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SCALAR CURVATURE, SIGMA CONSTANT AND THEIR RELATION WITH MINIMAL VOLUME

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0. Introduction

In Riemannian geometry, it is one of the important things to study the topological or geometric invariants of manifolds since they characterize the topology of manifolds.

In the early 80's, M. Gromov ([Gr]) introduced some invariants, so called *minimal volume* and *simplicial volume* and proved several important properties. Recently, Besson, Courtois and Gallot ([B-C-G]) proved Gromov's conjecture about the minimal volume, which is remarkable (See Theorem 1.4).

On the other hand, one can find a geometric invariant in [Sc2] and [Ko], called *sigma constant* (I don't know who first introduced or considered this invariant).

In this note, we will introduce some other characterizations and give elementary relation between these invariants.

1. Preliminaries and known results

In 1960, H. Yamabe ([Ya]) proposed and attempted to solve the following problem using techniques of the calculus of variations and the elliptic partial differential equations.

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THE YAMABE PROBLEM. *Given a compact Riemannian manifold (M,g) of dimension $n \geq 3$, find a metric conformal to g with constant scalar curvature.*

Suppose (M,g) is a compact Riemannian manifold of dimension $n \geq 3$. Any metric conformal to g can be written $\tilde{g} = u^{\frac{4}{n-2}} g$, where u is a positive function on M. If s_g and $\tilde{s}$ denote the scalar curvatures of g and $\tilde{g}$, respectively, they satisfy the transformation law:

$$\frac{4(n-1)}{n-2}\Delta u - s_g u + \tilde{s} u^{\frac{n+2}{n-2}} = 0, \tag{1}$$

where Δ is the Laplacian in the metric g. Thus $\tilde{g} = u^{\frac{4}{n-2}} g$ has a constant scalar curvature $\tilde{s}$ if and only if u satisfies the *Yamabe equation*:

$$Lu = -\frac{4(n-1)}{n-2}\Delta u + s_g u = \tilde{s} u^{\frac{n+2}{n-2}}. \tag{2}$$

This is a sort of nonlinear eigenvalue problem. Yamabe observed that the equation (1) or (2) is the Euler-Lagrange equation for the following functional restricted to conformal classes

$$\mathcal{S}(g) = \frac{\int_M s_g \, dv_g}{\left(\int_M dv_g\right)^{\frac{n-2}{n}}},$$

where dv_g is the volume element determined by the metric g. Thus, to solve the Yamabe equation is equivalent to find a minimum of the above functional, namely

$$\mu(M,[g]) = \inf_{g \in [g]} \frac{\int_M s_g \, dv_g}{\left(\int_M dv_g\right)^{\frac{n-2}{n}}}, \tag{3}$$

called the *Yamabe constant*, where $[g]$ denotes the conformal class containing g. Hence, by viewing the equations (1) and (3), we have

$$\mu(M,[g]) = \inf_{\phi \in C^\infty(M), \phi > 0} \frac{\frac{4(n-1)}{n-2}\int_M |d\phi|^2 \, dv_g + \int_M s_g \phi^2 \, dv_g}{\left(\int_M \phi^{\frac{2n}{n-2}} \, dv_g\right)^{\frac{n-2}{n}}}.$$

It is known by contribution of several people ([Au], [Sc1], [Tr], [Ya]) that the Yamabe problem is always solvable. In other words, for a given constant $\tilde{s}$, there always exists a solution $u > 0$ satisfying the equation (1) or (2). In fact, for a given conformal class $\mathcal{C}$, there exists a metric $g \in \mathcal{C}$ such that $s_g = \mu(M,\mathcal{C})vol(M,g)^{-2/n}$. We call such a metric a *Yamabe metric.* Now by using the standard minimax procedure, we can take the supremum. Namely, we get a geometric invariant $\sigma(M)$ of a smooth closed manifold M, which is called the *sigma constant* of M, defined as the supremum of $\mu(M,\mathcal{C})$ over all conformal classes $\mathcal{C}$ of Riemannian metrics on M,

$$\sigma(M) = \sup_{\mathcal{C}} \mu(M,\mathcal{C}).$$

Usually it is difficult to compute the sigma constant. There are very few manifolds on which the sigma constant is known. In fact, the only ones known to the author are the following.

