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VANISHING THEOREM ON SINGULAR MODULI SPACES

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1. Introduction

Let X be a smooth, simply connected and oriented closed four-manifold such that the dimension $b_2^+(X)$ of a maximal positive subspace for the intersection form is greater than or equal to 3. Suppose X is a connected sum $X_1 \sharp X_2$ with each $b_2^+(X_i) > 0$. Donaldson considered a sequence of connected sums

$$(X_1, g_n^1) \underset{\lambda_n}{\sharp} (X_2, g_n^2) = (X, g_{\lambda_n})$$

with a neck of radius λ_n, and studied the limiting behavior of the moduli space as $\lambda_n \to 0$. In [3] Donaldson got his celebrated theorem:

THEOREM. (Donaldson) *Suppose X is a smooth, simply connected and oriented closed four-manifold. If X is decomposed as a smooth connected sum $X = X_1 \sharp X_2$ with each $b_2^+(X_i) > 0$, $4c_2(E)[X] > 3(1 + b_2^+(X))$, then the polynomial invariant $q_{k,X}$ vanishes identically, where $q_{k,X}$ is defined by the moduli space of anti-self dual connections of an $SU(2)$-bundle E over X with $c_2(E)[X] = k$ and $b_2^+(X)$ is odd and not less than 3.*

In general a 2-dimensional homology class in a four manifold X can not be represented as a smoothly embedded sphere. This raises the problem of finding the smallest possible genus of the surfaces representing a given 2-dimensional homology class.

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To find a lower bound for the genus, Kronheimer and Mrowka considered the space $\mathcal{A}^\alpha$ of α-twisted, singular $SU(2)$ connections which have holonomy α along the embedded surface Σ in X. They used weighted Sobolev spaces and a singular metric with a cone like singularity along Σ to control effectively the moduli space. In [11, 12] they got the following theorem:

THEOREM. (Kronheimer, Mrowka) *Let X be a smooth closed, simply connected and oriented 4-manifold. Let $b_2^+(X) \geq 3$, odd and let X have non trivial polynomial invariants. Then the genus of any orientable, smoothly embedded surface Σ, other than a sphere of self-intersection -1 or 0, satisfies the inequality*

$$2g(\Sigma) - 2 \geq \Sigma \cdot \Sigma.$$

Suppose the cyclic group $\mathbb{Z}_p$ of order p acts on a smooth, orientable, closed 4-manifold X. Then an oriented surface Σ in X can be the fixed point set of the $\mathbb{Z}_p$-action on X.

In [1, 2] Cho considered the $\mathbb{Z}_p$-action on an $SU(2)$ bundle $E \to X$ and its quotient bundle $E' \to X'$. Then the fixed point set Σ of $\mathbb{Z}_p$-action on X is appeared in the quotient space X' as a singular set.

THEOREM. (Cho) (1) *Suppose $\pi_* : H_2(X, \mathbb{Z})^{\mathbb{Z}_p} \to H_2(X', \mathbb{Z})$ and $\pi_*(\alpha_i) = p\alpha_{i'}$, $i = 1, \cdots, d'$, then $q^{\mathbb{Z}_p}(\alpha_1, \cdots, \alpha_{d'}) = q'(\alpha'_1, \cdots, d'_{d'})$ where $q^{\mathbb{Z}_p}$ is the polynomial invariant defined on the invariant moduli space on X and q' is the polynomial invariant defined on the moduli space on the quotient setting X'.*

(2) *Let α_1 and α_2 be the holonomy parameters of the $\mathbb{Z}_p$-action along the fixed point set Σ. For regular values α_1 and α_2 the polynomial invariants $q_{k,\ell}^{\alpha_1} = q_{k,\ell}^{\alpha_2}$ are equal, where k is the instanton number and ℓ is the monopole number and the polynomial invariant $q_{k,\ell}^{\alpha_i}$ is defined on the singular moduli space of holonomy parameter α_i along the fixed point set Σ.*

Let X_1 and X_2 be smooth, closed, simply connected, oriented 4-manifolds, and let Σ_1 and Σ_2 be oriented embedded 2-dimensional surfaces with genus g_1 and g_2 in X_1 and X_2 respectively. Suppose $b_2^+(X_i) > 0$ for $i = 1, 2$, and each intersection number $\Sigma_i \cdot \Sigma_i$ is 0 for $i = 1, 2$.

Choose two points p_1 and p_2 in Σ_1 and Σ_2 respectively. By cutting out small open neighbourhoods of p_1 and p_2 in X_1 and X_2 and identifying their boundaries respectively, we have a connected sum of the form $(X_1, \Sigma_1)\sharp(X_2, \Sigma_2) = (X_1 \sharp X_2, \Sigma_1 \sharp \Sigma_2)$.

For a small positive number $\epsilon > 0$ choose a holonomy parameter $\alpha \in \left[\epsilon, \frac{1}{2} - \epsilon\right]$ around the surface Σ. Let $E \to X$ be an $SU(2)$-vector bundle on X and $N(\Sigma)$ be a small tubular neighbourhood of Σ in X and $E|_{N(\Sigma)} = L \oplus L^{-1}$ decomposed into complex line bundles. Let $k = c_2(E)[X]$ be the instanton number and $\ell = -c_1(L)[\Sigma]$ the monopole number. Choose an orbifold metric on X along Σ. We consider the α-twisted singular moduli space (over the bundle $E \to (X, \Sigma)$) $\mathfrak{M}^{\alpha}_{k,\ell,X}$ (for details see the next section). Then the moduli space has the formal dimension $8k - 3(1 + b_2^+(X)) + 4\ell - (2g - 2)$ where $g = g_1 + g_2$ is the genus of the surface Σ.

In this paper we would like to prove a kind of Donaldson's Vanishing theorem, that is, the polynomial invariant $q^{\alpha}_{k,\ell}$ (defined on the singular moduli space) vanishes under certain conditions. Roughly we can summarise as follow:

As Donaldson's case we consider $(X, \Sigma) = (X_1, \Sigma_1)\sharp_{\lambda}(X_2, \Sigma_2)$ and $g_\lambda = g^1_\lambda \sharp g^2_\lambda$ metrics on X depending on the neck parameter λ. Then the polynomial invariant, if $\dim \mathfrak{M}^{\alpha}_{k,\ell,X} = 2d$,

$$q^{\alpha}_{k,\ell,X}(g_\lambda)(\alpha_1, \cdots, \alpha_d) = \sharp(\mathfrak{M}^{\alpha}_{k,\ell,X}(g_\lambda) \cap V_1 \cap \cdots \cap V_d) = \sharp I(\lambda)$$

where the number is counted with sign, and V_i are the codimension 2 varieties defined by $\alpha_i = [\Sigma'_i]$, $i = 1, \cdots, d$.

For sufficiently small λ we have $I(\lambda) = I_1(\lambda) \cup I_2(\lambda)$ where the energy of the elements of $I_i(\lambda)$ are supported in $(X, \Sigma) \setminus (X_i, \Sigma_i)$, $i = 1, 2$.

By the arguments of perturbations of anti-self-dual equations and Euler number of odd dimension we have the following theorem.

THEOREM. *If* $(X, \Sigma) = (X_1, \Sigma_1)\sharp_{\lambda}(X_2, \Sigma_2)$ $(b_2^+(X_i) > 0)$ *and* $\Sigma_i \cdot \Sigma_i = 0$ *for* $i = 1, 2$, *and if* $b_2^+(X) \geq 3$, *odd and* $d \geq 2k + 1$ *where* $2d = \dim \mathfrak{M}^{\alpha}_{k,\ell,X}$. *Then for sufficiently small* λ, *and for generic metric* g_λ *the signed number* $\sharp I_i(\lambda)$ *is* 0 *for* $i = 1, 2$ *and hence the polynomial invariant*

$$q^{\alpha}_{k,\ell,X}(g_\lambda) \equiv 0 \quad \text{on} \quad H_2(X \setminus \Sigma, \mathbb{Z}).$$

In section 2 we will summerize the basic definitions, properties on the singular moduli spaces, the necks of connected sums, and compactification of the singular moduli spaces. In section 3 we will define a polynomial invariant on the singular moduli space. We discuss the behaviours of the moduli spaces when λ goes to 0, and the statement of our main theorem. In section 4, we will prove the main theorem by considering the given parameters and constructing Lie algebra valued self-dual 2-forms to cut out the $2g$-dimension from the moduli space which comes from the genus of Σ.

2. Preliminary steps

2.1. Singular moduli space

Let X_i be a smooth, compact, simply connected, oriented four-manifold and Σ_i be a closed oriented embedded 2-dimensional surface with genus g_i and we will assume that the self intersection number $\Sigma_i \cdot \Sigma_i$ is zero for $i = 1, 2$.

Let $N(\Sigma_i)$ be a tubular neighbourhood of $\Sigma_i \subset X$ diffeomorphic to the unit disk bundle of the normal bundle and Y_i be boundary of $N(\Sigma_i)$ which acquires the structure of a circle bundle over Σ_i via this diffeomorphism.

Consider an $SU(2)$-bundle E_i on X_i and choose a C^∞ decomposition of E_i on $N(\Sigma_i)$ as $E_i|_{N(\Sigma_i)} = L_i \oplus L_i^{-1}$ and L_i is a complex line bundle. We need not suppose that L_i is trivial bundle. See Diagram 2.1.1.

$$\begin{array}{ccc} L_i \oplus L_i^{-1} & & E_i \\ \downarrow & & \downarrow SU(2) \\ N(\Sigma_i) & \longrightarrow & X_i \end{array}$$

Diagram 2.1.1

There are two topological invariants in the bundle, which we write

$$k_i = c_2(E_i)[X_i] = \frac{1}{8\pi^2} \int_{X_i} tr(F_A \wedge F_A)$$
$$\ell_i = -c_1(L_i)[\Sigma_i]$$

where A is an $SU(2)$-connection on E_i.

For this bundle, we define an "α-twisted"and locally nontrivial connection near Σ_i; choose any $SU(2)$-connection $\overline{A}_i^\circ$ on E_i such that $\overline{A}_i^\circ|_{N(\Sigma_i)} = \begin{pmatrix} b_i & 0 \\ 0 & -b_i \end{pmatrix}$ where b_i is a smooth conection in $L_i, i = 1, 2$. Finally choose a number α in the range $\alpha \in \left[\epsilon, \frac{1}{2} - \epsilon\right]$ where $\epsilon > 0$ is a small positive number, and define an α-twisted connection A_i^α on $E_i|_{(X_i \setminus \Sigma_i)}$ by

$$A_i^\alpha = \overline{A}_i^\circ + \sqrt{-1}\beta_i(r) \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} \eta_i \quad (i = 1, 2)$$

where β_i is a smooth cut off function which equals to 1 in a neighbourhood of 0, and equals to 0 for $r \geq \frac{1}{2}$, and $\sqrt{-1}\eta_1$ and $\sqrt{-1}\eta_2$ are connection 1-forms for the circle bundle.

From now we will mean that (X_i, Σ_i) is a smooth, compact, simply connected, oriented four-manifold X_i which contains Σ_i and has our bundle structure mentioned above.

