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CONVERGENCE AND THE RIEMANN HYPOTHESIS

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ABSTRACT. For $1 < p \le 2$ it is shown that a certain sequence of functions converges to -1 in $L^{p-\varepsilon}(0,1)$ for any small $\varepsilon > 0$ if and only if the Riemann zeta function satisfies $\zeta(s) \ne 0$ for $\sigma = \mathrm{Re}\, s > 1/p$.

In this article, we investigate a certain aspect of the Riemann hypothesis on the location of the nontrivial zeros of the Riemann zeta function which is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for a complex variable $s = \sigma + it$ with $\sigma > 1$, and meromorphically continued on the whole plane. It is known that $\zeta(s)$ has no zero in the half-plane $\sigma \ge 1$ and has no nontrivial zero in $\sigma \le 0$. G.H. Hardy showed that $\zeta(s)$ has infinitely many zeros on the line $\sigma = 1/2$. Recall that the famous Riemann hypothesis states that all the zeros of $\zeta(s)$ in the strip $0 \le \sigma \le 1$ lie on the line $\sigma = 1/2$.

Let $\{x\}$ denote the fractional part of the real number x. Note that for any fixed real θ the function $\left\{\frac{\theta}{x}\right\}$ takes values in the interval $[0,1]$ and has only countably many discontinuities. Let C be the linear space of the bounded and measurable functions on the interval $(0,1)$ which is of the form

$$f(x) = \sum_{n=1}^{N} c_n \left\{ \frac{\theta_n}{x} \right\},$$

where N is a positive integer, c_n any complex numbers and θ_n real numbers with $0 < \theta_n \le 1$ such that

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$$\sum_{n\leq N} c_n\theta_n = 0.$$

Define a similar function space R by allowing θ_n to take only rational numbers in the interval $(0,1]$.

A theorem due to Beurling [2] states that for any $p>1$, the meromorphic function $\zeta(s)$ has no zeros with $\sigma > 1/p$ if and only if C is dense in $L^p(0,1)$. His proof depends on the existence of characters of the multiplicative semigroup $(0,1)$. See also [1].

In this article we prove a similar result using the Möbius function that is defined by

$$\mu(n) = \begin{cases} 1 & \text{if } n=1 \\ (-1)^s & \text{if } n \text{ is square free with prime factorization} \\ & \qquad n = p_1p_2\cdots p_s \\ 0 & \text{if } n \text{ is not square free.} \end{cases}$$

It is known that

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty}\frac{\mu(n)}{n^s} \quad (\sigma\geq 1),$$

hence $\sum_{n=1}^{\infty}\frac{\mu(n)}{n} = 0$. The Riemann hypothesis is equivalent to the statement that $\sum_{n\leq N}\mu(n) = O(N^{\frac{1}{2}+\epsilon})$. (See P.62 and P.370 [4].)

LEMMA. *Assume that $\zeta(s)$ has no zeros for $\sigma > \eta$. Then for any $\varepsilon > 0$ and any positive N,*

$$\sum_{n\leq N}\frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)} + O(N^{-\delta}|t|^{\varepsilon})$$

for $\sigma > \eta + 2\delta$.

PROOF OF LEMMA. By Perron's formula, we see that

$$\sum_{n\leq N}\frac{\mu(n)}{n^s} = \frac{1}{2\pi i}\int\frac{1}{\zeta(w)}\frac{N^{w-s}}{w-s}\,dw$$

where the integration is carried along an appropriate infinite vertical contour. Pushing the contour to the left,

$$\sum_{n\le N}\frac{\mu(n)}{n^s}=\frac{1}{\zeta(s)}+\frac{1}{2\pi i}\int\frac{1}{\zeta(w)}\frac{N^{w-s}}{w-s}\,dw$$

where the new contour is the vertical line $\mathrm{Re}(w)=\sigma-\delta$. Since $\frac{1}{\zeta(w)}\ll |\,\mathrm{Im}(w)|^{\varepsilon}$ on this line (see, for example, P.283 [4]), the last integral is

$$\ll N^{-\delta}\int_{-\infty}^{\infty}\frac{|v|^{\varepsilon}}{\delta+|v-t|}\,dv\ll N^{-\delta}|t|^{\varepsilon}.$$

Now we state an arithmetic version of Beurling's theorem.

THEOREM. *Let $1<p\le 2$. Then the following statements are equivalent:*

(a) $\zeta(s)\neq 0$ *for* $\sigma=Re\ s>1/p$,
(b) C *is dense in* $L^{p-\varepsilon}(0,1)$ *for any small* $\varepsilon>0$,
(c) R *is dense in* $L^{p-\varepsilon}(0,1)$ *for any small* $\varepsilon>0$,
(d) $f_N(x)=\sum_{n\le N}\mu(n)\left\{\frac{1}{nx}\right\}$ *converges to* -1 *as* N *tends to* ∞ *in* $L^{p-\varepsilon}(0,1)$ *for any small* $\varepsilon>0$.

REMARK. See P. 252–257, [3] for details on the proof on the equivalence of (a) and (b).

PROOF. Note that (c) implies (b) by definition.

