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BI-HERMITE POLYNOMIALS AND MATCHINGS IN COMPLETE GRAPHS

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ABSTRACT. Explicit formulas for bi-Hermite polynomials are found and their combinatorial model is considered. This combinatorial model is a generalization of the combinatorial model of Hermite polynomials as matching polynomials.

1. Introduction

Biorthogonal polynomials are generalizations of orthogonal polynomials. They are defined as follows [5, 7]. Let L be a linear functional on the vector space of polynomials in x with real coefficients. Let d be a positive integer. We consider two sets of polynomials, $\{R_n(x,d)\}_{n\geq 0}$ and $\{S_n(x,d)\}_{n\geq 0}$, such that $R_n(x,d)$ is a polynomial in x of degree n and $S_n(x,d)$ is a polynomial in x^d of degree n. (So $S_n(x,d)$ is a polynomial in x of degree dn.) These two sets of polynomials are said to be biorthogonal with respect to the linear functional L if

$$L(R_m(x,d)S_n(x,d)) = 0 \text{ if and only if } m \neq n.$$

If $d=1$, then the biorthogonality condition implies that $R_n(x,d)$ and $hsnd$ differ only by a scalar multiple and $\{R_n(x,d)\}_{n\geq 0}$ become ordinary orthogonal polynomials. If the linear functional L is given by the integral with respect to a weight function $\mu(x)$ over an interval $[\alpha,\beta]$, the above orthogonality condition can be written as

$$\int_\alpha^\beta R_m(x,d)S_n(x,d)\mu(x)\,dx = 0 \text{ if and only if } m \neq n.$$

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Let the linear functional L be given by the weight function

$$\mu(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

on $(-\infty, +\infty)$, which is the weight function of Hermite polynomials. Then,

$$L(x^n) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^n e^{-\frac{x^2}{2}}\, dx = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 1 \cdot 3 \cdots (n-1) & \text{otherwise,} \end{cases}$$

(for $n = 0$, the expression $1 \cdot 3 \cdots (2n-1) = \frac{(2n)!}{2^n n!}$ is assumed to be 1) and the orthogonality becomes

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} R_m(x,d) S_n(x,d) e^{-\frac{x^2}{2}}\, dx = 0 \text{ if and only if } m \neq n.$$

Let $R_n(x,d)$ and $S_n(x,d)$ denote the biorthogonal polynomials determined by L. These are uniquely determined by L up to non-zero constant multiples. We will call these bi-Hermite polynomials. Since the weight function $\mu(x)$ is even, the moments vanish for odd degree and $R_n(x,d)$ and $S_n(x,d)$ are orthogonal only if d is odd. For even d's, they are quasi-orthogonal.

As Hermite polynomials can be gotten from Laguerre polynomials by a suitable choice of parameters, bi-Hermite polynomials can be gotten from the biorthogonal polynomials associated to Laguerre polynomials, which are called Konhauser polynomials [6, 8]. Let $Y_n^{(a)}(x,d)$ and $Z_n^{(a)}(x,d)$ denote the Konhauser polynomials. They have the following explicit expressions:

$$Y_n^{(a)}(x,d) = (-1)^n \sum_{r=0}^{n} \frac{x^r}{r!} \sum_{s=0}^{r} (-1)^s \binom{r}{s} (a+s;d)_n$$

$$Z_n^{(a)}(x,d) = (-1)^n \sum_{t=0}^{n} (-1)^t \binom{n}{t} (a+dt)_{dn-dt} x^{dt},$$

where $(a;k)_n = \prod_{i=0}^{n-1}(a+ik)$ and $(a)_n = \prod_{i=0}^{n-1}(a+i)$.

