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A DIFFERENT PROOF OF THE EXISTENCE THEOREM ON THE NAVIER–STOKES EQUATIONS

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1. Introduction

The motions of homogeneous incompressible fluid flows in $\mathbf{R}^3$ are governed by the Navier-Stokes equations

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v = -\nabla p + \nu \Delta v + f \qquad \text{in } \mathbf{R}^3 \times \mathbf{R}_+ \tag{1}$$

$$\text{div } v = 0 \qquad \text{in } \mathbf{R}^3 \times \mathbf{R}_+ \tag{2}$$

$$v(\cdot, 0) = v_0 \qquad \text{in } \mathbf{R}^3 \tag{3}$$

Here $v = (v_1(x,t), v_2(x,t), v_3(x,t))$ is the velocity of the fluid flow, $p = p(x,t)$ is the scalar pressure, $f = (f_1(x,t), f_2(x,t), f_3(x,t))$ is the given external force on the fluid, $\nu > 0$ is the given kinematic viscosity, and v_0 is the initial velocity satisfying div $v_0 = 0$. We are concerned constructing a weak solution of the Navier-Stokes equations in the following formulation; by a weak solution of the Navier-Stokes equations with initial data $v_0 \in L^2(\mathbf{R}^3)$ we mean a vector field $v \in L^\infty(0, T; L^2(\mathbf{R}^3))$ satisfying the followings:

$$\int_{\mathbf{R}^3} \phi(x,0)v_0(x)dx + \int_0^T \int_{\mathbf{R}^3} (\phi_t \cdot v + \nabla\phi : v \otimes v + \nu\Delta\phi \cdot v)dxdt = \\ = -\int_0^T f \cdot \phi dxdt$$

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(4) $\forall\phi = (\phi_1, \phi_2, \phi_3) \in [C_0^\infty(\mathbf{R}^3 \times [0,T))]^3$ with $\text{div}\ \phi = 0$

(5) $$\int_0^T \int_{\mathbf{R}^3} \nabla\psi \cdot v\ dxdt = 0 \quad \forall\psi \in C_0^\infty(\mathbf{R}^3 \times (0,T))$$

where $v \otimes v = (v_i v_j)$, $\nabla\phi = (\frac{\partial\phi_i}{\partial x_j})$, and $A : B = \sum_{i,j=1}^3 A_{ij}B_{ij}$.The above definitions are obtained by multiplying test functions ϕ, ψ to (1), (2) respectively, and integrating by parts.

As is well-known the existence of weak solutions of the Navier-Stokes equations was established by J. Jeray[3] and E. Hopf[2](See also [1],or [5].). Their constructions are essentially based on a topological method, or a standard projection methods(Galerkin approximation scheme). In this note we construct a weak solution of the Navier-Stokes equations, using a completely different regularization method. This regularization was actually used in [4] to construct local in time smooth solution of the Navier-Stokes and the Euler equations. What we observed here is that the same regularization scheme works well in the construction of a global weak solution of the Navier-Stokes equations. Since there is no general uniqueness theorem established for weak solutions of the Navier-Stokes equations, our weak solution might turn out to be different from the previously constructed weak solutions. We are concerened here only with the 3-D case; extension of our result to the 2-D case is rather straightforward with minor modifications necessary due to the different Sobolev type imbeddings to be used in that case.

2. Preliminaries

Let ρ_ϵ be the standard mollifier in $\mathbf{R}^3$, i.e.

$$\rho_\epsilon(x) = \frac{1}{\epsilon^3}\rho(\frac{x}{\epsilon})$$

where $\rho \in C_0^\infty(\mathbf{R}^3)$, $\rho \geq 0$, supp $\rho \subset \{|x| \leq 1\}$, $\rho(x) = \rho(|x|)$, and $\int_{\mathbf{R}^3} \rho dx = 1$. We denote

$$J_\epsilon u = \rho_\epsilon * u = \frac{1}{\epsilon^3}\int_{\mathbf{R}^3} \rho(\frac{x-y}{\epsilon})u(y)dy \quad \forall u \in L^1_{loc}(\mathbf{R}^3)$$

Let us consider the system of integro-differential equations

$$(6) \qquad v_t^\epsilon + J_\epsilon((J_\epsilon v^\epsilon) \cdot \nabla(J_\epsilon v^\epsilon)) = -\nabla p^\epsilon + \nu J_\epsilon(\Delta J_\epsilon v^\epsilon) + J_\epsilon f$$

$$(7) \qquad \text{div } v^\epsilon = 0$$

$$(8) \qquad v^\epsilon|_{t=0} = v_0^\epsilon$$

For this system of equations we have the following:

LEMMA 1. *Given $v_0^\epsilon \in H^m(\mathbf{R}^3)$, div $v_0^\epsilon = 0$, $m > \frac{3}{2}$, and $f \in L^2(0,T; L^{\frac{6}{5}}(\mathbf{R}^3))$, then for any $\epsilon > 0$ the unique solution $v^\epsilon \in C^1([0,T]; H^m(\mathbf{R}^3))$ of the system (6)-(8) exists for any $T \in (0,\infty)$.*

REMARK. In the above lemma it is understood that the scalar field p^ϵ in (6) is determined by solving the Poisson equation

$$\Delta p^\epsilon = -\text{div } J_\epsilon((J_\epsilon v^\epsilon) \cdot \nabla(J_\epsilon v^\epsilon)) + \text{div } J_\epsilon f$$

once v^ϵ is obtained.

The proof of Lemma 1 in the case $f \equiv 0$ is given in [4]. The obvious modifications of the proof in [4] for our case of $f \neq 0$ provides us that of Lemma 1. (In particular an establishement of the uniform energy estimate

$$\sup_{0 \le t \le T} \|v^\epsilon(t)\| \le C(v_0, f, T)$$

can be obtained from (12) in the proof of Main Theorem in the next section.)

3. Main Result

We recall the following compactness lemma

LEMMA 2. *Let X_1, X_0, X_{-1} be Banach spaces with the compact imbedding $X_1 \to X_0$ and the continuous imbedding $X_0 \to X_{-1}$. Suppose $1 < p_1 < \infty$, $1 < p_2 < \infty$. Let $\{v^\epsilon\}$ be a bounded sequence in $L^{p_1}(0,T;X_1)$. Assume that $\{\frac{dv^\epsilon}{dt}\}$ is bounded in $L^{p_2}(0,T;X_{-1})$. Then, there exists a subsequence $\{v^{\epsilon_j}\}$ of $\{v^\epsilon\}$ and the limit v in $L^{p_1}(0,T;X_0)$.*

For the proof of the above Lemma see pp.69 of [1]. We now state and prove our main Theorem.

THEOREM 1. *Suppose $v_0 \in L^2(\mathbf{R}^3)$, div $v_0 = 0$.*

Let $f \in L^2(0,T; L^{\frac{6}{5}}(\mathbf{R}^3))$. Then, there exists a weak solution of the Navier-Stokes equations on $\mathbf{R}^3 \times (0,T)$ with initial data v_0. This solution v satisfies the followings:

$$v \in L^\infty(0,T; L^2(\mathbf{R}^3)) \cap L^2(0,T; H^1(\mathbf{R}^3)), \tag{9}$$

and

$$\|v(t)\|_2^2 + \nu \int_0^t \|\nabla v(s)\|^2 ds \le \|v_0\|_2^2 + \frac{C}{\nu}\int_0^t \|f(s)\|_{\frac{6}{5}}^2 ds \tag{10}$$

for almost every $t \in [0,T]$, where C is an absolute constant.

Proof. Let $v^\epsilon(t) \in H^m$, $t > 0$ be the global smooth solution of the system (6)-(8) associated with the initial data $v_0^\epsilon = J_\epsilon v_0$ constructed in Lemma 1. We firstly prove the uniform energy inequality for $J_\epsilon v^\epsilon$. We take the scalar product (6) with v^ϵ in $L^2(\mathbf{R}^3)$, and, integrating by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\|v^\epsilon(t)\|_2^2 + \nu\|\nabla J_\epsilon v^\epsilon(t)\|_2^2 \le |(J_\epsilon f, v^\epsilon)| \tag{11}$$

where we used in particular

$$\begin{aligned}(J_\epsilon(J_\epsilon v^\epsilon \cdot \nabla)J_\epsilon v^\epsilon, v^\epsilon) &= ((J_\epsilon v^\epsilon \cdot \nabla)J_\epsilon v^\epsilon, J_\epsilon v^\epsilon) = \\ &= \frac{1}{2}\int_{\mathbf{R}^3} J_\epsilon v^\epsilon \cdot \nabla |J_\epsilon v^\epsilon|^2 dx = 0\end{aligned}$$

due to div $v^\epsilon = 0$. We estimate, using the Hölder and the Sobolev inequalities,

$$\begin{aligned}|(J_\epsilon f, v^\epsilon)| &\le \|f\|_{\frac{6}{5}}\|J_\epsilon v^\epsilon\|_6 \le C\|f\|_{\frac{6}{5}}\|\nabla J_\epsilon v^\epsilon\|_2 \\ &\le \frac{\nu}{2}\|\nabla J_\epsilon v^\epsilon\|_2^2 + \frac{C}{\nu}\|f\|_{\frac{6}{5}}^2\end{aligned}$$