THEOREM 1.1 ([AU]). *$\sigma(S^n) = n(n-1)vol(S^n(1))^{2/n}$, and for any compact manifold M, $\sigma(M) \leq \sigma(S^n)$.*

THEOREM 1.2 ([KO], [SC2]).

$$\sigma(S^{n-1} \times S^1) = \sigma(S^n), \qquad \sigma(\#_k(S^{n-1} \times S^1)) = \sigma(S^n).$$

O. Kobayashi has proved the following theorem which is a little more general situation.

THEOREM 1.3 ([KO]). (a) *If M_1 and M_2 are compact manifolds of the same dimension ≥ 3, then*

$$\sigma(M_1 \# M_2) \geq \begin{cases} -(|\sigma(M_1)|^{n/2} + |\sigma(M_2)|^{n/2})^{2/n} \\ \qquad \text{if} \sigma(M_1) \leq 0 \ \text{ and } \ \sigma(M_2) \leq 0; \\ \min\{\sigma(M_1), \sigma(M_2)\} \qquad \text{otherwise} \end{cases}$$

(b) *If M is an S^{n-1} bundle over S^1 with $n \geq 3$, then $\sigma(M) = \sigma(S^n)$.*

On the other hand, in [Gr], M. Gromov introduced the following geometric invariant for a smooth manifold M, called the *minimal volume* of M,

$$Minvol(M) = \inf\{Vol(M,g) : |K(g)| \leq 1\},$$

where g is a Riemannian metric on M and $K(g)$ denotes the sectional curvature of the metric g.

EXAMPLE 1. If M is compact and admits a flat Riemannian metric, then we have $Minvol(M) = 0$.

Proof. Let g be a flat metric. Then any metric which is obtained from g by scaling is also flat. Note that $vol(M, \lambda^2 g) = \lambda^n vol(M, g)$, where $n = dim(M)$. Thus as $\lambda \to 0$, we get $vol(M, \lambda^2 g) \to 0$. □

EXAMPLE 2. Let M be a closed connected surface. Then

$$Minvol(M) = 2\pi|\chi(M)|.$$

In fact, by the Gauss -Bonnet theorem, for $|K(g)| \leq 1$,

$$vol(M, g) \geq \int_M |K(M, g)|\, dv_g \geq |\int_M K\, dv_g| = 2\pi|\chi(M)|$$

with equality for K constant, 1 or -1. In particular, metrics g of constant curvature ± 1 are extremal: $vol(M, g) = Minvol(M)$, while the torus and the Klein bottle, which have $\chi = 0$, carry no extremal metrics since their minimal volumes are zero.

EXAMPLE 3. (See [Gr]) If M admits a locally free S^1-action, then $Minvol(M) = 0$. In particular, $Minvol(M_o \times S^1) = 0$ and also $Minvol(S^3) = 0$.

Question Can one realize or compute the minimal volume ?

M. Gromov proved some important properties for the minimal volume and showed its relation with another invariant,e.g., simplicail volume (For more detail see [Gr]). Among them the most important thing is that if a compact manifold M admits a metric of negative sectional curvature, then $Minvol(M) > 0$.

In [B-C-G], Besson, Courtois and Gallot proved that in a compact hyperbolic manifold, the minimal volume is achieved by the hyperbolic metric.

THEOREM 1.4 ([B-C-G]). *Let (M, g_{-1}) be a closed hyperbolic Riemannian manifold, i.e., $K(g_{-1}) = -1$. Then $Minvol(M) = vol(M, g_{-1})$.*

In the next section, we will define some other concepts which characterize manifolds and prove elementary properties and relation with minimal volume and sigma constant.