Though bi-Hermite polynomials can be deduced from Konhauser polynomials as Hermite polynomials are deduced from Laguerre polynomials, it is not clear which value of a in $Y_n^{(a)}(x,d)$ and $Z_n^{(a)}(x,d)$ will work. It turns out that $a = 1/2$ for even n and $a = d/2+1$ for odd n work. To get the explicit expressions of bi-Hermite polynomials $R_n(x,d)$ and $S_n(x,d)$ for even n, we let $a = 1/2$ and substitute x with $x^2/2$. Then $2^n Y_n^{(\frac{1}{2})}(x,d)$ becomes $R_{2n}(x,d)$ and $2^{nd} Z_n^{(\frac{1}{2})}(x,d)$ becomes $S_{2n}(x,d)$. To get the explicit expressions of $R_n(x,d)$ and $S_n(x,d)$ for odd n, we let $a = d/2+1$ and substitute x with $x^2/2$. Then $2^n Y_n^{(\frac{d}{2}+1)}(x,d)$ becomes $R_{2n+1}(x,d)/x$ and $2^{nd} Z_n^{(\frac{d}{2}+1)}(x,d)$ becomes $S_{2n+1}(x,d)/x^d$. After some simplications, we see that $R_n(x,d)$, which is a polynomial of degree n in x, is of the form:

$$R_{2n}(x,d) = (-1)^n \sum_{r=0}^{n} \frac{x^{2r}}{2^r r!} \sum_{s=0}^{r} (-1)^s \binom{r}{s} (2s+1; 2d)_n \tag{1}$$

$$R_{2n+1}(x,d) = (-1)^n \sum_{r=0}^{n} \frac{x^{2r+1}}{2^r r!} \sum_{s=0}^{r} (-1)^s \binom{r}{s} (2s+2+d; 2d)_n \tag{2}$$

The other polynomial $S_n(x,d)$, which is a polynomial of degree n in x^d, is of the form:

$$\begin{aligned} S_{2n}(x,d) &= \sum_{i=0}^{n} \binom{n}{i} (-2dn+1; 2)_{di} x^{d(2n-2i)} \\ &= (-1)^n \sum_{i=0}^{n} (-1)^i \binom{n}{i} (2di+1; 2)_{d(n-i)} x^{2di} \end{aligned} \tag{3}$$

$$\begin{aligned} S_{2n+1}(x,d) &= \sum_{i=0}^{n} \binom{n}{i} (-2dn-d; 2)_{di} x^{d(2n-2i+1)} \\ &= (-1)^n \sum_{i=0}^{n} (-1)^i \binom{n}{i} (2di+d+2; 2)_{d(n-i)} x^{2di+d} \end{aligned} \tag{4}$$

In §2, we prove the explicit formulas in (1)–(4) are orthogonal and show how bi-Hermite polynomials are related to Konhauser polynomials (bi-Laguerre polynomials). In §3, we interpret $R_n(x,d)$'s combinatorially as the weight of appropriate matchings in the complete graph with n vertices.

We use the notation $[n]$ for the set $\{1,2,\ldots,n\}$

2. Bi-Hermite polynomials

Let $R_n(x,d)$ and $S_n(x,d)$ be the polynomials defined by the equations (1)–(4). In fact the polynomial $S_n(x,d)$ has a nicer representation:

$$S_n(x,d) = \sum_{i\geq 0} \frac{(-n;2)_i(-dn+1;2)_{di}}{2^i i!}(-1)^i x^{d(n-2i)}, \quad \text{if } n \text{ is even,}$$

$$S_n(x,d) = \sum_{i\geq 0} \frac{(-dn;2)_{di}(-n+1;2)_i}{2^i i!}(-1)^i x^{d(n-2i)}, \quad \text{if } n \text{ is odd.}$$

The above expresions are especially interesting since if we set $d=1$ in them both expressions reduce to the same expression.

Note that if $d=1$, both $R_n(x,1)$ and $S_n(x,1)$ become a Hermite polynomial

$$H_n(x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i}(1;2)_i(-1)^i x^{n-2i}.$$

We prove that the above expressions for $R_n(x,d)$ and $S_n(x,d)$ actually define bi-Hermite polynomials.