(In the above and hereafter we use the same notation C for the constants appearing in the inequalities.) This, combined with (11), provides us

$$\frac{1}{2}\frac{d}{dt}\|v^\epsilon(t)\|_2^2 + \frac{\nu}{2}\|\nabla J_\epsilon v^\epsilon\|_2^2 \le \frac{C}{\nu}\|f\|_{\frac{6}{5}}^2 \tag{12}$$

Integrating both sides of (12) over $[0,t]$, we have

$$(13) \quad \|J_\epsilon v^\epsilon(t)\|_2^2 + \nu \int_0^t \|\nabla J_\epsilon v^\epsilon(s)\|_2^2 ds \le \|v_0\|_0^2 + \frac{C}{\nu} \int_0^t \|f(s)\|_{\frac{6}{5}}^2 ds$$

where we used $\|J_\epsilon v^\epsilon(t)\|_2 \le \|v^\epsilon(t)\|_2$ for the first term of the left hand side. This energy inequality implies that the sequence $\{J_\epsilon v^\epsilon\}$ is uniformy bounded both in $L^\infty(0,T;L^2(\mathbf{R}^3))$ and in $L^2(0,T;H^1(\mathbf{R}^3))$.

We now estimate $\|(J_\epsilon v^\epsilon)_t\|_{H^{-1}}$. Operating both sides of (6) by J_ϵ we have

$$(14) \quad (J_\epsilon v^\epsilon)_t = -J_\epsilon^2\{(J_\epsilon v^\epsilon)\cdot \nabla J_\epsilon v^\epsilon\} - J_\epsilon \nabla p^\epsilon + \nu J_\epsilon^2 \Delta J_\epsilon v^\epsilon + J_\epsilon^2 f$$

where $J_\epsilon^2(\cdot)$ denotes $J_\epsilon(J_\epsilon(\cdot))$. We estimate $H^{-1}-$ norm for each term of the right hand side of (14). Let $\phi \in C_0^\infty(\mathbf{R}^3)$. Then, integratiing by parts, and using the Hölder inequality and the Gagliardo-Nirenberg inequality, we obtain that

$$(15) \quad \begin{aligned} &\left|\int_{\mathbf{R}^3} \phi J_\epsilon^2\{(J_\epsilon v^\epsilon)\cdot\nabla J_\epsilon v^\epsilon\}dx\right| = \left|\int_{\mathbf{R}^3} J_\epsilon^2\phi(J_\epsilon v^\epsilon)\cdot\nabla J_\epsilon v^\epsilon dx\right| \\ \le& \|J_\epsilon^2\nabla\phi\|_2\|J_\epsilon v^\epsilon\|_4^2 \le \|\nabla\phi\|_2\|J_\epsilon v^\epsilon\|_2^{\frac{1}{2}}\|\nabla J_\epsilon v^\epsilon\|_2^{\frac{3}{2}} \\ \le& \|\nabla\phi\|_2\|v_0\|_2^{\frac{1}{2}}\|\nabla J_\epsilon v^\epsilon\|_2^{\frac{3}{2}} \end{aligned}$$

Using the standard density argument for $H^1(\mathbf{R}^3)$ we obtain

$$(16) \quad \|J_\epsilon^2\{(J_\epsilon v^\epsilon)\cdot\nabla J_\epsilon v^\epsilon\}\|_{H^{-1}} \le \|v_0\|_2^{\frac{1}{2}}\|\nabla J_\epsilon v^\epsilon\|_2^{\frac{3}{2}}$$

To estimate $\|J_\epsilon\nabla p^\epsilon\|_{H^{-1}}$ we take div operation on the both sides of (6) to obtain

$$(17) \quad \Delta J_\epsilon p^\epsilon = -\text{div } J_\epsilon^2\{(J_\epsilon v^\epsilon\cdot\nabla)J_\epsilon v^\epsilon\} + \text{div } J_\epsilon^2 f$$

Thus we have

$$(18) \quad J_\epsilon\nabla p^\epsilon = -\nabla\Delta^{-1}\text{div } J_\epsilon^2\{(J_\epsilon v^\epsilon\cdot\nabla)J_\epsilon v^\epsilon\} + \nabla\Delta^{-1}\text{div } J_\epsilon^2 f$$

Here the notation Δ^{-1} was used in the following sense: Let $P(D)$ be any homogeneous differential operator with constant coefficients, and let φ be any tempered distribution. Then, we define

$$P(D)\Delta^{-1}\varphi = -\mathcal{F}^{-1}\left\{\frac{P(i\xi)}{|\xi|^2}\hat{\varphi}(\xi)\right\}$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform.