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2. Basic results and problems

We start with some curvature functional. Let $\mathcal{M}$ be the space of all Riemannian metrics on a compact manifold M and let

$$\mathcal{M}_1 = \{g \in \mathcal{M} : vol(M, g) = 1\}.$$

For $g \in \mathcal{M}_1$, define the L^∞-*functional* by

$$\mathcal{R}^\infty(g) = |R(g)|_{L^\infty},$$

where $R(g)$ is the full Riemannian curvature tensor of the metric g and $| \ |_{L^\infty}$ denotes the sup-norm. Then consider the following quantity, called the *minimal curvature*,

$$Mincur(M) = \inf_{g \in \mathcal{M}_1} \mathcal{R}^\infty(g).$$

Note that $Mincur(M)$ is the smallest pinching of the sectional curvature among metrics of volume 1. Dually (by rescaling) this is the same problem as minimizing the volume functional on the space of metrics with $|K(g)| \leq 1$. In fact, assume $Mincur(M)$ is realized by a metric $g \in \mathcal{M}_1$, i.e., $Mincur(M) = \mathcal{R}^\infty(g) = |R(g)|_{L^\infty}$. Denoting $\lambda = \max_M |K(g)|$ and defining $\tilde{g} = \lambda \cdot g$, we have $|K(\tilde{g})| \leq 1$ and

$$vol(\tilde{g}) = \lambda^{n/2} vol(g) = \lambda^{n/2} = Minvol(M), \quad n = dim(M).$$

Suppose $vol(\tilde{g}) > Minvol(M)$. Then there is a metric g' with $|K(g')| \leq 1$ and $vol(g') < vol(\tilde{g}) = \lambda^{n/2}$. Define $h = vol(g')^{-2/n} g'$ so that $vol(h) = 1$. Then we have

$$|K(h)| = vol(g')^{2/n} |K(g')| < \lambda,$$

which is a contradiction to the fact that $Mincur(M) = \mathcal{R}^\infty(g)$.

The first observation is the following.

LEMMA 2.1. *Let M be a compact smooth manifold. Then we have*

$$Minvol(M) = 0 \qquad \text{if and only if} \qquad Mincur(M) = 0.$$

Proof. Suppose $\{g_i\}$ is a sequence of metrics such that $|K(g_i)| \leq 1$ and $vol(g_i) := v_i \to 0$. Let $h_i = v_i^{-2/n} g_i, n = dim(M)$ so that $h_i \in \mathcal{M}_1$. Then we have

$$|K(h_i)| = v_i^{2/n}|K(g_i)| \leq v_i^{2/n} \to 0.$$

So, $|R(h_i)| \to 0$ and so $Mincur(M) = 0$.

The converse is similar. □

REMARK 1. This lemma shows that

$$Minvol(M) > 0 \qquad \text{if and only if} \qquad Mincur(M) > 0.$$

LEMMA 2.2. *Let M be a smooth compact manifold. If $Minvol(M) = 0$, then the sigma constant is non-negative, i.e., $\sigma(M) \geq 0$.*

Proof. By Lemma 2.1 above, there is a sequence of metrics $\{g_i\} \subset \mathcal{M}_1$ such that $|R(g_i)|_{L^\infty} = \epsilon_i \to 0$. In particular, the scalar curvature satisfies $|s_{g_i}| \leq n(n-1)\epsilon_i$.

Consider the Sobolev quotient with respect to g_i

$$\begin{aligned}
&\frac{\frac{4(n-1)}{n-2}\int_M |d\phi|^2\, dv_{g_i} + \int_M s_{g_i}\phi^2\, dv_{g_i}}{\left(\int_M \phi^{\frac{2n}{n-2}}\, dv_{g_i}\right)^{\frac{n-2}{n}}} \\
&\geq \frac{\frac{4(n-1)}{n-2}\int_M |d\phi|^2\, dv_{g_i}}{\left(\int_M \phi^{\frac{2n}{n-2}}\, dv_{g_i}\right)^{\frac{n-2}{n}}} - n(n-1)\epsilon_i \frac{\int_M \phi^2\, dv_{g_i}}{\left(\int_M \phi^{\frac{2n}{n-2}}\, dv_{g_i}\right)^{\frac{n-2}{n}}} \\
&\geq \frac{\frac{4(n-1)}{n-2}\int_M |d\phi|^2\, dv_{g_i}}{\left(\int_M \phi^{\frac{2n}{n-2}}\, dv_{g_i}\right)^{\frac{n-2}{n}}} - n(n-1)\epsilon_i \geq -n(n-1)\epsilon_i.
\end{aligned}$$