THEOREM 2.1. *Let L be the linear functional for Hermite polynomials, i.e.*

$$L(x^n) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^n e^{-\frac{x^2}{2}}\,dx = \begin{cases} 0 & \textit{if } n \textit{ is odd,} \\ 1\cdot 3\cdots(n-1) & \textit{otherwise.} \end{cases}$$

For a positive odd integer d, $R_n(x,d)$ and $S_n(x,d)$ defined in the equations (1)–(4) are orthogonal with respect to L, i.e.

$$L(R_m(x,d)S_n(x,d)) = 0 \textit{ if and only if } m \neq n.$$

PROOF. If m and n have a different parity, then the orthogonality is trivial since the moments vanish for odd degrees and d is odd. So it suffices to prove the orthogonality for the cases where m and n have the same parity. We will prove the case where m and n are odd. We evaluate the sum

$$
\begin{aligned}
&L(R_{2m+1}(x,d)S_{2n+1}(x,d)) = \\
&\quad (-1)^m \sum_{r=0}^{m} \frac{1}{2^r r!} \sum_{s=0}^{r} (-1)^s \binom{r}{s} (2s+2+d;2d)_m \sum_{i=0}^{n} \binom{n}{i} \\
&\quad (-d(2n+1);2)_{dn-di}(1;2)_{di+r+\frac{d+1}{2}} \\
&= (-1)^{m+n} 2^m (1;2)_{dn+\frac{d+1}{2}} \sum_{r=0}^{m} \frac{1}{r!} \sum_{s=0}^{r} (-1)^s \binom{r}{s} \left(s+\frac{d}{2}+1;d\right)_m \\
&\quad \sum_{i=0}^{n} (-1)^i \binom{n}{i} \left(di+\frac{d}{2}+1\right)_r \\
&= (-1)^{m+n} 2^m (1;2)_{dn+\frac{d+1}{2}} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \\
&\quad \sum_{r=0}^{m} \frac{(di+\frac{d}{2}+1)_r}{r!} \sum_{s=0}^{r} (-1)^s \binom{r}{s} \left(s+\frac{d}{2}+1;d\right)_m \\
&= (-1)^{m+n} 2^m (1;2)_{dn+\frac{d+1}{2}} \sum_{i=0}^{n} (-1)^i \binom{n}{i} (-di;d)_m \\
&= (-1)^{n} 2^m d^m (1;2)_{dn+\frac{d+1}{2}} \sum_{i=m}^{n} (-1)^i \binom{n}{i} \frac{i!}{(i-m)!} \\
&= 2^m d^m n! (1;2)_{dn+\frac{d+1}{2}} \delta_{m,n}
\end{aligned}
$$

The case where m and n are even is similar.

3. Combinatorial model for bi-Hermite polynomials

We look at the known combinatorial interpretation for Hermite polynomials. Hermite polynomials $H_n(x)$ are orthogonal polynomials with respect to moments,

$$
L(x^n) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 1 \cdot 3 \cdots (n-1) & \text{otherwise,} \end{cases}
$$

and they satisfy the recurrence relation

$$H_{n+1}(x) = xH_n(x) - nH_{n-1}(x), \quad n \geq 1,$$

with $H_0 = 1$ and $H_{-1} = 0$ [2]. In the combinatorics of $H_n(x)$ the moment $L(x^n)$ is interpreted as the weight of all perfect matchings of K_n. A q-analogue of Hermite polynomials is possible and its combinatorics has been studied in [4].

Since the moments for Hermite polynomials vanish for odd degrees, there is a theory connecting these polynomials to some other orthogonal polynomials gotten by replacing x^2 with x in $H_{2n}(x)$ and $H_{2n+1}(x)/x$. In fact $H_n(x)$'s are specializations of Laguerre polynomials [2 p. 43].