Using the Planchard identity we have

$$\|J_\epsilon \nabla p^\epsilon\|_{H^{-1}} \leq \|J_\epsilon^2\{(J_\epsilon v^\epsilon \cdot \nabla) J_\epsilon v^\epsilon\}\|_{H^{-1}} + \|J_\epsilon^2 f\|_{H^{-1}} \tag{19}$$

Thus, using the esimate (16), we have

$$\|J_\epsilon \nabla p^\epsilon\|_{H^{-1}} \leq \|v_0\|_2^{\frac{1}{2}} \|\nabla J_\epsilon v^\epsilon\|_2^{\frac{3}{2}} + \|J_\epsilon^2 f\|_{H^{-1}} \tag{20}$$

To estimate $\|J_\epsilon^2 \Delta(J_\epsilon v^\epsilon)\|_{H^{-1}}$ we observe

$$\begin{aligned}\left|\int_{\mathbf{R}^3} \phi J_\epsilon^2 \Delta(J_\epsilon v^\epsilon) dx\right| &\leq \int_{\mathbf{R}^3} |\nabla(J_\epsilon^2 \phi)||\nabla J_\epsilon v^\epsilon| dx \\ &\leq \|\nabla(J_\epsilon^2 \phi)\|_2 \; \|\nabla J_\epsilon v^\epsilon\|_2 \leq \|\nabla \phi\|_2 \; \|\nabla J_\epsilon v^\epsilon\|_2\end{aligned}$$

where we used

$$\|\nabla(J_\epsilon^2 \phi)\|_2 = \|J_\epsilon^2 \nabla \phi\|_2 \leq \|\nabla \phi\|_2$$

Thus, using the density argument as before,

$$\|J_\epsilon^2 \Delta(J_\epsilon v^\epsilon)\|_{H^{-1}} \leq \|\nabla J_\epsilon v^\epsilon\|_2 \tag{21}$$

We finally estimate $\|J_\epsilon^2 f\|_{H^{-1}}$. The Sobolev inequality provides us

$$\left|\int_{\mathbf{R}^3} \phi J_\epsilon^2 f dx\right| \leq \|\phi\|_6 \|J_\epsilon^2 f\|_{\frac{6}{5}} \leq C\|\nabla \phi\|_2 \|f\|_{\frac{6}{5}} \tag{22}$$

Thus,

$$\|J_\epsilon^2 f\|_{H^{-1}} \leq C\|f\|_{\frac{6}{5}} \tag{23}$$

Combining the results (16),(19),(21) and (23), we have

$$\|(J_\epsilon v^\epsilon)_t\|_{H^{-1}} \leq C\|v_0\|_2^{\frac{1}{2}} \|\nabla J_\epsilon v^\epsilon\|_2^{\frac{3}{2}} + C\nu\|\nabla J_\epsilon v^\epsilon\|_2 + C\|f\|_{\frac{6}{5}}$$

We thus obtain

$$
\begin{aligned}
&\int_0^T \|(J_\epsilon v^\epsilon)_t\|_{H^{-1}}^{\frac{4}{3}} dt \\
\leq & C\|v_0\|_2^{\frac{2}{3}} \int_0^T \|\nabla J_\epsilon v^\epsilon\|_2^2 dt + C\nu^{\frac{4}{3}} \int_0^T \|\nabla J_\epsilon v^\epsilon\|_2^{\frac{4}{3}} dt + C \int_0^T \|f\|_{\frac{6}{5}}^{\frac{4}{3}} dt \\
\leq & C\|v_0\|_2^{\frac{2}{3}} \int_0^T \|\nabla J_\epsilon v^\epsilon\|_2^2 dt + C\nu^{\frac{4}{3}} T^{\frac{1}{3}} \left(\int_0^T \|\nabla J_\epsilon v^\epsilon\|_2^2 dt \right)^{\frac{2}{3}} \\
& + CT^{\frac{1}{3}} \left(\int_0^T \|f\|_{\frac{6}{5}}^2 dt \right)^{\frac{2}{3}} \\
\leq & C\|v_0\|_2^{\frac{2}{3}} \left(\frac{\|v_0\|_2^2}{\nu} + \frac{1}{\nu} \int_0^T \|f\|_{\frac{6}{5}}^2 dt \right) + CT^{\frac{1}{3}} \left(\nu\|v_0\|_2^2 + \int_0^T \|f\|_{\frac{6}{5}}^2 dt \right)^{\frac{2}{3}} \\
& + CT^{\frac{1}{3}} \left(\int_0^T \|f\|_{\frac{6}{5}}^2 dt \right)^{\frac{2}{3}}
\end{aligned}
$$

