|^6$ and $g(\tilde{z}) \leq (5\beta-9)|\tilde{z}|^6 \leq 0$ when $\tilde{z}^4 > 0$ imply $l = 6$ and $c \neq 0$.

If $\tilde{z}_n = re^{i\frac{n\pi}{4}}$, $n = 0, 2$, then $\tilde{z}_n^4 = r^4 > 0$ and

$$g(\tilde{z}_n) = Re\, F(\tilde{z}_n) = r^6 Re\, ce^{i\frac{3n\pi}{2}} + O(r^7).$$

Hence

$$g(\tilde{z}_1) + g(\tilde{z}_2) = r^6 Re\, c(1 + e^{i3\pi}) + O(r^7) = O(r^7),$$

which contradicts the fact that $g(\tilde{z}) \leq (5\beta - 9)|\tilde{z}|^6$ when $\tilde{z}^4 > 0$. This completes the proof.

The following Proposition gives us another Kohn–Nirenberg example in $\mathbb{C}^2$ given by a polynomial defining function whose degree in z and $\overline{z}$ is 6 and so lower than that of the defining function for *the* Kohn–Nirenberg example.

PROPOSITION 4. *Let*

$$\rho_3(z, w) = Re\, w + \beta|z|^6 - \frac{9}{5}|z|^2 Re\, z^4, \quad \beta \in \mathbb{R}.$$

and the pseudoconvex domain Ω_β in $\mathbb{C}^2$ defined by $\rho_3(z, w) < 0$. Then

(1) *If $\beta \geq \frac{9}{5}$, Ω_β has a holomorphic support function at* 0.

(2) *If $1 \leq \beta < \frac{9}{5}$, Ω_β has no holomorphic support function at* 0.

REMARK. When $\beta < 1$, the Levi–form of ρ_3 at (z, w) with $Re\, w = 0 = Im\, z$, $Re\, z \neq 0$ is strictly negative if $|z|$ is small.

PROOF. Let $T = \{(z, w) \in \mathbb{C}^2 \mid w = z^8\}$. It suffices to show that T is a holomorphic support manifold at 0. Let $z = re^{i\theta}$ with $r > 0$. Then

$$\rho_3(z, z^8) = 2r^8 \cos^2 4\theta - r^8 + r^6(\beta - \frac{9}{5}\cos 4\theta).$$

If $\cos 4\theta < -\frac{\sqrt{2}}{2}$, we have $\rho_3(z, w) \geq 2r^8 \cos^2 4\theta - r^8 > 0$ for all $\beta \geq \frac{9}{5}$. On the other hand, if $\cos 4\theta \geq -\frac{\sqrt{2}}{2}$, then $\rho_3(z, z^8) \geq -r^8 + r^6(\beta - \frac{9}{5}\cos 4\theta) > 0$ for small $r > 0$ if $\beta > \frac{9}{5}$.

When $\beta = \frac{9}{5}$; suppose $\cos 4\theta \neq 1$, then

$$\rho_3(z, z^8) > -r^8 + r^6(\beta - \frac{9}{5}\cos 4\theta) > 0.$$

If $\cos 4\theta = 1$, $\rho_3(z, z^8) > r^8 > 0$. Hence (1) holds.

(2) When the complex tangent space T_0T has a component in the z-direction;

M can be written as a graph $\{w = \phi(z)\}$ over the z-axis. In particular, suppose $\phi(z) = \alpha z^6 + O(|z|^7)$ where $\alpha \neq 0$. Let $z_1 = r$ and $z_2 = ir$, $r > 0$. Then

$$\rho_3(z_1, w_1) + \rho_3(z_2, w_2) = 2(\beta - \frac{9}{5})r^6 + O(r^7) < 0,$$

which contradicts the fact that $(z_n, w_n) \in \Omega_\beta$ for $n = 1, 2$ and $T \cap \overline{\Omega}_\beta = \{0\}$. All other cases when $\phi(z) = \alpha z^l + O(z^{l+1})$ with $l \neq 6$ or when $T_o T$ is the w-axis can be proved by imitating the proof of [5, page 119 or 3, Theorem C].

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