We used the Hölder inequality in the second inequality. Hence, by definition, we get

$$\mu(M, [g_i]) \geq -n(n-1)\epsilon_i.$$

Since $\epsilon_i \to 0$, we have finally $\sigma(M) \geq 0$. □

REMARK 2. We can also show this property by using the following functional

$$\mathcal{S}^{n/2}(g) = \int_M |s_g|^{n/2}\, dv_g$$

defined in [B-C-G]. Lemma 2.2 implies that if the sigma constant is negative for a compact manifold, then the minimal volume is positive. So it is interesting to know whether there is a metric minimizing the minimal volume or not and for which manifold the sigma constant is negative. In $dim(M) = 2$, it is known by the Gauss-Bonnet theorem. So we may assume $dim(M) \geq 3$. To the author, Theorem 1.4 is the first result which gives the minimal volume of a manifold explicitly.

Now assume that a manifold M admits a polarized F-structure (For this definition, see [C-G]) which is a topological concept. In [Pa], P. Pansu showed that if M admits a polarized F-structure, then $Minvol(M) = 0$. Thus, we have

COROLLARY 2.3. *If a compact manifold M admits a polarized F-structure, then $\sigma(M) \geq 0$.*

Next we will see the relation of the minimal volume with the sigma constant. First we have

THEOREM 2.4 ([B-C-G]). *If g is a metric of negative constant scalar curvature on M and $\tilde{g}$ a metric conformal to g, then*

$$\int_M |\tilde{s}|^{n/2}\, dv_{\tilde{g}} \geq \int_M |s_g|^{n/2}\, dv_g$$

and the equality holds if and only if $g = \tilde{g}$, where $n = dim(M)$.

LEMMA 2.5. *Let M^n be a compact smooth manifold with $\sigma(M) \leq 0$. Then*

$$|\sigma(M)|^{n/2} = \inf_{g \in \mathcal{M}_1} \int_M |s_g|^{n/2}\, dv_g.$$

Proof. Let $h \in \mathcal{M}_1$ be any metric and let $g \in [h] \cap \mathcal{M}_1$ be the Yamabe metric with $s_g = \mu(M, [h]) \leq 0$. Then, from Theorem 2.4, we have

$$\begin{aligned}\int_M |s_h|^{n/2}\, dv_h &\geq \int_M |s_g|^{n/2}\, dv_g \\ &= |\mu(M, [h])|^{n/2} = |\mu(M, [g])|^{n/2}.\end{aligned}$$

Thus

$$\inf_{g\in\mathcal{M}_1}\int_M |s_g|^{n/2}\,dv_g \geq \inf_{g\in\mathcal{M}_1}|\mu(M,[g])|^{n/2} \geq |\sigma(M)|^{n/2}.$$

Now let $\{g_i\}$ be a sequence of Yamabe metrics with $vol(g_i)=1$ and $s_{g_i}=\mu(M,[g_i])\to\sigma(M)$. Then it is easy to see

$$\int_M |s_{g_i}|^{n/2}\,dv_{g_i}\to|\sigma(M)|^{n/2}.$$

□

PROPOSITION 2.6. *There exists a constant $C_n>0$, depending only on dimension n, such that for any compact smooth n-manifold M with $\sigma(M)\leq 0$,*

$$|\sigma(M)|\leq C_n Mincur(M). \tag{4}$$

Proof. This property follows from the decomposition of curvature tensor and Lemma 2.5 above. In fact, if $n\geq 4$, then we have