A matching M in a graph G is a subset of the edge set $E(G)$ of G such that each vertex in G is incident to at most one edge in M. If each vertex in G has exactly one edge in M, then M is called a perfect matching or 1-factor of G. We assign a weight to each matching as follows. If M is a matching of G and G has n vertices, then define the weight $\omega(M)$ of M to be $(-1)^{|M|}x^{n-2|M|}$. Note that $(n-2|M|)$ is the number of vertices which are incident to no edge in M. If G is the complete graph K_n with vertices $\{1, 2, \ldots, n\}$, then each matching corresponds to an involution in S_n, where S_n denote the group of all permutation of $\{1, 2, \ldots, n\}$ [3, 9]. It is known that the Hermite polynomial $H_n(x)$ is the matching polynomial for K_n, i.e.

$$H_n(x) = \sum_{M:\text{ a matching of } K_n} (-1)^{|M|}x^{n-2|M|}.$$

[1, 3].

As we interpret $H_n(x)$ as a weight generating function of matchings of the complete graph K_n, we will also interpret $R_n(x,d)$ as the weight of (partial) matchings of K_n using different weight for matchings. Since $R_n(x,d)$ contains a factor d, the weight of a matching must contain some power of d. We describe a new weight ω for matchings in K_n below. For convenience' sake, we label vertices of K_{2m} by $\{1, \underline{1}, 2, \underline{2}, \ldots, m, \underline{m}\}$ and vertices of K_{2m+1} by $\{0, 1, \underline{1}, 2, \underline{2}, \ldots, m, \underline{m}\}$. Let I_{2m} denote the set $\{1, \underline{1}, 2, \underline{2}, \ldots, m, \underline{m}\}$ and I_{2m+1} the set $\{0, 1, \underline{1}, 2, \underline{2}, \ldots, m, \underline{m}\}$. Let M be a partial matching in the complete graph K_n whose vertices are labeled by I_n. We define a weight $\omega(M)$ of M to be $(-1)^{|M|}d^{\alpha}x^{n-2|M|}$, where the exponent α is explained below. We identify the vertices labeled i and $\underline{i}$ for each $i \in [n]$ in K_n. The matching M is reduced to a disjoint union of cycles and paths with vertices $\{1, 2, \ldots, \frac{n}{2}\}$, if n is even, and with vertices $\{0, 1, 2, \ldots, \frac{n-1}{2}\}$, if n is odd. Let M' denote the result of the reduction of M. Note that an edge $i - \underline{i} \in M$ becomes a 1-cycle in M' and two fixed points $\{i, \underline{i}\}$ in M become a fixed point i in M'. This implies that we distinguish a fixed point from a 1-cycle. Figures 1 and 2 show a matching M and its reduction M'. We will explain below how the arrows are directed.

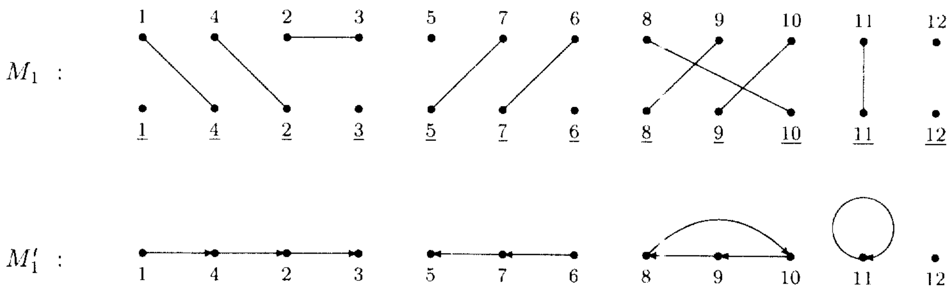

Figure 1: A matching M_1 in K_{24} and M_1'.