To prove that (b) implies (a) we briefly follow Beurling's original argument for the sake of completeness. For $\sigma>0$, the Riemann zeta function has the representation

$$\zeta(s)=\frac{s}{s-1}-s\int_1^{\infty}\{u\}u^{-s-1}\,du.$$

For a fixed θ, $0<\theta\le 1$, we make the change of variable $x=\theta/u$ in the last integral and multiply both sides by θ^s/s. Then we have

$$\frac{\zeta(s)\theta^s}{s}=\frac{\theta}{s-1}-\int_0^1\left\{\frac{\theta}{x}\right\}x^{s-1}\,dx.$$

For $f(x) = \sum_{n \le N} c_n\{\theta_n/x\} \in C$, we will have

$$\frac{\zeta(s)}{s} \sum_{n \le N} c_n \theta_n^s = -\int_0^1 f(x) x^{s-1}\, dx,$$

and consequently

$$\frac{1}{s}\left(1 - \zeta(s) \sum_{n \le N} c_n \theta_n^s\right) = \int_0^1 (1 + f(x)) x^{s-1}\, dx, \tag{1}$$

for $\sigma > 0$. Suppose that $\zeta(\rho) = 0$ for some $\rho = \beta + i\gamma$ with $\beta > 1/p$. Choose any real number a such that $1/p < 1/a < \beta$, and b satisfying $1/a + 1/b = 1$. Then by Hölder's inequality

$$\begin{aligned}\frac{1}{|\rho|} = \left|\int_0^1 (1 + f(x)) x^{\rho-1}\, dx\right| &\le \|1 + f\|_a \left(\int_0^1 x^{(\beta-1)b}\, dx\right)^{1/b} \\ &= \left(\frac{a-1}{a\beta - 1}\right)^{(a-1)/a} \|1 + f\|_a.\end{aligned}$$

By the hypothesis we may choose $f \in C$ so that $\|1 + f\|_a$ is arbitrarily small. This gives a contradiction.

Now we prove that (d) implies (c). Fix any real a, $1 < a < p$. Since the set of step functions is dense in $L^a(0,1)$, it is enough to show that any characteristic function $\chi_{(0,r)}$ on the interval $(0, r)$, where r is a rational number in $(0, 1)$, can be approximated by the elements in R. Let

$$g_N(x) = \sum_{n \le N} \mu(n) \left\{\frac{1}{nx}\right\} - \sum_{n \le N} \frac{\mu(n)}{n} \left\{\frac{1}{x}\right\}. \tag{2}$$

Then $g_N \in R$. Since $\sum_{n \le N} \mu(n)/n = o(1)$, g_N converges to -1 in $L^a(0,1)$. Clearly $g_N(x/r) \in R$ and $g_N(x/r) = 0$ if $x > r$.

Now we have

$$\begin{aligned}\int_0^1 |\chi_{(0,r)}(x) + g_N(x/r)|^a\, dx &= \int_0^r |1 + g_N(x/r)|^a\, dx \\ &= r \int_0^1 |1 + g_N(y)|^a\, dy.\end{aligned}$$

This goes to 0 as N tends to ∞.

Finally we show that (a) implies (d). Let $g_N(x)$ be as in (2). Let a, $1 < a < p$, be given. It is enough to show that g_N converges to -1 in $L^a(0,1)$ since $\sum_{n\le N}\mu(n)/n = o(1)$. By the change of variable $e^y = x$, we have

$$\int_0^1 (1+g_N(x))x^{s-1}\,dx = \int_{-\infty}^0 (1+g_N(e^y))e^{y\sigma}e^{iyt}dy = \int_{-\infty}^{\infty} G_N(y)e^{iyt}dy$$

where $G_N(y) = 1 + g_N(e^y)e^{y\sigma}$ for $y < 0$ and $G_N(y) = 0$ for $y > 0$. Let b be such that $1/a + 1/b = 1$. Then from (1) and the Hausdorff-Young inequality (see, for example, P.254, [5]),

$$\begin{aligned} &\left(\int_{-\infty}^0 |(1+g_N(e^y))e^{y\sigma}|^a\,dy\right)^{1/a} \\ (3)\qquad &= \left(\int_0^1 |1+g_N(x)|^a x^{\sigma a-1}\,dx\right)^{1/a} \\ &\ll \left(\int_{-\infty}^{\infty}\left|1-\zeta(s)\left(\sum_{n\le N}\frac{\mu(n)}{n^s} - \sum_{n\le N}\frac{\mu(n)}{n}\right)\right|^b \frac{dt}{|s|^b}\right)^{1/b} \end{aligned}$$

Taking $\sigma = 1/a$ in (3), we see by Minkowski's inequality that $\|1+g_N\|_a$ is

$$(4)\quad \ll \left(\int_{-\infty}^{\infty}\left|1-\zeta(s)\sum_{n\le N}\frac{\mu(n)}{n^s}\right|^b\frac{dt}{|s|^b}\right)^{1/b} + \left|\sum_{n\le N}\frac{\mu(n)}{n}\right|\left(\int_{-\infty}^{\infty}\frac{|\zeta(s)|^b}{|s|^b}\,dt\right)^{1/b}$$

Using the Lemma with $\delta = 1/2a - 1/2p$, we see that the first term is

$$\ll \left(\int_{-\infty}^{\infty} N^{-\delta b}\,|t|^{\epsilon b}\,\frac{dt}{|s|^b}\right)^{1/l}$$

If ϵ is picked so that $\epsilon < 1 - 1/b$, this is $\ll N^{-\delta}$. The second term in (4) tends to 0 since $\sum_{n\le N}\mu(n)/n = o(1)$. This completes the proof.

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