We now define an affine space of connections modelled on A_i^α by choosing some $p > 2$ and setting $\mathcal{A}_i^\alpha = \{A_i^\alpha + a_i | \nabla_{A_i^\alpha} a_i, \ \ a_i \in L^p(X_i \setminus \Sigma_i)\}$.

Similarly we define a gauge group $\mathcal{G}_i = \{g \in Aut(E_i) | \nabla_{A_i^\alpha} g, \nabla^2_{A_i^\alpha} g \in L^p(X_i \setminus \Sigma_i)\}$.

Then we can consider a Banach space $\mathcal{B}^\alpha_{k_i, \ell_i, X_i} = \mathcal{A}_i^\alpha / \mathcal{G}_i$ and a Banach manifold $(\mathcal{B}^\alpha_{k_i, \alpha_i, X_i})^*$. Here $(\mathcal{B}^\alpha_{k_i, \ell_i, X_i})^*$ is the space of irreducible α-twisted connections which is open in $\mathcal{B}^\alpha_{k_i, \ell_i, X_i}$ where k_i and ℓ_i are two topological invariants of our bundle.

We have been using a smooth metric on X_i to define a moduli space but this is not the only possibility. We can take a metric which, near to Σ_i, is modelled on

$$g^i = du_i^2 + dv_i^2 + dr_i^2 + \left(\frac{r_i}{\nu}\right)^2 d\theta_i^2$$

where (u_i, v_i) are coordinates on Σ_i and ν is a real parameter not less than 1 and (r_i, θ_i) are polar coordinates in some local trivialisation of the disk bundle.

Over $N(\Sigma_i)$ the metric g^i has a cone-angle of $\frac{2\pi}{v}$ in the normal planes to Σ_i and equal to a smooth one on the complement of $N(\Sigma_i)$, $i = 1, 2$.

Now consider an α-twisted singular moduli space (over the bundle $E_i \to (X_i, \Sigma_i)$), $\mathfrak{M}^{\alpha}_{k_i,\ell_i,X_i} = \{A \in \mathcal{A}^{\alpha}_i | F^+(A) = 0,\ F^+(A)$ is self-dual part of $F(A)$ with respect to the orbifold metric $g^i\}/\mathcal{G}_i \subset \mathcal{B}^{\alpha}_{k_i,\ell_i,X_i}$. Then Kronheimer and Mrowka computed the dimension of the α-twised singular moduli space.

$$\dim \mathfrak{M}^{\alpha}_{k_i,\ell_i,X_i} = 8k_i - 2(1 + b_2^+(X_i)) + 4\ell_i - (2g_i - 2),$$

$$\text{and } \frac{1}{8\pi^2}\int_{X_i \setminus \Sigma_i} tr(F_A \wedge F_A) = k_i + 2\alpha\ell_i - \alpha^2 \cdot \Sigma_i \cdot \Sigma_i$$

$(i = 1, 2)$. (For details see [11])

We suppose that X_i is given a homology orientation Ω; such a homology orientation is fixed, by choosing an orientation for the line $(\overset{\max}{\wedge} H^1(X_i))^{-1} \otimes (\overset{\max}{\wedge} H^+(X_i))$, where $H^+(X_i)$ is any maximal positive subspace of $H^2(X_i)$, $i = 1, 2$. Then the moduli spaces have orientations.

2.2. Connected Sums

In this section we will consider a smooth compact, simply connected, oriented four-manifold X with $b_2^+(X) \geq 3$, odd and a closed oriented embedded 2-dimensional surface Σ such that (X, Σ) can be decomposed as a smooth oriented connected sum $(X, \Sigma) = (X_1, \Sigma_1) \sharp (X_2, \Sigma_2)$, where $b_2^+(X_i) > 0$ and (X_i, Σ_i) is four-manifold as in (2.1) $(i = 1, 2)$; fix points p_1, p_2 in Σ_1, Σ_2 respectively and put $Z_i(r) = (X_i, \Sigma_i) \setminus B(p_i, r)$ for $r < 1$. The ball $B(p_i, r)$ denote the image of the 4-dimensional ball with radius r under the exponential map at p_i and it is contained in $N(\Sigma_i)$, $i = 1, 2$.

Let $Z(r)$ be the disjoint union of $Z_1(r)$ and $Z_2(r)$. Choose an orientation reversing isometry $I : T_{p_1}X_1 \to T_{p_2}X_2$. Then the map f_λ between punctured tangent space given by $f_\lambda(\xi) = \frac{\lambda}{|\xi|^2} \cdot I(\xi)$ identifies the annulus in $N(\Sigma_1)$, centered on p_1, inner radius $N^{-1}\sqrt{\lambda}$ and outer radius $N\sqrt{\lambda}$ with the corresponding annulus in $N(\Sigma_2)$. (Here we take N such that $N > 1$ and $N\sqrt{\lambda} \ll 1$.)

We form the oriented connected sum $(X, \Sigma) = (X_1, \Sigma_1) \underset{\lambda}{\sharp} (X_2, \Sigma_2)$ as a quotient $Z(N^{-1}\sqrt{\lambda}) = Z_1(N^{-1}\sqrt{\lambda}) \amalg Z_2(N^{-1}\sqrt{\lambda})$ by the gluing map f_λ for small λ.

Now define an orbifold metric g_λ on the connected sum $(X, \Sigma) = (X_1, \Sigma_1) \sharp_\lambda (X_2, \Sigma_2)$; the key property is that it should agree with g^i on a large open set, for example on $Z_i(2\sqrt{\lambda})$, $i = 1, 2$. Over the neck in (X, Σ) we can define g_λ to be a weighted average of metrics g^1, g^2 on (X_1, Σ_1), (X_2, Σ_2) respectively, compared via the identification map f_λ.

Consider an $SU(2)$ bundle E over the connected sum (X, Σ) such that we have a C^∞ decomposition of E on $N(\Sigma)$ as $E|_{N(\Sigma)} = L \oplus L^{-1}$ where L is a complex line bundle. Then there are two toplological invariants in our bundle $E \to (X, \Sigma)$ which we write $k = k_1 + k_2$ and $\ell = \ell_1 + \ell_2$ where k_i and ℓ_i are two topological invariants in the given bundle $E_i \to (X_i, \Sigma_i)$ as in (2.1), $i = 1, 2$. As (2.1) we now define an affine space of connections on $(X \setminus \Sigma)$ and a gauge group by choosing some $p > 2$ and setting

$$\begin{aligned} &\mathcal{A}^\alpha = \{A^\alpha + a \mid a, \quad \nabla_{A^\alpha} a \in L^p(X \setminus \Sigma)\} \\ \text{and} \quad &\mathcal{G} = \{g \in Aut(E) \mid \nabla_{A^\alpha} g, \quad \nabla^2_{A^\alpha} g \in L^p(X \setminus \Sigma)\}, \\ &\alpha \in \left[\epsilon, \frac{1}{2} - \epsilon\right]. \end{aligned}$$

Then the quotient $\mathcal{A}^\alpha / \mathcal{G} = \mathcal{B}^\alpha_{k,\ell,X}$ is a Banach manifold over $(X \setminus \Sigma)$ except at points corresponding to the reducible connections. Now we consider an α-twisted singular moduli space (over the bundle $E \to (X, \Sigma)$) $\mathfrak{M}^\alpha_{k,\ell,X}(g_\lambda) = \{A \in \mathcal{A}^\alpha \mid F^+(A) = 0,\ F^+(A)$ is self-dual part of $F(A)$ with respect to the metric $g_\lambda\}/\mathcal{G}$. Then $\dim(\mathfrak{M}^\alpha_{k,\ell,X}(g_\lambda))$ is $8k - 3(1 + b_2^+(X)) + 4\ell - (2g - 2)$ where the genus g of Σ is the sum of the genus g_1 of Σ_1 and g_2 of Σ_2. Since $b_2^+(X)$ is odd, $\dim(\mathfrak{M}^\alpha_{k,\ell,X}(g_\lambda))$ becomes even. Now let $\dim(\mathfrak{M}^\alpha_{k,\ell,X}(g_\lambda))$ be $2d$ where $d \in \mathbb{Z}^+$.

2.3. Compactification of the singular moduli space

In (2.2) we constructed a smooth oriented connected sum (X, Σ) which depends on the gluing map f_λ. When λ is small the gluing part becomes small and the connected sum $(X, \Sigma) = (X_1, \Sigma_1) \sharp_\lambda (X_2, \Sigma_2)$

tends to $(X_1, \Sigma_1) \amalg (X_2, \Sigma_2) = Y$ and the metric $g_\lambda = g^1 \underset{\lambda}{\sharp} g^2$ tends to (g^1, g^2) over Y as $\lambda \to 0$. We have the following proposition

PROPOSITION 2.3.1. *If $\lambda n \to 0$ and A_n is a sequence of $g_{\lambda n}$-anti-self-dual connections on a bundle $E \to (X, \Sigma) = (X_1, \Sigma_1) \underset{\lambda n}{\sharp} (X_2, \Sigma_2)$ then there is a bundle $E' \to Y$, a connection A' on E' with chern number k' and monopole number ℓ' and a multi set $(z_1, \cdots, z_n)$ in $Y \setminus \{p_1, p_2\}$ such that a subsequence $[A'_n]$ of $[A_n]$ converges to a limit $[A']$ over $Y \setminus \{p_1, p_2, z_1, \cdots, z_n\}$. In this case we have $k = k'_1 + k'_2 + \Sigma_{i=1}^r k_i + \Sigma_{j=1}^s k_j$ and $\ell = \ell'_1 + \ell'_2 + \Sigma_{j=1}^s \ell_j$ where $k'_i = k'|_{(X_i, \Sigma_i)}$ and $\ell'_i = \ell'|_{(X_i, \Sigma_i)}$ $(i = 1, 2)$. And k_i is an associated positive integer for points of concentration z_i in $X \setminus \Sigma$, $i = 1, \cdots, r$, and (k_j, ℓ_j) is an associated pair for points of concentration z_j in Σ, $j = 1, \cdots, s$, where r and s be the number of points concentration in $X \setminus \Sigma$ and Σ respectively $(r + s = n)$.*

Proof. By the Uhlenbeck's compactness theorem and the gluing construction for the connected sum (X, Σ) it is clear.

Now we have the compactification of the singular moduli space.

LEMMA 2.3.2. *We have a compactification of the singular moduli space $\mathfrak{M}^\alpha_{k,\ell,X}(g_\lambda)$ over the connected sum (X, Σ) such that*

$$\overline{\mathfrak{M}^\alpha_{k,\ell,X}(g_\lambda)} \subset \bigcup_{r+s \geq 0} \mathfrak{M}^\alpha_{k-(r+s), \ell - \Sigma_{j=1}^s \ell_j, X} \times S^{r+s}(X)$$

where $\mathfrak{M}^\alpha_{k-(r+s), \ell-\Sigma_{j=1}^s \ell_j, X}$ is an α-twisted singular moduli space with chern number $k - (r + s)$ and monopole number $\ell - \Sigma_{j=1}^s \ell_j$ over (X, Σ). And $S^{r+s}(X)$ is a multiset of degree $r + s$ (unordered $(r + s)$-tuple) of points of X.