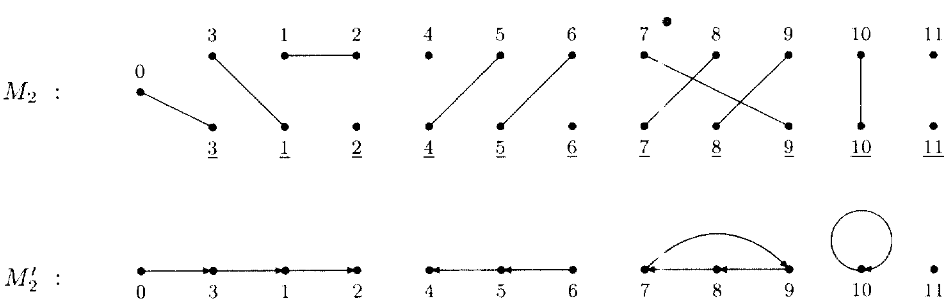

Figure 2: A matching M_2 in K_{23} and M_2'.

Note that we have used arrows in M'. We explain how we get the connected components of M' and their weight from M.

Even case: If $n = 2m$, then there are four kinds of components in M'.

1. Type-1 path: For $i_1 < i_k$, a path $i_1 \to i_2 \to \cdots \to i_k$ in M' is a connected component in M', if i_1 is matched to either i_2 or $\underline{i_2}$ in M and $\hat{\imath}_j$ is matched to $\hat{\imath}_{j+1}$ in M for $j = 2, \ldots, k-1$ where $\hat{\imath}_j$ denotes either i_j or $\underline{i_j}$. In this case we regard the path as a permutation $i_1 i_2 \ldots i_k$ of the set $\{i_1, i_2, \ldots, i_k\} = \{j_1, j_2, \ldots, j_k\}_<$ mapping j_t to i_t for each t. Let β be the number of disjoint cycles of this permutation. Then the weight of the path is defined to be $(-1)^{k-1} x^2 d^{k-\beta}$. The first connected component in M'_1 corresponds to a permutation $\binom{1\ 2\ 3\ 4}{1\ 4\ 2\ 3} = (1)(2\,4\,3)$ whose weight is $-d^2 x^2$.
2. Type-2 path: For $i_1 < i_k$, a path $i_1 \leftarrow i_2 \leftarrow \cdots \leftarrow i_k$ in M' is a connected component in M', if $\underline{i_1}$ is matched to either i_2 or $\underline{i_2}$ in M and $\hat{\imath}_j$ is matched to $\hat{\imath}_{j+1}$ in M for $j = 2, \ldots, k-1$, where $\hat{\imath}_j$ denotes either i_j or $\underline{i_j}$. In this case we regard the path as a permutation $i_k i_{k-1} \ldots i_1$ of the set $\{i_1, i_2, \ldots, i_k\} = \{j_1, j_2, \ldots, j_k\}_<$ mapping j_t to i_{k+1-t} for each t. Let β be the number of disjoint cycles of this permutation. Then the weight of the path is defined to be $(-1)^{k-1} x^2 d^{k-\beta}$. The second connected component in M'_1 corresponds to a permutation $\binom{5\ 6\ 7}{7\ 6\ 5} = (5\,7)(6)$ whose weight is dx^2.
3. Cycle: A cycle $i_1 \to i_2 \to \cdots \to i_k \to i_1$ in M' is a connected component in M', if $\hat{\imath}_j$ is matched to $\hat{\imath}_{j+1}$ in M for $j = 1, \ldots, k$ where $\hat{\imath}_j$ denotes i_j or $\underline{i_j}$ and $i_{k+1} = i_1$. In this case we represent the cycle as $(i_1\, i_2\, \ldots\, i_k)$ and give a weight $(-1)^k d^{k-1}$. The length k may be 1. The third component in M'_1 corresponds to a permutation (in a cycle notation) $(8\,10\,9)$ whose weight is $-d^2$. The fourth component in M'_1 corresponds to a permutation (in a cycle notation) (11), a 1-cycle, whose weight is -1.
4. Point: If none of i and $\underline{i}$ is matched to any vertex in M, then i is a fixed point in M' whose weight is x^2. Note that a point is regarded as a path consisting of one point.

Odd case: If $n = 2m + 1$, then there are five kinds of components in M'.