$$|R|^2=\frac{2}{n(n-1)}s^2+\frac{4}{n-2}|Z|^2+|W|^2$$

and if $n=3$, then

$$|R|^2=\frac{1}{3}s^2+4|Z|^2,$$

where Z is the traceless Ricci tensor, i.e., $Z=r-\frac{s}{n}g$, r is the Ricci curvature tensor, s is the scalar curvature and W is the Weyl tensor. So,

$$|s|\leq c'_n|R|.$$

□

REMARK 3. It is really interesting to find an explicit number for C_n and when the equality does hold. In fact, I don't know whether the reverse inequality in (4) holds if M admits a metric of negative sectional curvature.

Next, we will show that the minimal volume is bounded by the sigma constant for the non-positive sigma constant case.

PROPOSITION 2.7. *Let $dim(M) = n \geq 3$. If $\sigma(M) \leq 0$, then*

$$Minvol(M) \geq \left(\frac{|\sigma(M)|}{n(n-1)}\right)^{n/2}.$$

Proof. Let g be a metric with $|K(g)| \leq 1$ so that $|s_g| \leq n(n-1)$. Let $g_o \in [g]$ be a *Yamabe metric*, i.e., $s_{g_o} = \mu(M,[g])vol(g_o)^{-2/n} \leq 0$. Then Theorem 2.4 implies that

$$\begin{aligned}|\mu(M,[g])|^{n/2} = \int_M |s_{g_o}|^{n/2}\, dv_{g_o} &\leq \int_M |s_g|^{n/2}\, dv_g \\ &\leq (n(n-1))^{n/2} vol(M,g).\end{aligned}$$

Thus,

$$vol(M,g) \geq \left(\frac{|\mu(M,[g])|}{n(n-1)}\right)^{n/2} \geq \left(\frac{|\sigma(M)|}{n(n-1)}\right)^{n/2}.$$

Hence

$$Minvol(M) \geq \left(\frac{|\sigma(M)|}{n(n-1)}\right)^{n/2}.$$

□

REMARK 4. By Theorems 1.1 and Proposition 2.7, we have

$$-n(n-1)Minvol(M)^{2/n} \leq \sigma(M) \leq n(n-1)vol(S^n(1))^{2/n}.$$

Moreover, it follows from Theorem 1.4 that $Minvol(M) = vol(g_{hyp})$ for the hyperbolic metric g_{hyp}. Thus, we can conjecture that if a manifold M admits a hyperbolic metric g_{hyp}, then

$$Minvol(M) = vol(g_{hyp}) = \left(\frac{|\sigma(M)|}{n(n-1)}\right)^{n/2}.$$

In [B-C-G], Besson, Courtois and Gallot have proved a more general argument than Theorem 1.4. Namely, they showed that if (M,h) is a compact hyperbolic n-manifold($n \geq 3$), then for any Riemannian metric g with $Ric(g) = r_g \geq -(n-1)$, one has $vol(M,g) \geq vol(M,h)$ and the equality holds if and only if (M,g) is isometric to (M,h). We have a similar property for the negative sigma constant case in replacing the Ricci tensor by the scalar curvature if the sigma constant is realized by some conformal class. To do this, we need a theorem due to O. Kobayashi.

THEOREM 2.8 ([Ko]). *Suppose $dim(M) \geq 3, \sigma(M) \leq 0$ and $\sigma(M) = \mu(M,\mathcal{C})$ for some conformal class $\mathcal{C}$. Then $f \in C^\infty(M)$ is the scalar curvature of some metric g with $vol(M,g) = 1$ if and only if either $\inf_M f < \sigma(M)$ or $f = const. = \sigma(M)$.*

PROPOSITION 2.9. *Suppose $dim(M) = n \geq 3$ and $\sigma(M) < 0$. Assume that $\sigma(M) = \mu(M,\mathcal{C})$ for some conformal class $\mathcal{C}$. Then for any metric g on M*

$$s_g \geq s_{g_1} \Longrightarrow vol(M,g) \geq vol(M,g_1),$$

where $g_1 \in \mathcal{C}$ is the Yamabe metric with $|K(g_1)| \leq 1$.