Proof. If $[A_n]$ is a sequence of $\mathfrak{M}^\alpha_{k,\ell,X}(g_\lambda)$ for small λ then a subsequence $[A'_n]$ of $[A_n]$ converges weakly to a limit $([A'], \{z_1, \cdots, z_n\})$. (That is $[A'_n]$ converges to a limit $[A'] \in \mathfrak{M}^\alpha_{k', \ell', X}$ over $(X, \Sigma) \setminus \{z_1, \cdots, z_n\}$.)

The instanton number k' and monopole number ℓ' of $[A']$ have properties such that $k = k' + \Sigma_{i=1}^r k_i + \Sigma_{j=1}^s k_j$ and $\ell = \ell' + \Sigma_{j=1}^s \ell_j$. Then we have $k \geq k' + r + s$. Here r and s be the number of points of concentration in $X \setminus \Sigma$ and Σ $(r+s=n)$. So the infinite sequence in $\mathfrak{M}^\alpha_{k,\ell,X}(g_\lambda)$ has a weakly convergent subsequence with a limit point in

$$\bigcup_{r+s\geq 0} \mathfrak{M}^\alpha_{k-(r+s),\ell-\Sigma^s_{j=1}\ell_j,X} \times S^{r+s}(X). \quad \square$$

3. A polynomial invariant over the connected sum (X, Σ)

In this section we will define an invariant $q^\alpha_{k,\ell,X}$, a polynomial of degree d in $H^2(X \setminus \Sigma; \mathbb{Z})$, assuming that k is in a "stable range" $d \geq 2k+1$.

Fix a generic orbifold metric g_λ on $(X,\Sigma) = (X_1,\Sigma_1)\underset{\lambda}{\sharp}(X_2,\Sigma_2)$ and choose compact 2-dimensional surfaces $\Sigma'_1, \cdots, \Sigma'_d$, such that Σ'_i is embedded in $X \setminus \Sigma$ and $N(\Sigma'_i) \cap \Sigma = \phi$ $(i = 1, \cdots, d)$. We can choose Σ'_i, $i = 1, \cdots, d$ such that $N(\Sigma'_i) \cap N(\Sigma'_j) \cap N(\Sigma'_k) = \phi$ for distinct i, j, k.

Let $\mathcal{B}^\alpha_{N(\Sigma'_i)} = \mathcal{B}^\alpha_{k,\ell,X}|_{N(\Sigma'_i)}$ be the space of gauge equivalence classes of α-twisted $SU(2)$ connections on $N(\Sigma'_i)$ and $(\mathcal{B}^\alpha_{N(\Sigma'_i)})^* \subset \mathcal{B}^\alpha_{N(\Sigma'_i)}$ be the Banach manifold of irreducible connections.

Let $\mathcal{L}_{N(\Sigma'_i)} \to (\mathcal{B}^\alpha_{N(\Sigma'_i)})^*$ be the determinant line bundle with $c_1(\mathcal{L}_{N(\Sigma'_i)}) = \mu([\Sigma'_i])$ in $H^2((\mathcal{B}^\alpha_{N(\Sigma'_i)})^*; \mathbb{Z})$ and fiber $[A] \times det(indD_A)$ $= [A] \times (\overset{\max}{\wedge} kerD_A \otimes (\overset{\max}{\wedge} cokerD_A)^*)$ where $\mu; H_2(X \setminus \Sigma; \mathbb{Z}) \to H^2((\mathcal{B}^\alpha_{k,\ell,X})^*; \mathbb{Z})$, $i = 1, \cdots, d$.

Let s_i be any smooth section of $\mathcal{L}_{N(\Sigma'_i)}$ and let V_i be $s_i^{-1}(0)$. Since the elements in $\mathfrak{M}^\alpha_{k,\ell,X}(g_\lambda)$ are α-twisted, anti-self-dual connections on (X,Σ), there is a well-defined restriction map $r_i; \mathfrak{M}^\alpha_{k,\ell,X}(g_\lambda) \to \mathcal{B}^\alpha_{N(\Sigma'_i)}$ and image of r_i is contained in $(\mathcal{B}^\alpha_{N(\Sigma'_i)})^*$ by the unique continuation theorem. (For details see [6]). We shall write $\mathfrak{M}^\alpha_{k,\ell,X}(g_\lambda) \cap V_i$ in the place of $\{[A] \in \mathfrak{M}^\alpha_{k,\ell,X}(g_\lambda) | r_i([A]) \in V_i\}$, $i = 1, \cdots, d$.

The smooth section s_i can be choosen such that it is transverse to r_i. Then $\mathfrak{M}^\alpha_{k,\ell,X}(g_\lambda) \cap V_i$ is a smooth codimension 2-submanifold of $\mathfrak{M}^\alpha_{k,\ell,X}(g_\lambda)$. We can further arrange transversality for $\mathfrak{M}^\alpha_{k,\ell,X}(g_\lambda) \cap V_{i_1} \cap \cdots \cap V_{i_c}$ $(c \leq d)$ such that it is a smooth $(2d-2c)$-dimensional manifold.

Specially we consider a 0-dimensional space $\mathfrak{M}^\alpha_{k,\ell,X}(g_\lambda) \cap V_1 \cap \cdots \cap V_d$ and let this be $I(\lambda)$ for small values of λ. Then $I(\lambda)$ is a collection of signed points. For $I(\lambda)$, we have the following Lemma.

LEMMA 3.1. *We can find λ_0 such that for all $\lambda \geq \lambda_0$ the intersection $I(\lambda) = \mathfrak{M}^{\alpha}_{k,\ell,X}(g_\lambda) \cap V_1 \cap \cdots \cap V_d$ is compact.*

Proof. Suppose $[A_n]$ is a sequence in $I(\lambda)$ and there is no strongly convergent subsequence. Then there is a subsequence $[A'_n]$ of $[A_n]$ such that $[A'_n]$ converges to a limit $[A'] \in \mathfrak{M}^{\alpha}_{k',\ell',X}(g_\lambda)$ with chern number k' and monopole number ℓ' over $(X \setminus \Sigma) \setminus \{z_1, \cdots, z_n\}$ where ρ is a small positive real number and $z_i \in (X, \Sigma)$, $i = 1, \cdots, n$. Let the number of points z_i of concentration in $X \setminus \Sigma$ and Σ be r and s respectively $(r + s = n)$. Then we have

$$k = k' + \Sigma^r_{i=1} k_i + \Sigma^s_{j=1} k_j = k'_1 + k'_2 + \Sigma^r_{i=1} k_i + \Sigma^s_{j=1} k_j$$
$$\ell = \ell' + \Sigma^s_{j=1} \ell_j = \ell'_1 + \ell'_2 + \Sigma^s_{j=1} \ell_j$$

where $k'_i = k'|_{(X_i,\Sigma_i)}$ and $\ell'_i = \ell'|_{(X_i,\Sigma_i)}$, $i = 1, 2$.

Let the set of tubular neighbourhoods $N(\Sigma'_j)$ which contain no point z_i of intersection be $\{N(\Sigma'_{i1}), \cdots, N(\Sigma'_{ic})\}$ $(c \leq d)$. Then we conclude that $c \geq d - 2r$ and $\mathfrak{M}^{\alpha}_{k',\ell',X}(g_\lambda) \cap V_{i1} \cap \cdots \cap V_{ic}$ is non empty.

Suppose first that $0 < n < k$.

Then we have $\dim(\mathfrak{M}^{\alpha}_{k',\ell',X}(g_\lambda) \cap V_{i1} \cap \cdots \cap V_{ic}) = \dim \mathfrak{M}^{\alpha}_{k',\ell',X}(g_\lambda) - 2c = \dim \mathfrak{M}^{\alpha}_{k,\ell,X}(g_\lambda) - 8\Sigma^r_{i=1} k_i - 4\Sigma^s_{j=1}(2k_j + \ell_j) - 2c \leq 2d - 8r - 4s - 2c \leq 2d - 8r - 4s - 2d + 4r = -4(r + s) = -4n < 0$. So we obtain a contradiction.

Now consider the case when $n = k$.

In this case the limit $[A']$ is a flat α-twisted connection over $(X \setminus \Sigma)$.

Now we use the following Alternative;

ALTERNATIVE 3.2. [6]. *For each i, either*

(i) *$[A']$ is non trivaial and $[A'] \in V_i$ or*
(ii) *$N(\Sigma'_i)$ $(i = 1, \cdots, d)$ contains one of the points z_j, $j = 1, \cdots, n$.*

By Alternative 3.2 each $N(\Sigma'_i)$ contains one of the points z_j in our case, $j = 1, \cdots, n$, $i = 1, \cdots, d$.

But there is a $N(\Sigma'_k)$, which contains none of the points $z_j, j = 1, \cdots, n$; if we let the number of such $N(\Sigma'_k)'s$ be c then we have

$$c \geq d - 2r \geq d - 2n = d - 2k \geq 1 \quad \text{(by stable range } d \geq 2k + 1).$$

Thus there is a $N(\Sigma'_k)$, which contains none of the point z_j. So we have a contradiction and the only possibility is $n = 0$. And we conclude that suppose $[A_n]$ is a sequence in $I(\lambda)$ then $[A_n]$ converge strongly to a limit $[A']$ in $I(\lambda)$. Hence $I(\lambda)$ is a compact 0-dimensional space for all $\lambda \leq \lambda_0$. □

DEFINITION 3.3. By Lemma 3.1 we know that the intersection $I(\lambda)$ is a finite set of points. So we can define a polynomial invariant $q^{\alpha}_{k,\ell,X}$: $\overset{d}{\otimes} H_2(X \setminus \Sigma; \mathbb{Z}) \to \mathbb{Z}$ such that

$$q^{\alpha}_{k,\ell,X}([\Sigma'_1], \cdots, [\Sigma'_d]) = \sharp(\mathfrak{M}^{\alpha}_{k,\ell,X} \cap V_1 \cap \cdots \cap V_d).$$

where $[\Sigma'_i] \in H_2(X \setminus \Sigma; \mathbb{Z})$, $i = 1, \cdots, d$.

REMARK 3.4. As long as α remains in an interval $\left[\epsilon, \frac{1}{2} - \epsilon\right]$ the invariant $q^{\alpha}_{k,\ell,X}$ defined by above is independent of α, the choice of the orbifold metric, and the choice of the smooth section s_i. And $q^{\alpha}_{k,\ell,X}$ depends on the surface $[\Sigma'_i]$, $i = 1, \cdots, d$, only through their homology class in $(X \setminus \Sigma)$ and as a function $q^{\alpha}_{k,\ell,X}$ is multi linear and symmetric.

Proof. Refer to [12].

To understand the properties of $q^{\alpha}_{k,\ell,X}$, we consider the following; let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be smooth monotone function with $f(x) = x^2$ for small x and $f(x) = x$ for large x.