1. Type-1 path: Same as the even case. The first connected component in M'_2 corresponds to a permutation $\binom{1\ 2\ 3}{3\ 1\ 2} = (1)(1\,3\,2)$

whose weight is $-d^2x^2$.

2. Type-2 path: Same as the even case.
3. Type-3 path: A path $0 \to i_1 \to i_2 \to \cdots \to i_k$ in M' is a connected component in M', if 0 is matched to either i_1 or $\underline{i_1}$ in M and $\hat{\imath}_j$ is matched to $\hat{\imath}_{j+1}$ in M for $j = 1, \ldots, k-1$, where $\hat{\imath}_j$ denotes either i_j or $\underline{i_j}$. In this case we regard the path as a permutation $i_1 i_2 \ldots i_k$, where the set $\{i_1, i_2, \ldots, i_k\} = \{j_1, j_2, \ldots, j_k\}_<$, mapping j_t to i_t for each t. Let β be the number of disjoint cycles of this permutation. Then the weight of the path is defined to be $(-1)^k x d^{k-\beta}$. If $k = 0$, then the path becomes a point with weight x. The first connected component in M_2' corresponds to a permutation $\binom{1\ 2\ 3}{3\ 1\ 2} = (1\,3\,2)$ whose weight is $-d^2x$.
4. Cycle: Assume $0 \notin \{i_1, i_2, \ldots, i_k\}$. A cycle $i_1 \to i_2 \to \cdots \to i_k \to i_1$ in M' is a connected component in M', if $\hat{\imath}_j$ is matched to $\hat{\imath}_{j+1}$ in M for $j = 1, \ldots, k$ where $\hat{\imath}_j$ denotes i_j or $\underline{i_j}$ and $i_{k+1} = i_1$. A cycle of this type is given a weight $(-1)^k d^k$. The length k may be 1. Note that the weight of a cycle is different from that in the even case. The third component in M_2' corresponds to a permutation (in a cycle notation) $(7\,9\,8)$ whose weight is $-d^3$. The fourth component in M_2' corresponds to a permutation (in a cycle notation) (10), a 1-cycle, whose weight is $-d$.
5. Point: If $i > 0$ and none of i and $\underline{i}$ is matched to any vertex in M, then i is a fixed point in M' whose weight is x^2. Note that a point is regarded as a path consisting of one point. If 0 is not matched to any vertex in M, then 0 is a fixed point in M' whose weight is x.

The weight $\omega(M)$ of M is defined as the product of weights of cycles and paths in M'. The weight of M_1 in Figure 1 is $(-d^2x^2)(dx^2)(-d^2)(-1)(x^2) = -d^5x^6$ and the weight of M_2 in Figure 2 is $(-d^2x)(dx^2)(-d^2)(-1)(x^2) = -d^5x^5$. We can show that $R_n(x,d)$ is the weight generating function of the set of all matchings of K_n.

THEOREM 3.1. *$R_n(x,d)$ defined in the equations (1)–(2) is the weight of the set of all matchings of K_n, i.e.*

$$R_n(x,d) = \sum_M \omega(M),$$

where M is a matching in K_n and its weight ω is defined as the product of weights of cycles and paths in M'.

PROOF. We use the interpretation of $Y_n^{(a)}(x,d)$ in [6, Definition 2.8 and Theorem 2.9]. We explain the case where n is odd. Even case is similar.

Equation (2) can be written as

$$R_{2n+1}(x,d) = (-1)^n \sum_{r=0}^{n} \frac{2^{n-r}x^{2r+1}}{r!} \sum_{s=0}^{r} (-1)^s \binom{r}{s} (s+1+\frac{d}{2};d)_n.$$

The expression

$$\frac{(-1)^n}{r!} \sum_{s=0}^{r} (-1)^s \binom{r}{s} (s+1+\frac{d}{2};d)_n \tag{5}$$

can be interpreted as the weight generating funtion of some 'partitioned permutation'. We define 'partitioned partition' as follows [6, Definition 2.8]:

A permutation σ is a partitioned permutation if its cycles are divided into several blocks, at most one of which is empty. We distinguish a block, called the principal block, from other blocks, called secondary blocks. Secondary blocks are assumed to be indistinguishable, i.e. they are distinguished by their contents only. Assume that only the principal block may be empty but none of the secondary blocks can be empty.