Thus the sequence $\{(J_\epsilon v^\epsilon)_t\}$ is also uniformly bounded in $L^{\frac{4}{3}}(0,T;H^{-1}(\mathbf{R}^3))$. Due to the Banach-Alaoglu theorem and Lemma 2 we have a subsequence, $\{J_\epsilon v^\epsilon\}$, which we labeled by the same ϵ, and the limit v having the following properties

(24)

$J_\epsilon v^\epsilon \to v$ weak-* in $L^\infty(0,T;L^2(\mathbf{R}^3))$ and weakly $L^2(0,T;H^1(\mathbf{R}^3))$

and, for any fixed $R > 0$

$$J_\epsilon v^\epsilon \to v \text{ stronly in } L^2(0,T;L^2(B_R)) \tag{25}$$

where $B_R = \{x \in \mathbf{R}^3, |x| \leq R\}$. The energy inequality (10) follows from (13), combined with the convergence (24). By the weak lower semicontinuity of the norms. We now show that the limit v satisfies (4),(5),i.e. v is actually a weak solution of the Navier-Stokes equations in $\mathbf{R}^3$. (5) is immediate from (24). Below we prove (4) for the v. Let $\phi \in C_0^\infty(\mathbf{R}^3)^3$, div $\phi = 0$ be given. We choose R so that

$$\cup_{0\leq t\leq T} \text{supp } \phi(\cdot,t) \subset B_R$$

We multiply both sides of (6) by $J_\epsilon \phi$ and integrate over $\mathbf{R}^3 \times [0,T)$, and integrate by parts to obtain

$$\int_0^T \int_{B_R} \phi_t \cdot J_\epsilon v^\epsilon dxdt + \int_0^T \int_{B_R} J_\epsilon^2(\nabla\phi) : (J_\epsilon v^\epsilon) \otimes (J_\epsilon v^\epsilon) dxdt + \\ + \int_{B_R} J_\epsilon(\phi(\cdot,0)) \cdot v_0 dx$$

$$= \nu \int_0^T \int_{B_R} J_\epsilon^2(\Delta\phi) \cdot (J_\epsilon v^\epsilon) dxdt - \int_0^T \int_{B_R} J_\epsilon^2 \phi \cdot f dxdt \tag{26}$$

Since

$$\begin{aligned} &\|J_\epsilon^2\phi - \phi\|_{L^\infty(0,T;L^\infty(B_R))} \\ &\le \|J_\epsilon^2\phi - J_\epsilon\phi\|_{L^\infty(0,T;L^\infty(B_R))} + \|J_\epsilon\phi - \phi\|_{L^\infty(0,T;L^\infty(B_R))} \\ &\le 2\|J_\epsilon\phi - \phi\|_{L^\infty(0,T;L^\infty(B_R))} \to 0, \end{aligned}$$

$$\begin{aligned} &\|(J_\epsilon v^\epsilon) \otimes (J_\epsilon v^\epsilon) - v \otimes v\|_{L^1(0,T;L^1(B_R))} \\ &\le (\|J_\epsilon v^\epsilon\|_{L^2(0,T;L^2(B_R))} \\ &\quad + \|v\|_{L^2(0,T;L^2(B_R))})\|J_\epsilon v^\epsilon - v\|_{L^2(0,T;L^2(B_R))} \\ &\le C(T)\|v_0\|_2 \|J_\epsilon v^\epsilon - v\|_{L^2(0,T;L^2(B_R))} \to 0, \end{aligned}$$

and, finally

$$\begin{aligned} &\|J_\epsilon^2\phi \cdot f - \phi \cdot f\|_{L^1(0,T;L^1(B_R))} \\ &\le \|J_\epsilon^2\phi - \phi\|_{L^2(0,T;L^6(B_R))} \|f\|_{L^2(0,T;L^{\frac{6}{5}}(B_R))} \to 0 \end{aligned}$$

as $\epsilon \to 0$, we can pass to limit $\epsilon \to 0$ for each term of (26) to obtain (4) for the limit v of $\{J_\epsilon v^\epsilon\}$. This completes the proof of Theorem 1.

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