If A is any α-twisted, $SU(2)$-connection over $X \setminus \Sigma$, we put

$$E_1(A) = \int_{Z_1(\rho)} f(|F(A)|) + |F^+(A)|^p d\mu \quad \text{where} \quad Z_1(\rho) = X_1 \setminus B(p_1, \rho)$$

and $B(p_1, \rho)$ is a a 4-dimensional ball with radius ρ centered at p_i which is contained in $N(\Sigma_1)$. And ρ is a small real number such that $N^{-1}\sqrt{\lambda} < \rho < 1$.

For $\epsilon > 0$ we let $U_1(\epsilon) \subset \mathcal{B}^{\alpha}_{k,\ell,X}$ be the open sets $E_1^{-1}[0, \epsilon]$. The function E_1 does measure the distance to the flat connection over $(X, \Sigma)|_{(X_1, \Sigma_1)}$. (Also we define $U_2(\epsilon)$ as $E_2^{-1}[0, \epsilon]$ where $E_2(A) = \int_{Z_2(\rho)} f(|F(A)|) + |F^+(A)|^p d\mu$.)

PROPOSITION 3.5. *For any ρ, λ_0 there is an $\epsilon_0(\rho, \lambda_0)$ such that $I(\lambda) \cap U_1(\epsilon) \cap U_2(\epsilon) = \emptyset$ for all $\lambda \leq \lambda_0$ and $\epsilon \leq \epsilon_0(\rho, \lambda_0)$.*

Proof. If the result were false we could find a sequence $[A_n]$ in $I(\lambda_n) \cap U_1(\epsilon_n) \cap U_2(\epsilon_n)$ with $\epsilon_n \to 0$ and $\lambda_n \in (0, \lambda_0)$ with $\lambda_n \to 0$. Then we can suppose that a subsequence $[A'_n]$ of $[A_n]$ such that $[A'_n]$ converges to a trivial α-twisted flat connection $[A']$ over $Y(= (X_1, \Sigma_1) \amalg (X_2, \Sigma_2)) \setminus \{p_1, p_2, z_1, \cdots, z_n\}$ as $n \to \infty$. (That is $[A'_n]$ converges weakly to a limit $([A']; (z_1, \cdots, z_n))$ and $E(A'_n) \to 0$ as $\lambda_n \to 0$ over $Y \setminus \{p_1, p_2, z_1, \cdots, z_n\}$ where $z_i \in Y \setminus \{p_1, p_2\}$, $i = 1, \cdots, n$.) Let $[A']$ be $[\theta_1, \theta_2]$ and the number of the points of the concentration in $X \setminus \Sigma$ and Σ be r and s where θ_i is a flat α-twisted connection over (X_i, Σ_i), $i = 1, 2$. Since θ_1 and θ_2 are α-twisted trivial flat connections, each $N(\Sigma'_i)$ contains one of the points $z_i \in (X_1 \setminus \Sigma_1) \amalg (X_2 \setminus \Sigma_2)$, $i = 1, \cdots, r$. (See Alternative 3.2.) But there must be a tubular neighbourhood $N(\Sigma'_k)$, which does not contain any of these points. Thus we have a contradiction. □

PROPOSITION 3.6. *For any fixed ϵ, ρ there is a $\lambda_1(\epsilon, \rho)$ such that $I(\lambda)$ is contained in $U_1(\epsilon) \cup U_2(\epsilon)$ for all $\lambda \leq \lambda_1(\epsilon, \rho)$.*

Proof. If the statement were false there would be an ϵ, a sequence $\lambda_n \to 0$ as $n \to \infty$ and g_{λ_n}-anti self dual connection $[A_n]$ in $I(\lambda_n)$ but $[A_n] \notin U_1(\epsilon) \cup U_2(\epsilon)$. We can suppose the sequence $[A_n]$ converges weakly to a limit $([A']; (z_1, \cdots, z_n))$ and $[A'] \in \mathfrak{M}^\alpha_{k', \ell', Y}$ is not an α-twisted flat connection over either component $(X_1 \setminus \Sigma_1), (X_2 \setminus \Sigma_2)$ since $[A_n] \notin U_1(\epsilon) \cup U_2(\epsilon)$. Let the number of points of concentration in $X \setminus \Sigma$ and Σ be r and s and the chern number k' of the limit $[A']$ has component $k'_1 = k'|_{(X_1, \Sigma_1)}$, $k'_2 = k'|_{(X_2, \Sigma_2)}$ and the monopole number ℓ' of $[A']$ has component $\ell'_1 = \ell'|_{(X_1, \Sigma_1)}$, $\ell'_2 = \ell'|_{(X_2, \Sigma_2)}$ respectively.

Then we have $k = k'_1 + k'_2 + \Sigma^r_{i=1} k_i + \Sigma^s_{j=1} k_j = k' + \Sigma^r_{i=1} k_i + \Sigma^s_{j=1} k_j$ and $\ell = \ell'_1 + \ell'_2 + \Sigma^s_{j=1} \ell_j = \ell' + \Sigma^s_{j=1} \ell_j$. Let $[A'] \in \mathfrak{M}^\alpha_{k', \ell', Y} \cong \mathfrak{M}^\alpha_{k'_1, \ell'_1, X_1} \times \mathfrak{M}^\alpha_{k'_2, \ell'_2, X_2}$ be $[A_1, A_2]$ and the number of the surface Σ'_i in $X_1 \setminus \Sigma_1$ and $X_2 \setminus \Sigma_2$ be d_1, d_2 ($d = d_1 + d_2$). And let the number of the points of concentration in $X_1 \setminus \Sigma_1$ and $X_2 \setminus \Sigma_2$ be r_1, r_2 respectively ($r_1 + r_2 = r$).

If $[A_1] \in \mathfrak{M}^\alpha_{k'_1, \ell'_1, X_1} \cap V_{i1} \cap \cdots \cap V_{ip}$ then $p \geq d_1 - 2r_1$ and if $[A_2] \in \mathfrak{M}^\alpha_{k'_2, \ell'_2, X_2} \cap V_{j1} \cap \cdots \cap V_{jq}$ then $q \geq d_2 - 2r_2$. Since $[A_1]$ and $[A_2]$ are not α-twisted flat connections, both $\dim(\mathfrak{M}^{\alpha}_{k'_1,\ell'_1,X_1} \cap V_{i1} \cap \cdots \cap V_{ip})$ and $\dim(\mathfrak{M}^{\alpha}_{k'_2,\ell'_2,X_2} \cap V_{j1} \cap \cdots \cap V_{jq})$ are non negative.

Thus $\dim(\mathfrak{M}^{\alpha}_{k'_1,\ell'_1,X_1}) - 2p \geq 0$ and $\dim(\mathfrak{M}^{\alpha}_{k'_2,\ell'_2,X_2}) - 2q \geq 0$. Then

(3.7)

$$d_1 \leq p + 2r_1 \leq 2r_1 + \frac{1}{2}\dim(\mathfrak{M}^{\alpha}_{k'_1,\ell'_1,X_1}),$$

$$d_2 \leq q + 2r_2 \leq 2r_2 + \frac{1}{2}\dim(\mathfrak{M}^{\alpha}_{k'_2,\ell'_2,X_2}) \quad \text{and}$$

$$\dim(\mathfrak{M}^{\alpha}_{k,\ell,X}(g_{\lambda_n})) = 2d \leq 4r + \dim(\mathfrak{M}^{\alpha}_{k'_1,\ell'_1,X_1}) + \dim(\mathfrak{M}^{\alpha}_{k'_2,\ell'_2,X_2}).$$

Since $k = k' + \sum_{i=1}^{r} k_i + \sum_{j=1}^{s} k_j$, $\ell = \ell' + \sum_{j=1}^{s} \ell_j$ and α remains in an interval $\left[\epsilon, \frac{1}{2} - \epsilon\right]$, we have an associated pair (k_j, ℓ_j) for a point of concentration z_j in $\Sigma_1 \underset{\lambda}{\sharp} \Sigma_2$, $j = 1, \cdots, s$, and now we can use the following Lemma.

LEMMA 3.8. [11]. *For each $\epsilon > 0$, there is a ν such that, provided α in the interval $\left[\epsilon, \frac{1}{2} - \epsilon\right]$, $k_j + 2\epsilon\ell_j \geq 0$ and $k_j + (1 - 2\epsilon)\ell_j \geq 0$ where (k_j, ℓ_j) is an associated pair for a point of concentration z_j in $\Sigma = \Sigma_1 \underset{\lambda}{\sharp} \Sigma_2$ and ν is a real parameter not less than 1 which is associated with the metric $g_\lambda = du^2 + dv^2 + dr^2 + \left(\frac{r}{\nu}\right)^2 d\theta^2$. The two inequalities yield $2k_j + \ell_j \geq 0$.*

Using Lemma 3.8 we have

(3.9)

$$\begin{aligned}
2d &= \dim(\mathfrak{M}^{\alpha}_{k,\ell,X}(g_{\lambda_n})) \\
&= 8k + 4\ell - 3(1 + b_2^+(X)) - (2g - 2) \\
&= 8k' + 8\Sigma_{i=1}^{r} k_i + 8\Sigma_{j=1}^{s} k_j + 4\ell' + 4\Sigma_{j=1}^{s}\ell_j \\
&\qquad - 3(1 + b_2^+(X)) - (2g - 2) \\
&= \dim(\mathfrak{M}^{\alpha}_{k'_1,\ell'_1,X_1}) + \dim(\mathfrak{M}^{\alpha}_{k'_2,\ell'_2,X_2}) + 1 + 8\Sigma_{i=1}^{r} k_i \\
&\qquad + 4\Sigma_{j=1}^{s}(2k_j + \ell_j) \\
&\geq \dim(\mathfrak{M}^{\alpha}_{k'_1,\ell'_1,X_1}) + \dim(\mathfrak{M}^{\alpha}_{k'_2,\ell'_2,X_2}) + 8r + 1.
\end{aligned}$$

By (3.7) and (3.9), $\dim(\mathfrak{M}^{\alpha}_{k'_1,\ell'_1,X_1}) + \dim(\mathfrak{M}^{\alpha}_{k'_2,\ell'_2,X_2}) + 8r + 4s + 1 \leq 2d \leq \dim(\mathfrak{M}^{\alpha}_{k'_1,\ell'_1,X_1}) + \dim(\mathfrak{M}^{\alpha}_{k'_2,\ell'_2,X_2}) + 4r.$

Then we have $1+8r+4s \leq 4r$ and so $1+4(r+s)=4n+1 \leq 0$. So we have a contradiction. □

REMARK 3.10. Proposition 3.5 and 3.6 imply that when ϵ and λ are small we can write $I(\lambda)=\mathfrak{M}^{\alpha}_{k,\ell,X}(g_\lambda)\cap V_1\cap\cdots\cap V_d$ as the union of $I_1(\lambda)\subset U_1(\epsilon)$ and $I_2(\lambda)\subset U_2(\epsilon)$ respectively. Here $I_1(\lambda)$ is corresponding to $(\mathfrak{M}^{\alpha}_{k,\ell,X}(g_\lambda)\cap U_1(\epsilon))\cap V_1\cap\cdots\cap V_d$ and $I_2(\lambda)$ is corresponging to $(\mathfrak{M}^{\alpha}_{k,\ell,X}(g_\lambda)\cap U_2(\epsilon))\cap V_1\cap\cdots\cap V_d$.