The weight of a partitioned permutation σ is $(-1)^k a^l d^{n-\beta}$, where k denotes the number of secondary blocks and l denotes the number of cycles in the principal block and β denotes the number of cycles in all blocks. (Figure 3)

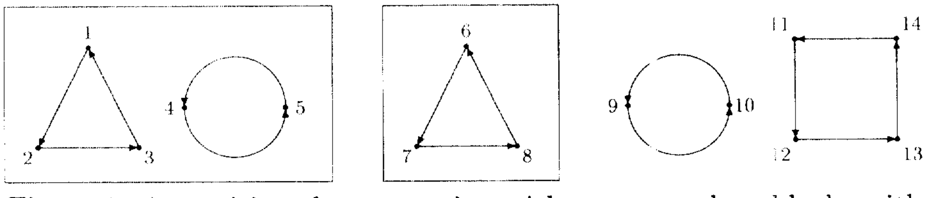

Figure 3: A partitioned permutation with two secondary blocks with weight $(-1)^2a^2d^9$. The principal block consists of last two cycles.

The expression (5) is the weight generating function of all partitioned permutations σ of S_n with one principal block and r secondary blocks; the weight of σ is $(-1)^{n-r}a^l d^{n-\beta}$ where $a = 1 + d/2$ and l is the number of cycles in the principal block and β is the number of cycles in all blocks [6, Theorem 2.9]. Note that the principal block may be empty but none of the secondary blocks is empty. Since a is the sum of 1 and $d/2$, we may regard the expression as the weight generating function of all partitioned permutations σ of S_n into two distinct principal blocks, called B_1 and B_2, and r secondary blocks; the weight of σ is $(-1)^{n-r}(d/2)^{l_1} d^{n-\beta}$ where l_1 is the number of cycles in the principal block B_1 and β is the number of cycles in all blocks. We will associate to each partitioned partition an M' defined earlier. Each cycle in the principal block B_1 corresponds to a cycle in M'; the cycles in each secondary block form a path of type-1 or type-2; the cycles in the principal block B_2 form a path of type-3. If a partition permutation has r secondary blocks, then the corresponding M' has r paths of type-1 or type-2. Note that a cycle of length k in the block B_1 has weight $d^k/2$ but the corresponding cycle in M' has weight d^k. So multiplying the expression $(-1)^{n-r}(d/2)^{l_1} d^{n-\beta}$ by 2^{n-r} corresponds to multiplying M' by 2^{n-r-l_1}. It suffices to show that if M' has weight $(-1)^{n-r} d^{l_1} d^{n-\beta}$, then there are exactly 2^{n-r-l_1} matchings M which reduce to the same M'. This can be done componentwise:

1. For each path $i_1 \to i_2 \to \cdots \to i_k$ of type-1 in M', there are 2^{k-1} different choices in M, corresponding to the path, since each arrow $i_j \to i_{j+1}$ in M' has two possibilities $\hat{\imath}_j \to i_{j+1}$ and $\hat{\imath}_j \to \underline{i_{j+1}}$ for $j = 1, 2, \ldots, k-1$ where $(\hat{\imath}_1 \to \hat{\imath}_2) = (i_1 \to \hat{\imath}_2)$. If the path is of type-2, i.e. $i_1 > i_k$, then each arrow $i_j \to i_{j+1}$ in M' has two possibilities $i_j \to \hat{\imath}_{j+1}$ and $\underline{i_j} \to \hat{\imath}_{j+1}$ for $j = 1, 2, \ldots, k-1$ where $(\hat{\imath}_{k-1} \to \hat{\imath}_k) = (\hat{\imath}_{k-1} \to \underline{i_k})$.
2. For each path $0 = i_0 \to i_1 \to i_2 \to \cdots \to i_k$ in M', there are 2^k different choices in M, corresponding to the the path, since each arrow $i_j \to i_{j+1}$ in M' has two possibilities $\hat{\imath}_j \to i_{j+1}$ and $\hat{\imath}_j \to \underline{i_{j+1}}$ for $j = 0, 1, \ldots, k-1$.
3. For each cycle $i_1 \to i_2 \to \cdots \to i_k \to i_1$ in M', there are 2^{k-1} different choices in M, corresponding to the cycle, since each arrow $i_j \to i_{j+1}$ in M' has two possibilities $\hat{\imath}_j \to i_{j+1}$ and $\hat{\imath}_j \to \underline{i_{j+1}}$ for $j = 1, 2, \ldots, k-1$ where $(\hat{\imath}_1 \to \hat{\imath}_2) = (i_1 \to \hat{\imath}_2)$.