Proof. Let $h \in \mathcal{C}$ be the normalized Yamabe metric, i.e., $vol(M,h) = 1$ and $s_h = \mu(M,\mathcal{C}) = \sigma(M)$. Then we have $g_1 = (\max_M |K(h)|) \cdot h$ (Note that $\max_M |K(h)| > 0$ since $\sigma(M) < 0$). So one has

$$vol(M,g_1) = \left(\max_M |K(h)|\right)^{n/2} \tag{5}$$

Now assume $s_g \geq s_{g_1}$ for a metric g. Define

$$\tilde{g} = vol(g)^{-2/n} g \qquad \text{so that} \qquad vol(\tilde{g}) = 1.$$

It follows from Theorem 2.8 that $\tilde{s} = vol(g)^{2/n} s_g = \sigma(M)$ or $\min \tilde{s} < \sigma(M)$. Note that

$$\tilde{s} = vol(g)^{2/n} s_g \geq vol(g)^{2/n} s_{g_1} = \frac{vol(g)^{2/n}}{\max_M |K(h_1)|} \sigma(M).$$

and recall that $\sigma(M) < 0$ by assumption.

If $\tilde{s} = \sigma(M)$, then we have

$$\frac{vol(g)^{2/n}}{\max_M |K(h_1)|} \sigma(M) \geq 1.$$

and if $\min_M \tilde{s} < \sigma(M)$, then

$$\frac{vol(g)^{2/n}}{\max_M |K(h_1)|} \sigma(M) > 1.$$

So in any case, we get, from (5), $vol(M,g) \geq vol(M,g_1)$ or $vol(M,g) > vol(M,g_1)$. □

REMARK 5. As mentioned above, Besson, Courtoi and Gallot ([B-C-G]) showed that if M admits a hyperbolic metric, then the minimal volume is achieved by it *among the metrics of Ricci curvature* $\geq -(n-1)$. Thus, together with the conjecture in Remark 2, Proposition 2.9 implies that the minimal volume is realized by the hyperbolic metric *among the metrics of scalar curvature* $\geq -n(n-1)$.

DEFINITION. Let (M, g) be a closed Riemannian manifold of dimension $n \geq 3$. The differential operator L_g defined by

$$L_g = -\frac{4(n-1)}{n-2}\Delta_g + s_g$$

is called the *conformal Laplacian*, where Δ_g is the Laplacian with respect to the metric g and s_g scalar curvature. Denote by $\lambda_1(g)$ the lowest eigenvalue for L_g.

Note that both L_g and $\lambda_1(g)$ depend on the metric g. Recall that the conformal invariant of $[g]$ is

$$\begin{aligned}\mu(M,[g]) &= \inf_{\phi \in C^\infty(M), \phi>0} \frac{\frac{4(n-1)}{n-2}\int_M |d\phi|^2\, dv_g + \int_M s_g \phi^2\, dv_g}{\left(\int_M \phi^{\frac{2n}{n-2}}\, dv_g\right)^{\frac{n-2}{n}}} \\ &= \inf_{\phi \in C^\infty(M), \phi>0} \frac{\langle L_g\phi, \phi\rangle}{\left(\int_M \phi^{\frac{2n}{n-2}}\, dv_g\right)^{\frac{n-2}{n}}}.\end{aligned}$$

The first observation is the following.