Let $\sharp I_1(\lambda)$ be $i_1(\lambda)$ and $\sharp I_2(\lambda)$ be $i_2(\lambda)$. Then our polynomial invariant $q^{\alpha}_{k,\ell,X}(g_\lambda)\,([\Sigma'_1],\cdots,[\Sigma'_d])=\sharp(\mathfrak{M}^{\alpha}_{k,\ell,X}(g_\lambda)\cap V_1\cap\cdots\cap V_d)=\sharp I(\lambda)$ is equal to $i_1(\lambda)+i_2(\lambda)$ for all sufficiently small values of λ. The integers $i_1(\lambda)$ and $i_2(\lambda)$ are independent of the parameters λ,ϵ,ρ provided these are suitably small.

Thus we can define a polynomial invariant $q^{\alpha}_{k,\ell,X}$. From now we will prove our main result for $q^{\alpha}_{k,\ell,X}$ by establishing the following theorem.

THEOREM 3.11. *Suppose (X,Σ) is a connected sum $(X_1,\Sigma_1)\underset{\lambda}{\sharp}(X_2,\Sigma_2)$ and the self intersection number $\Sigma_i\cdot\Sigma_i$ is 0 for $i=1,2$. Also suppose that $b_2^+(X_i)>0$, $i=1,2$, and $b_2^+(X)\geq 3$, odd and $d\geq 2k+1$ where $2d=\dim(\mathfrak{M}^{\alpha}_{k,\ell,X}(g_\lambda))$. Then for sufficiently small λ, the signed number $\sharp I_i(\lambda)=i_i(\lambda)$ is 0 for $i=1,2$ and hence the polynomial invariant $q^{\alpha}_{k,\ell,X}(g_\lambda)\equiv 0$ on $H_2(X\setminus\Sigma,\mathbb{Z})$.*

4. The proof of Theorem 3.11

4.1. Preliminary works for the proof

LEMMA 4.1.1. *If $[A_n]$ is a sequence of $I(\lambda_n)=\mathfrak{M}^{\alpha}_{k,\ell,X}(g_{\lambda_n})\cap V_1\cap\cdots\cap V_d$ with $\lambda_n\to 0$ then there is a subsequence $[An']$ of $[A_n]$ such that $[An']$ converges to a limit $[A']=[A_1,A_2]\in\mathfrak{M}^{\alpha}_{k,\ell,Y}$ over $((X_1\setminus\Sigma_1)\amalg(X_2\setminus\Sigma_2))\setminus\{z_1,\cdots,z_n\}$ and the limit $[A']$ is of the form $[\theta_1,A_2]$ or $[A_1,\theta_2]$ where θ_1 and θ_2 are α-twisted flat connections over $(X_1\setminus\Sigma_1)$, $(X_2\setminus\Sigma_2)$ and A_1,A_2 are non trivial, anti-self-dual, α-twisted connections over $(X_1\setminus\Sigma_1)$, $(X_2\setminus\Sigma_2)$ respectively.*

Proof. First suppose that the limit $[A']$ is of the form $[A_1,A_2]$ where both A_1 and A_2 are non trivial α-twisted anti-self-dual connections.

Let the Chern number k' of $[A']$ has components $k'_1 = k'|_{(X_1,\Sigma_1)}$, $k'_2 = k'|_{(X_2,\Sigma_2)}$ and the monopole number ℓ' of $[A']$ has components $\ell'_1 = \ell'|_{(X_1,\Sigma_1)}$, $\ell'_2 = \ell'|_{(X_2,\Sigma_2)}$.

Also let each number of points z_i of concentration in $X \setminus \Sigma$ and Σ be r and s, respectively $(r + s = n)$.

Then we have

$$\begin{aligned}
&k = k'_1 + k'_2 + \Sigma_{i=1}^{r} k_i + \Sigma_{j=1}^{s} k_j \\
&\ell = \ell'_1 + \ell'_2 + \Sigma_{j=1}^{s} \ell_j \quad \text{and} \\
&[A'] = [A_1, A_2] \in \mathfrak{M}^{\alpha}_{k'_1,\ell'_1,X_1} \times \mathfrak{M}^{\alpha}_{k'_2,\ell'_2,X_2}.
\end{aligned}$$

Let each number of the surface Σ'_i, $i = 1, \cdots, d$, in $X_1 \setminus \Sigma_1$ and $X_2 \setminus \Sigma_2$ be d_1, d_2 respectively $(d = d_1 + d_2)$ and each number of the points z_i of concentration in $X_1 \setminus \Sigma_1$ and $X_2 \setminus \Sigma_2$ be r_1, r_2, respectively $(r = r_1 + r_2)$.

If $[A_1] \in \mathfrak{M}^{\alpha}_{k'_1,\ell'_1,X_1} \cap V_{i_1} \cap \cdots \cap V_{ip} \Rightarrow p \geq d_1 - 2r_1$ and $[A_2] \in \mathfrak{M}^{\alpha}_{k'_2,\ell'_2,X_2} \cap V_{j_1} \cap \cdots \cap V_{jq} \Rightarrow q \geq d_2 - 2r_2$. Since $[A_1]$ and $[A_2]$ are not α-twisted flat connections,

$$\dim(\mathfrak{M}^{\alpha}_{k'_1,\ell'_1,X_1} \cap V_{i1} \cap \cdots \cap V_{ip}) \quad \text{and} \quad \dim(\mathfrak{M}^{\alpha}_{k'_2,\ell'_2,X_2} \cap V_{j1} \cap \cdots \cap V_{jq})$$

are non negative.

Thus $\dim(\mathfrak{M}^{\alpha}_{k'_1,\ell'_1,X_1}) - 2p \geq 0$ and $\dim(\mathfrak{M}^{\alpha}_{k'_2,\ell'_2,X_2}) - 2q \geq 0$. Then we have

$$d_1 \leq p + 2r_1 \leq 2r_1 + \frac{1}{2}\dim(\mathfrak{M}^{\alpha}_{k'_1,\ell'_1,X_1})$$

$$d_2 \leq q + 2r_2 \leq 2r_2 + \frac{1}{2}\dim(\mathfrak{M}^{\alpha}_{k'_2,\ell'_2,X_2}).$$

So

(4.1.2)

$$\begin{aligned}
2d =\ & \dim(\mathfrak{M}^{\alpha}_{k,\ell,X}(g_{\lambda_n})) \\
&= 2d_1 + 2d_2 \leq 4r_1 + 4r_2 + \dim(\mathfrak{M}^{\alpha}_{k'_1,\ell'_1,X_1}) + \dim(\mathfrak{M}^{\alpha}_{k'_2,\ell'_2,X_2}) \\
&= 4r + \dim(\mathfrak{M}^{\alpha}_{k'_1,\ell'_1,X_1}) + \dim(\mathfrak{M}^{\alpha}_{k'_2,\ell'_2,X_2}).
\end{aligned}$$

And, by Lemma 3.8, we have

(4.1.3)
$$\begin{aligned}2d &= \dim(\mathfrak{M}^{\alpha}_{k,\ell,X}(g_{\lambda_n}))\\ &= \dim(\mathfrak{M}^{\alpha}_{k'_1,\ell'_1,X_1}) + \dim(\mathfrak{M}^{\alpha}_{k'_2,\ell'_2,X_2}) + 1 + 8\Sigma^{r}_{i=1}k_i\\ &\quad + 4\Sigma^{s}_{j=1}(2k_j + \ell_j)\\ &\geq \dim(\mathfrak{M}^{\alpha}_{k'_1,\ell'_1,X_1}) + \dim(\mathfrak{M}^{\alpha}_{k'_2,\ell'_2,X_2}) + 8r + 4s + 1.\end{aligned}$$

Thus, by (4.1.2) and (4.1.3), we have

$$\begin{aligned}&\dim(\mathfrak{M}^{\alpha}_{k'_1,\ell'_1,X_1}) + \dim(\mathfrak{M}^{\alpha}_{k'_2,\ell'_2,X_2}) + 1 + 8r + 4s\\ &\leq 2d \leq 4r + \dim(\mathfrak{M}^{\alpha}_{k'_1,\ell'_1,X_1}) + \dim(\mathfrak{M}^{\alpha}_{k'_2,\ell'_2,X_2}).\end{aligned}$$

Then $8r + 4s + 1 \leq 4r$ and $4n + 1 \leq 0$. Thus we have a contradition. Secondly suppose that the limit $[A']$ is of the form $[\theta_1, \theta_2]$ where θ_i is an α-twisted flat connection over (X_i, Σ_i), $i = 1, 2$. Then each $N(\Sigma'_i)$, $i = 1, \cdots, d$, contains one of the point z_j, $j = 1, \cdots, n$, by Alternative 3.2 and n is equal to k. But there is a $N(\Sigma'_k)$ such that $N(\Sigma'_k)$ contains none of the point z_j, $j = 1, \cdots, n$. (See the proof of Lemma 3.1.) Thus we have a contradiction. By above two steps, we conclude that the limit $[A']$ is of the form $[\theta_1, A_2]$ or $[A_1, \theta_2]$ where θ_i is an α-twisted flat connection and A_i is an α-twisted, non trivial, anti-self-dual connection over (X_i, Σ_i), $i = 1, 2$. □

4.2. Simple case

From now we will fix attention on $[A']$ of the form $[\theta_1, A_2]$. (Similary for the form $[A_1, \theta_2]$.) If $[A_2] \in \mathfrak{M}^{\alpha}_{k'_2,\ell'_2,X_2} \cap V_{i1} \cap \cdots \cap V_{ic}$ $(c \leq d)$ then dimension formula gives

(4.2.1)
$$\begin{aligned}&\dim(\mathfrak{M}^{\alpha}_{k,\ell,X}(g_{\lambda_n})) \cap V_1 \cap \cdots \cap V_d) - \dim(\mathfrak{M}^{\alpha}_{k'_2,\ell'_2,X_2} \cap V_{i1} \cap \cdots \cap V_{ic})\\ &\quad = 8k'_1 + 8\Sigma^{r}_{i=1}k_i + 8\Sigma^{s}_{j=1}k_j + 4\ell'_1 + 4\Sigma^{s}_{j=1}\ell_j\\ &\qquad - 3b^{+}_2(X_1) - 2g_1 - 2d + 2c.\end{aligned}$$

Now we use the following theorem.

THEOREM 4.2.2. [11]. *Let the self intersection number of* Σ *be* $n \neq 0$ *and let* A *be a flat* α*-twisted connection over* $(X \setminus \Sigma)$. *Then the holonomy parameter* α *is of the form* $\frac{a}{n}$ *and the instanton number and the monopole number are given by* $\ell = a$, $k = -\frac{a^2}{n}$. *If on the other hand* $\Sigma \cdot \Sigma$ *is zero then* k *and* ℓ *are zero.*

Applying Theorem 4.2.2 to surface Σ_1, since $\Sigma_1 \cdot \Sigma_1$ is 0, we conclude that k'_1 and ℓ'_1 are zero and the dimension formular (4.2.1) becomes $8\Sigma_{i=1}^r k_i + 4\Sigma_{j=1}^s (2k_j + \ell_j) - 3b_2^+(X_1) - 2g_1 - 2d + 2c$.