Since there are r paths of type-1 or type-2 and l_1 cycles in M', it follows that if M' has weight $(-1)^r d^{l_1} d^{n-\beta}$, then there are exactly 2^{n-r-l_1} matchings M which reduce to the same M'.

Since each path of type-1 or of type-2 contributes x^2 and the path of type-3 contributes x to the weight,

$$\sum_M \omega(M) = (-1)^n \frac{2^{n-r} x^{2r+1}}{r!} \sum_{s=0}^{r} (-1)^s \binom{r}{s} (s+1+\frac{d}{2}; d)_n,$$

where M is a matching in K_{2n+1} whose reduction M' has r paths of type-1 and type-2. By summing for $r = 0, 1, \ldots, n$, we get

$$\begin{aligned} R_{2n+1}(x,d) =&(-1)^n \sum_{r=0}^{n} \frac{2^{n-r} x^{2r+1}}{r!} \sum_{s=0}^{r} (-1)^s \binom{r}{s} (s+1+\frac{d}{2}; d)_n \\ =& \sum_M \omega(M), \end{aligned}$$

where M is a matching in K_{2n+1}.

4. Remark

The other polynomial $S_n(x,d)$ can be interpreted as a weight of some kind of matchings in a complete graph with dn vertices. The description of these objects is complicated. A 'simple' combinatorial proof of the orthogonality might be possible as well.

We now state some facts for the case $d = 3$. Let $d = 3$ and let $R_n^{(3)}(x) = R_n(x,3)$ for a typographical purpose. Then the following recurrence relation holds:

$$\begin{aligned} R_{n+3}^{(3)}(x) =& x^3 R_n^{(3)}(x) + (3n+3+(-1)^n) R_{n+1}^{(3)}(x) \\ &+ 3n(3n - 3/2 + (-1)^n/2) R_{n-1}^{(3)}(x). \end{aligned}$$

The orthogonality is

$$L(R_m(x,3), S_n(x,3)) = \begin{cases} 0, & \text{if } m \neq n \\ \frac{(3n)!}{(2;6)_{\lfloor \frac{n+1}{2} \rfloor} (4;6)_{\lfloor \frac{n}{2} \rfloor}}, & \text{if } m = n. \end{cases}$$

Note that d must be odd for $R_n(x,d)$ and $S_n(x,d)$ to be a set of biorthogonal polynomials. If d is even, then the resulting polynomials are quasi-orthogonal. For instance, if $d = 2$ then the polynomials are quasi-orthogonal and

$$L(R_m(x,2), S_n(x,2)) = \begin{cases} 0, & \text{if } m \neq n \text{ or } n \text{ is odd,} \\ \frac{(2n)!}{(2;4)_{n/2}(3;4)_{n/2}}, & \text{if } m = n \text{ and } n \text{ is even.} \end{cases}$$

A q-analogue of Hermite polynomials and its combinatorics have been studied in [4]. Any q-analogue of $R_n(x,d)$ and $S_n(x,d)$ and their combinatorics will be very interesting.

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