LEMMA 2.10. *For any conformal class* $\mathcal{C}$,

$$\mu(M, \mathcal{C}) = \inf_{g \in \mathcal{C}} \lambda_1(g) vol(M, g)^{2/n}.$$

Proof. The variational charaterization implies that for any $g \in \mathcal{C}$

$$
\begin{aligned}
\lambda_1(g) &= \inf_{\phi} \frac{\langle L_g\phi, \phi\rangle}{\int_M \phi^2\, dv_g} \\
&= \inf_{\phi\in C^\infty(M),\phi>0} \frac{\frac{4(n-1)}{n-2}\int_M |d\phi|^2\, dv_g + \int_M s_g\phi^2\, dv_g}{\left(\int_M \phi^{\frac{2n}{n-2}}\, dv_g\right)^{\frac{n-2}{n}}} \cdot \frac{\left(\int_M \phi^{\frac{2n}{n-2}}\, dv_g\right)^{\frac{n-2}{n}}}{\int_M \phi^2\, dv_g} \\
&\geq vol(M,g)^{-2/n} \inf_{\phi\in C^\infty(M),\phi>0} \frac{\frac{4(n-1)}{n-2}\int_M |d\phi|^2\, dv_g + \int_M s_g\phi^2\, dv_g}{\left(\int_M \phi^{\frac{2n}{n-2}}\, dv_g\right)^{\frac{n-2}{n}}} \\
&= vol(M,g)^{-2/n}\mu(M,\mathcal{C})
\end{aligned}
$$

where the inequality follows from the Hölder inequality.

On the other hand, the Yamabe problem is always solvable, i.e., there exists a metric $g_o \in \mathcal{C}$ such that $s_{g_o} = \mu(M,\mathcal{C})vol(M,g_o)^{-2/n}$. Then considering L_{g_o} and $\lambda_1(g_o)$, we have $\lambda_1(g_o) = s_{g_o}$ and so $\mu(M,) = \lambda_1(g_o)vol(g_o)^{-2/n}$. This completes the proof. $\square$

Next we consider $\lambda_1(g)vol(g)^{2/n}$ in the variational aspect. Define a functional $\mathcal{F}$ as

$$\mathcal{F} : \mathcal{M} \to \mathbf{R}, \qquad \mathcal{F}(g) = \lambda_1(g)vol(M,g)^{n/2},$$

where $n = dim(M)$. First, note that this is scale invariant. To compute the variational formula, we assume $vol(M,g) = 1$ and the scalar curvature s_g of g is constant. Let h be a symmetric bilinear form and consider a variation $g_t = g + th$. Then, from [K-W2], we have $\lambda' \cdot h = -\langle h, r_g\rangle$, where $'$ denotes the derivative with respect to t and $r_g = r$ is the Ricci curvature of the metric g. An easy calculation shows that

$$(6)\quad \mathcal{F}' \cdot h = vol(M,g)^{\frac{n}{2}-1}\left[\frac{1}{n}\lambda_1(g)\int_M Tr_g(h)\, dv_g - vol(M,g)\langle h, r\rangle\right],$$

where Tr_g denotes the trace with respect to the metric g. We used the fact $vol'(g) \cdot h = \frac{1}{2}\int_M Tr_g h\, dv_g$. Thus, we have

LEMMA 2.11. *If g is a critical point for $\mathcal{F}$, then g is Einstein and*

$$Ric(g) = r_g = \frac{\lambda_1(g)}{n \cdot vol(g)} g.$$

The converse is also true.

Proof. From the equation (6), we get

$$\mathcal{F}' \cdot h = 0 \Longleftrightarrow \int_M (r \cdot vol(g) - \frac{\lambda_1(g)}{n} g, h)\, dv_g = 0$$

for all h. Thus $r = \frac{\lambda_1(g)}{n \cdot vol(g)} g$ and so g is an Einstein metric.
It is easy to see the converse is also true. □

REMARK 6. The author does not know whether $\inf_{|K(g)|\leq 1} |\lambda_1(g)|$ $vol(g)^{2/n}$ is strictly positive or not if M admits a metric of negative sectional curvature. In fact, the author would like to conjecture that

$$\inf_{|K(g)|\leq 1} |\lambda_1(g)| vol(g)^{2/n} > 0$$

if M admits a metric of negative sectional curvature. If one can figure out the second variational formula for $\mathcal{F}$, it can be a help in understanding this quantity.

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