So we have $\dim(\mathfrak{M}^{\alpha}_{k,\ell,X}(g_\lambda) \cap V_1 \cap \cdots \cap V_d) - \dim(\mathfrak{M}^{\alpha}_{k'_2,\ell'_2,X_2} \cap V_{i1} \cap \cdots \cap V_{ic}) = 8\Sigma_{i=1}^r k_i + 4\Sigma_{j=1}^s (2k_j + \ell_j) - 3b_2^+(X_1) - 2g_1 - 2d + 2c$.

Since $\dim(\mathfrak{M}^{\alpha}_{k,\ell,X}(g_\lambda) \cap V_1 \cdots \cap V_d)$ is 0, $\dim(\mathfrak{M}^{\alpha}_{k'_2,\ell'_2,X_2} \cap V_{i1} \cap \cdots \cap V_{ic}) = 3b_2^+(X_1) + 2g_1 + 2d - 2c - 8\Sigma_{i=1}^r k_i - 4\Sigma_{j=1}^s (2k_j + \ell_j)$. Since $\dim(\mathfrak{M}^{\alpha}_{k'_2,\ell'_2,X_2} \cap V_{i1} \cap \cdots \cap V_{ic}) \geq 0$, $3b_2^+(X_1) + 2g_1 + 2d - 2c \geq 8\Sigma_{i=1}^r k_i + 4\Sigma_{j=1}^s (2k_j + \ell_j)$.

If $b_2^+(X_1) = 1$, $g_1 = 0$ and $d = c$ then $8\Sigma_{i=1}^r k_i + 4\Sigma_{j=1}^s (2k_j + \ell_j) = 0$ and the sequence $[A_n] \in \mathfrak{M}^{\alpha}_{k,\ell,X}(g_{\lambda_n}) \cap V_1 \cap \cdots \cap V_d$ converges strongly to a limit $[A'] = [\theta_1, A_2]$. In this case we have $k'_2 = k$ and $\ell'_2 = \ell$ where k'_2 is Chern number and ℓ'_2 is monopole number of $[A_2]$. So $\mathfrak{M}^{\alpha}_{k'_2,\ell'_2,X_2}$ is corresponding to $\mathfrak{M}^{\alpha}_{k,\ell,X_2}$ and $\mathfrak{M}^{\alpha}_{k,\ell,X_2} \cap V_1 \cap \cdots \cap V_d$ becomes a compact 3-dimensional manifold. In this case we can apply the following proposition.

PROPOSITION 4.2.3. [6]. *For* λ *sufficiently small, the moduli space* $\mathfrak{M}_{k,X}(g_\lambda)$ *for the Riemannian metric* g_λ *over an oriented, compact, simply connected, 4-dimensional Riemannian manifold* X *can be identified with the zero set of a smooth section* $\overline{\psi}$ *of the bundle* $\mathcal{H} \to \mathfrak{M}_{k,X_2}$ *where* $\mathcal{H} \to \mathfrak{M}_{k,X_2}$ *is an associated vector bundle with fiber* $H^+_{X_1} \otimes so(3)$ *and* $H^+_{X_1}$ *is the space of self dual harmonic forms on* X_1 *and* $\mathfrak{M}_{k,X_2}$ *is an anti-self-dual moduli space over an oriented, smooth, simply connected, compact, four manifold* X_2 *with Chern number* k.

In our case the space $\mathfrak{M}_{k,X}(g_\lambda)$ is corresponding to $\mathfrak{M}^{\alpha}_{k,\ell,X}(g_\lambda) \cap U_1(\epsilon)$ and $\mathfrak{M}_{k,X_2}$ is corresponding to $\mathfrak{M}^{\alpha}_{k,\ell,X_2}$. Thus we can construct an associated vector bundle $\mathcal{H} \to \mathfrak{M}^{\alpha}_{k,\ell,X_2}$ with fiber $H^+_{X_1} \otimes so(3)$ and the local model for the space $(\mathfrak{M}^{\alpha}_{k,\ell,X}(g_\lambda) \cap U_1(\epsilon)) \cap V_1 \cap \cdots \cap V_d$ is the zero set of a local section $\overline{\psi} : U \to \mathcal{H}$ where U is a neighbourhood in a compact 3-dimensional submanifold $\mathfrak{M}^{\alpha}_{k,\ell,X_2} \cap V_1 \cap \cdots \cap V_d$ of $\mathfrak{M}^{\alpha}_{k,\ell,X_2}$.

Then $i_1(\lambda) = \sharp I_1(\lambda) = \sharp(\mathfrak{M}^{\alpha}_{k,\ell,X}(g_\lambda) \cap U_1(\epsilon)) \cap V_1 \cap \cdots \cap V_d)$ is equal to the Poincare dual of the rational Euler class of the bundle $\mathcal{H} \to \mathfrak{M}^{\alpha}_{k,\ell,X_2} \cap V_1 \cap \cdots \cap V_d$ for all suitably small values of λ. But the rational Euler class of an odd-dimensional vector bundle is always zero.

So we deduce that $i_1(\lambda)$ is zero. Thus we can show that $i_1(\lambda) = \sharp I_1(\lambda) = 0$ when $b_2^+(X_1) = 1$ and $g_1 = 0$. (Similarly $i_2(\lambda) = \sharp I_2(\lambda) = 0$ when $b_2^+(X_2) = 1$ and $g_2 = 0$.)

From now we must show that $i_1(\lambda)$ is zero in general case (i.e. $b_2^+(X_1) \geq 0$ and $g_1 \geq 0$). Then we can prove that $i_1(\lambda)$ is zero for all case (similarly for $i_2(\lambda)$) and we conclude that our polynomial invariant $q^{\alpha}_{k,\ell,X}(g_\lambda) \equiv 0$ for suitably small values of λ.

4.3. Extended equations in general case

Next we will prove that $i_1(\lambda) = 0$ in general case ($b_2^+(X_1) \geq 0$, $g_1 \geq 0$). To prove this, we will construct a larger family of equation and show how the anti-self-dual equations $F_A^+ = 0$ over $U_1(\epsilon) = E_1^{-1}[0, \epsilon]$ can be embedded in a larger family of equations and we complete the proof of Theorem 3.11 using the extended equations. (Similarly for $U_2(\epsilon)$)

To construct a larger family of equation which contains the anti-self-dual equation $F_A^+ = 0$ over $U_1(\epsilon)$, we will consider a 3-dimensional subspace $\mathcal{S}_A$ in $\Omega^0_{z_1(\rho)}(\mathcal{G}_E)$, a self-dual 2-form $w \in H^+_{X_1} \subset \Omega^+_{X_1}$ (it is possible since $b_2^+(X_1) > 0$ by assumption) which are considered by Donaldson. And we will construct a self dual 2-form $\tau_t(A) \in \Omega^+_{X_1}(\mathcal{G}_E)$ supported in $N(\Sigma_1) \setminus \Sigma_1$. Let $\sigma > 0$ be the first eigenvalue of the laplacian Δ on the functions on X_1. And define a function R on (X_1, Σ_1), equal to 2σ on $B(p_1, r)$ and supported in $B(p_1, 2r)$ and r is a real number such that $Vol(B(p_1, 2r)) < \frac{1}{8} Vol(X_1)$ where $\rho < \frac{1}{2} r$. We define a form g on the sections s which lie in $\Omega^0_{z_1(\rho)}(\mathcal{G}_E)$ and vanish on the boundary of $Z_1(\rho)$ as

$$g(s) = \int_{Z_1(\rho)} |\nabla_A s|^2 + R|s|^2 d\mu, \quad s \in \Omega^0_{Z(\rho)}(\mathcal{G}_E)$$

where $A = A^\alpha + a \in \mathcal{A}^\alpha$, $\nabla_{A^\alpha} a \in L^p(X \setminus \Sigma)$, $p > 2$.

The associated eigenvalue problem is to find sections and constant τ such that $\Delta_A s + Rs = \tau s$. Let $\mathcal{S}_A$ be the space of sections in $\Omega^0_{Z_1(\rho)}(\mathcal{G}_E)$ spanned by equations s belong to the eigenvalues τ with $\tau < \frac{1}{2}\sigma$, vanishing on $\partial Z_1(\rho)$.

LEMMA 4.3.1. *There is ρ_0 with $\frac{1}{2}r > \rho_0 > 0$ and a function $\epsilon(\rho)$ such that if $\rho < \rho_0$, $N^{-1}\sqrt{\lambda} < \rho < 1$, $\epsilon < \epsilon(\rho)$ and $[A]$ is an α-twisted connection in $U_1(\epsilon) \subset \mathcal{B}^{\alpha}_{k,\ell,X}$ then $\mathcal{S}_A$ is 3-dimensional.*

Proof. For the proof see the paper [4], he proved the same results for the case [A] is not an α-twised cannection in $U_1(\epsilon) \subset \mathcal{B}_{k,X}$, from our all discussion we can go over word for word, as does the analogue of [4]. By Lemma 4.3.1 we have a 3-dimensional space $\mathcal{S}_A$ for all $[A] \in U_1(\epsilon)$. Next we consider a self-dual 2-form $w \in H^+_{X_1}$ and a $3b_2^+(X_1)$-dimensional space W_A for all $[A] \in U_1(\epsilon)$. Let W_A be the space $\{s \otimes w | s \in \mathcal{S}_A, w \in H^+_{X_1}\}$ for all $[A] = [A^\alpha + a] \in U_1(\epsilon)$. Then $\dim(W_A)$ is $3b_2^+(X_1)$ and we can regard $s \otimes w$ as a $\mathcal{G}_E$-valued self dual 2-form over the connected sum $(X, \Sigma) = (X_1, \Sigma_1) \sharp (X_2, \Sigma_2)$, extending by 0 outside $Z_1(1) \setminus \Sigma_1$ (for details see [4]). Finally we will construct a self-dual 2-form $\tau_t(A)$ for all $[A] \in U_1(\epsilon)$ supported on $N(\Sigma_1) \setminus \Sigma_1$; since Σ_1 is a closed, oriented, 2-dimensional surface with genus g_1 and $\Sigma_1 \setminus p_1$ is homotopic to a $2g_1$-leaved rose G_{2g_1}, $\pi_1(\Sigma_1 \setminus p_1)$ is represented by independent $2g_1$-loops $\gamma_1, \cdots, \gamma_{2g_1}$ in Σ_1 representing G_{2g_1}.

If we deform $\gamma_i \subset \Sigma_1$ into a loop γ_i' in $N(\Sigma_1) \setminus \Sigma_1$ then we may think it as a map $\gamma_i' : S^1 \to N(\Sigma_1) \setminus \Sigma_1$, $i = 1, \cdots, 2g_1$.

For each α-twisted connection $A = A^\alpha + a \in \mathcal{A}^\alpha$, $\alpha \in \left[\epsilon, \frac{1}{2} - \epsilon\right]$, we have a holonomy element $h_{\gamma_i'}(A)$ of A along the curve $\gamma_i' \subset N(\Sigma_1) \setminus \Sigma_1$, $i = 1, \cdots, 2g_1$.

For all $[A] \in U_1(\epsilon) \subset \mathcal{B}^{\alpha}_{k,\ell,X}$ define a map $h : \{\gamma_1, \cdots, \gamma_{2g_1}\} \underset{\text{homotopic}}{\cong} \{\gamma_1', \cdots, \gamma_{2g_1}'\} \to SU(2)$ such that $h(\gamma_i') = h_{\gamma_i'}(A)$ for all $[A] \in U_1(\epsilon)$. If A is a flat α-twisted connection over (X_1, Σ_1) then the map h is well-defined since $h_{\gamma_i'}(A)$ is independent of the choice of γ_i' which is homotopic to γ_i, $i = 1, \cdots, 2g_1$. However each α-twisted connection $[A] \in U_1(\epsilon)$ is close to a flat α-twisted connection over (X_1, Σ_1) part of (X, Σ) and $[A]$ converge weakly to a limit $[A']$ which is splitted in $(X_1, \Sigma_1) \amalg (X_2, \Sigma_2)$ and is an α-twisted flat connection over (X_1, Σ_1) part. Thus the map h is well-defined. Suppose $h_{\gamma_i'}(A)$ is not $-1 \in$

$SU(2)$. This is possible since we have the holonomy parameter α in the region $\left[\epsilon, \frac{1}{2} - \epsilon\right]$; by the definition of holonomy, the holonomy of each α-twisted connection $A \in \mathcal{A}^\alpha$ along the curve γ_i' is approximately $\exp 2\pi i \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}$, $\alpha \in \left[\epsilon, \frac{1}{2} - \epsilon\right]$, we have

$$h_{\gamma_i'}(A) \approx \exp 2\pi i \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} \neq \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -1 \in SU(2)$$

for all $\alpha \in \left[\epsilon, \frac{1}{2} - \epsilon\right]$.

Now we consider a map $\exp^{-1} : SU(2) \to su(2)$ which invert the exponential map when restricted to the complement of $-1 \in su(2)$. The map $\exp^{-1}$ sends $h_{\gamma_i'}(A) \in SU(2)$ to $\exp^{-1}(h_{\gamma_i'}(A)) \in SU(2)$ for all $[A] \in U_1(\epsilon)$ and $\exp^{-1}(h_{\gamma_i'}(A))$ is defined over $\gamma_i' \subset N(\Sigma_1) \setminus \Sigma_1$, $i = 1, \cdots, 2g_1$. Next consider a surface $\Sigma_1'' \subset N(\Sigma_1) \setminus \Sigma_1$ which is represented by $\{\gamma_1', \cdots, \gamma_{2g_1}'\}$ and a small tubular neighbourhood $N\epsilon'(\gamma_i')$ of γ_i' in $N(\Sigma_1) \setminus \Sigma_1$ which is diffeomorphic to a disk $D_{\epsilon'}$-bundle over γ_i' where $D_{\epsilon'}$ is a disk with small radius $\epsilon' > 0$.

We can extend the value $\exp^{-1}(h_{\gamma_i'}(A))$ to a small tubular neighbourhood of γ_i' using the parallel transport along the radial geodesics.

Now we consider a self-dual 2-form v_i such that $\mathrm{supp}(v_i) \subset N_{\epsilon'}(\gamma_i')$, $i = 1, \cdots, 2g_1$. Then $v_1, \cdots, v_{2g_1}$ are linearly independent. Using the 2-forms, define a self-dual 2-form v as $v = \Sigma_{i=1}^{2g_1} \varphi_i v_i$ where φ_i is partition of unity supported in $N_{\epsilon'}(\gamma_i')$. Then we can define a section $\tau(v_i, \gamma_i, A) \in \Omega_X^+(\mathcal{G}_E)$ by $\tau(v_i, \gamma_i, A) = \exp^{-1}(h_{\gamma_i'}(A)) \otimes v$ where the loop $\gamma_i' \in N(\Sigma_1) \setminus \Sigma_1$ is a deformation of γ_i $(i = 1, \cdots, 2g_1)$ and the set $\{\gamma_1, \cdots, \gamma_{2g_1}\}$ is the standard basis of the free group $\pi_1(\Sigma_1 \setminus p_1)$.

The section $\tau(v_i, \gamma_i, A)$ is a $\mathcal{G}_E$-valued self-dual 2-form, supported in a small tubular neighbourhood $N_{\epsilon'}(\gamma_i')$ of γ_i' in $N(\Sigma_1) \setminus \Sigma_1$.

For a vector $t = (t_1, \cdots, t_{2g_1})$ in a compact $2g_1$-dimensional ball $B^{2g_1}(\delta)$ with small radius $\delta > 0$ we consider

$$\tau_t(A) = \Sigma_{i=1}^{2g_1} t_i \tau(v_i, \gamma_i, A) \quad \text{for all} \quad [A] \in U_1(\epsilon).$$

Then $\tau_t(A)$ becomes a $\mathcal{G}_E$-valued self-dual 2-form supported in $N(\Sigma_1) \setminus \Sigma_1$.

Now we can construct a larger family of equation which contains the anti-self dual equation $F^+(A) = 0$ over $U_1(\epsilon)$ as follows;

(4.3.2)

$$F^+(A) + s \otimes w + \tau_t(A) = 0 \quad \text{in the three variables} \quad (A, s, t)$$
$$\text{where} \quad s \otimes w \in W_A, \quad t = (t_1, \cdots, t_{2g_1}) \in \mathcal{B}^{2g_1}(\delta).$$

The extended equation $F^+(A) + s \otimes w + \tau_t(A) = 0$ is gauge invariant and we can pass to the corresponding quotient space. Consider a space $\mathcal{C}$ whose points consist of $([A], s, t)$ where $[A] \in U_1(\epsilon)$, $s \in \mathcal{S}_A$ and $t = (t_1, \cdots, t_{2g_1}) \in \mathcal{B}^{2g_1}(\delta) \subset \mathbb{R}^{2g_1}$. Then there is a projection map $\pi : \mathcal{C} \to U_1(\epsilon) \subset \mathcal{B}^{\alpha}_{k,\ell,X}$ sending $([A], s, t)$ to $[A]$.

Recall that the anti-self dual equation $F^+(A) = 0$ can be viewed as the zero set of a section of an infinite dimensional bundle $\mathcal{F} = \mathcal{A}^\alpha \underset{\mathcal{G}}{\times} \Omega^+_X(\mathcal{G}_E)$ over $\mathcal{B}^{\alpha}_{k,\ell,X}$. (Here $\mathcal{F} = \mathcal{A}^\alpha \underset{\mathcal{G}}{\times} \Omega^+_X(\mathcal{G}_E) = (\mathcal{A}^\alpha \times \Omega^+_X(\mathcal{G}_E))/\sim$ where $(Ag, g^{-1}\psi) \sim (A, \psi)$ for all $\psi \in \Omega^+_X(\mathcal{G}_E)$, $A \in \mathcal{A}^\alpha$, and $g \in \mathcal{G}$.)

The fiber of $\mathcal{F}$ over $[A] \in U_1(\epsilon)$ is a copy of $\Omega^+_X(\mathcal{G}_E)$ and we let $\phi([A], s, t) = ([A], F^+(A) + s \otimes w + \tau_t(A))$ then we can regard ϕ as a section of $\pi^*(\mathcal{F})$ over $\mathcal{C}$. This is possible because $\phi([A], s, t) = ([A], F^+(A) + s \otimes w + \tau_t(A)) = (gAg^{-1},\ g(F^+(A) + s \otimes w + \tau_t(A))g^{-1}) = (Ag^{-1}, g(F^+(A) + s \otimes w + \tau_t(A))) = (A,\ F^+(A) + s \otimes w + \tau_t(A))$ over $\pi^*\mathcal{F} \to \mathcal{C} \cong \bigcup_{[A]} ([A] \times \mathcal{S}_A \times \mathcal{B}^{2g_1}(\delta))$.

$$\begin{array}{ccc} \pi^*\mathcal{F} & & \mathcal{F} \\ \downarrow & & \downarrow \\ \phi^{-1}(0) \subset \mathcal{C} & \xrightarrow{\ \pi\ } & U_1(\epsilon), \\ ([A], s, t) & & [A] \end{array}$$

Diagram 4.3.3.

LEMMA 4.3.4. *The section ϕ of $\pi^*\mathcal{F}$ over $\mathcal{C}$ is Fredholm of index $2d + 3 + 2g_1$.*

Proof. We can construct local models for $\mathcal{C}$. We write the elements in $U_1(\epsilon)$ in a neighbourhood of $[A^\alpha]$ using the standard slice;

$$A = A^\alpha + a, \quad d^*_{A^\alpha} a = 0, \quad A^\alpha = A^0 + \sqrt{-1}\beta(r) \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} \eta,$$

where A^0 is an $SU(2)$ connection on the bundle $E \to (X, \Sigma)$ and β is a smooth cut off function equal to 1 in a neighbourhood of 0, equal to 0 for $r \geq \frac{1}{2}$, and $\sqrt{-1}\eta$ is a connection 1-form for the circle bundle. (For details see 2.1 and 2.2.)

Let $\mathcal{S}$ be a bundle over $U_1(\epsilon)$ with fiber $\mathcal{S}_A$ for all $[A]$ in $U_1(\epsilon)$. Define γ_A as a spectral formula $\int_\wedge (\Delta_A + R - z)^{-1} dz$ where $\wedge$ is a contour in the complex plane running around the interval $[0, \frac{1}{2}\sigma]$ and not meeting the spectrum of $\Delta_A + R$ and $(\Delta_A + R - z)^{-1}$ is a Green's operator. (That is $(\Delta_A + R - z)^{-1}(\zeta)$ is equal to the unique solution x of $\Delta x = \zeta - H(\zeta)$ in $(H^0)^\perp$ for all ζ in $\Omega^0_{Z_1(\rho)}(\mathcal{G}_E)$ to $\mathcal{S}_A$; to prove this we must show that $(\Delta_A + R)(\gamma_A(\zeta)) < \frac{1}{2}\sigma \cdot \gamma_A(\zeta)$ for all ζ in $\Omega^0_{Z_1(\rho)}(\mathcal{G}_E)$. This is true for all contours in the complex plane running around the interval $[0, \frac{1}{2}\sigma]$ and not meeting the spectrum of $\Delta_A + R$. (See [4]). Now we identify the fibers of $\mathcal{S}$ in a small neighbourhood with $\mathcal{S}_{A^\alpha}$ using the restriction of γ_{A^α} to $\mathcal{S}_{A^\alpha + a}$ for all $A = A^\alpha + a$ in the small neighborhood of A^α. (That is $\gamma_{A^\alpha}|_{\mathcal{S}_{A^\alpha+a}} : \mathcal{S}_{A^\alpha + a} \to \mathcal{S}_{A^\alpha}$, $\mathcal{S}_{A^\alpha+a}$, $\mathcal{S}_{A^\alpha} \subset \Omega^0_{Z_1(\rho)}(\mathcal{G}_E)$.)

Consider

$$\begin{aligned}
\phi([A], s, t) &= F^+(A) + \gamma_A(s) \otimes w + \tau_t(A) \\
&= F^+(A^\alpha + a) + \gamma_{A^\alpha + a}(s) \otimes w + \tau_t(A^\alpha + a) \\
&= F^+(A^\alpha) + d^+_{A^\alpha\, a} + [a, a]^+ + \gamma_A(s) \otimes w + \tau_t(A)
\end{aligned}$$

where $s \in \mathcal{S}_{A^\alpha}$, $t \in B^{2g_1}(\delta)$.

Let $\delta\gamma_A$ be the derivative of γ_A with respect to a, evaluated at $a = 0$. This gives

$$\begin{aligned}
\delta\gamma_A &= \frac{d}{dt}|_{t=0}\Big(\int_\wedge (\Delta_{A^\alpha + ta} + R - z)^{-1} dz\Big) \\
&= \int_\wedge \frac{d}{dt}|_{t=0}(\Delta_{A^\alpha + ta} + R - z)^{-1} dz \\
&= \int_\wedge -(\Delta_{A^\alpha} + R - z)^{-2} \cdot \frac{d}{dt}|_{t=0}(\Delta_{A^\alpha + ta}) dz \\
&= -\int_\wedge (\Delta_{A^\alpha} + R - z)^{-2} \delta\Delta_A dz = -\int_\wedge G_z \delta\Delta_A G_z dz
\end{aligned}$$

where $G_z = (\Delta_{A^\alpha} + R - Z)^{-1}$ and $\delta\Delta_A = \frac{d}{dt}|_{t=0}(\Delta_{A^\alpha+ta})$ is the derivative of Δ_A with respect to a evaluated at $a = 0$.

We then have

$$(4.3.5)\quad \begin{aligned}(d\phi)_0(a,s,t) &= d^+_{A^\alpha}a + s\otimes w - \int_\wedge G_z\delta\Delta_A G_z(s)dz\otimes w + \Sigma_{i=1}^{2g_1}t_i\tau(v_i,\gamma_i,A^\alpha)\\ &= d^+_{A^\alpha}a + s\otimes w - \int_\wedge G_z(d^*_{A^\alpha} + a^* d_{A^\alpha})G_z(s)dz\otimes w\\ &\quad + \Sigma_{i=1}^{2g_1}t_i\tau(v_i,\gamma_i,A^\alpha)\end{aligned}$$

where $a\in ker d^*_{A^\alpha}, s\in\mathcal{S}_{A^\alpha}$ and $t = (t_1,\cdots,t_{2g_1})\in\mathcal{B}^{2g_1}(\delta)$, $\delta > 0$.

For fixed z in $\wedge$ and s in $\mathcal{S}_{A^\alpha}$ the map

$$(4.3.6)\qquad a\mapsto G_z(d^*_{A^\alpha}a(G_zs) + a^*d_{A^\alpha}(G_zs))$$

is compact.

We call a section Fredholm if it is represented in the local trivialisations by maps with Fredholm derivatives and we have a fact that a sum $F+K$ of a Fredholm operator F and a compact operator K is also Fredholm and index

$$(4.3.7)\qquad \text{index}(F+K) = \text{index}(F).$$

By (4.3.5), (4.3.6) and (4.3.7) we have index $(d\phi)$ = index(ϕ) = index(h) where h is the map $(a,s,t)\to d^+_{A^\alpha}a + s\otimes w + \tau_t(A^\alpha)$. And the map h is Fredholm of index(h) = index$(d^+_{A^\alpha}) + 3 + 2g_1 = 2d + 3 + 2g_1$. Thus we conclude that the section ϕ of $\pi^*\mathcal{F}$ over $\mathcal{C}$ is Fredholm of index $2d + 3 + +2g_1$. □

To achieve transversality we must construct a family of perturbations - section of $\mathcal{F}$.

Let h be a monotone cut off function, equal to 1 on $[0,B]$ and supported in $[0,2B]$ for a finite real number. For all $[A]\in U_1(\epsilon)$ define $h_i(A) = h(\int_{G'_i}|F(A)|^p d\mu)$. We are now able to define a family of perturbations parametrised by a ball $B^s(\delta') \subset \mathbb{R}^s$; For a vector $x = (x_1, \cdots, x_s) \in B^s(\delta')$ we let

$$\sigma_x(A) = \Sigma_{i=1}^s x_i h_i(A) p_i(A)$$

where $p_1(A), \cdots, p_s(A)$ are sections of $\pi^*\mathcal{F}$ which are supported on a set $\{G'_1, \cdots, G'_s | G'_i$ is a neighbourhood of a loop ℓ_i in $(X \setminus \Sigma), G'_i \cap G'_j = 0$ for $i \neq j\}$. And $p_1(A), \cdots, p_s(A)$ generates the harmonic representitive of $H^2_{A,s,t} = \Omega^+_X(\mathcal{G}_E)/Imd\phi$.

So we conclude that $\sigma_x(A)$ is a well-defined self-dual 2-form which generates $H^2_{A,s,t} \subset \Omega^+_X(\mathcal{G}_E)$. (For details see [4])

REAMRK 4.3.8.

(1) The loops $\ell_1, \cdots, \ell_s$ can, by general position, be taken to avoid the surfaces $\Sigma'_1, \cdots, \Sigma'_d$.

(2) Let ϕ' be a section of $\pi^*\mathcal{F}$ over $\mathcal{C}$ such that $\phi'([A], s, t) = F^+(A) + s \otimes w + \tau_t(A) + \sigma_x(A)$. Since ϕ' differ from ϕ by a compact perturbation term $\sigma_x(A)$, the section ϕ' has Fredholm of index $2d + 3 + 2g_1$.

4.4. End of the proof of Theorem 3.11

In this section we will complete the proof of Theorem 3.11 and hence a Vanishing theorem. We again consider the space $U_1(\epsilon)$ of α-twisted connections over $(X, \Sigma) = (X_1, \Sigma_1) \natural_\lambda (X_2, \Sigma_2)$ which are almost flat over most of $Z_1(\rho)$, and the bundle $\mathcal{C} \xrightarrow{\pi} U_1(\epsilon) \subset \mathcal{B}^\alpha_{k,\ell,X}$. With $w \in H^+_{X_1}$ and $x \in B^s(\delta')$ fixed, we can now consider the extended equation $F^+(A) + s \otimes w + \tau_t(A) + \sigma_x(A) = 0$ over the extended space $\mathcal{C}$.

We denote the solution space by $\mathcal{L}^\alpha_{k,\ell,X}(\lambda)$. Then $\mathcal{L}^\alpha_{k,\ell,X}(\lambda)$ is a $2d + 3 + 2g_1$-dimensional manifold. If we consider the intersection of $\mathcal{L}^\alpha_{k,\ell,X}(\lambda)$ with the zero section in the bundle $\mathcal{C} \xrightarrow{\pi} U_1(\epsilon)$, we can regard it as being obtained from the singular moduli space $\mathfrak{M}^\alpha_{k,\ell,X}(g_\lambda) \cap U_1(\epsilon)$ by perturbing the anti-self dual equation $F^+(A) = 0$ by the term $\sigma_x(A)$.

With V_i fixed, $i = 1, \cdots, d$, we consider the intersection $\mathcal{L}^\alpha_{k,\ell,X}(\lambda) \cap V_1 \cap \cdots \cap V_d$ and denote it by $S(\lambda)$. Then $S(\lambda)$ is a $3 + 2g_1$-dimensional manifold.

Let $I'_1(\lambda)$ be the intersection of $S(\lambda)$ with the zero section in the bundle $\mathcal{C} \xrightarrow{\pi} U_1(\epsilon)$; it is obtained from the intersection $I_1(\lambda) = (\mathfrak{M}^{\alpha}_{k,\ell,X}(g_\lambda) \cap U_1(\epsilon)) \cap V_1 \cap \cdots \cap V_d$ by perturbing the equation $F^+(A) = 0$ by the term $\sigma_x(A)$.

By these constructions we conclude that $S(\lambda)$ is a $(3+2g_1)$-dimensional manifold which contains 0-dimensional space $I'_1(\lambda)$ and $\sharp I'_1(\lambda)$ is equal to $\sharp I_1(\lambda) = i_1(\lambda)$ for all small values of λ.

$$
\begin{array}{ccccc}
 & \pi^*\mathcal{F} & & \mathcal{F} = \mathcal{A}^{\alpha} \times_{\mathcal{G}} \Omega^+_X(\mathcal{G}_E) & \\
 & \downarrow & & \downarrow & \\
 & \mathcal{C} & \xrightarrow{\pi} & U_1(\epsilon) & \\
 & \cup & & \cup & \\
\mathcal{L}^{\alpha}_{k,\ell,x}(\lambda) & = (\phi')^{-1}(0) & & (F^+_A)^{-1}(0) = \mathfrak{M}^{\alpha}_{k,\ell,X}(g_\lambda) \cap U_1(\epsilon) & \\
 & \cup & & \cup & \\
 & S(\lambda) & & I_1(\lambda) & \\
 & \cup & & & \\
 & I'_1(\lambda) & & &
\end{array}
$$

Diagram 4.4.1

To complete the proof of Theorem 3.11, we will apply the Euler number argument. But it can not be applied if our manifold containing $I'_1(\lambda)$ is not compact. In general the $3+2g_1$-dimensional manifold $S(\lambda)$ which contains $I'_1(\lambda)$ is not compact and so we must show that, if λ is small enough, there is a compact $3+2g_1$-dimensional submanifold of $S(\lambda)$ which contains $I'_1(\lambda)$. Finally let $S^*(\lambda)$ be the union of the path components of $S(\lambda)$ which contain points of $I'_1(\lambda)$.

Remark 4.4.2 For small values of λ, $I'_1(\lambda)$ is contained in the space whose elements consist of equivalence classes $[A] \in \mathfrak{M}^{\tilde{\alpha}}_{k,\ell,X}(g_\lambda) \cap U_1(\epsilon)$ such that $E_1(A) \in \left[0, \frac{1}{4}\epsilon\right]$ where $\mathfrak{M}^{\tilde{\alpha}}_{k,\ell,X}(g_\lambda)$ is obtained from the moduli space $\mathfrak{M}^{\alpha}_{k,\ell,X}(g_\lambda)$ by perturbing the anti-self-dual equation by the term $\sigma_x(A)$.

Now we have the following key Lemma.

LEMMA 4.4.3. *For small values of* λ, $S^*(\lambda)$ *is contained in the space* $\{([A], s, t) \in S(\lambda) | E_1(A) \in [0, \frac{1}{2}\epsilon]\}$.

Proof. The proof is similar with Donaldson's construction (see [4]) but we have some difference. We will check the difference caused by our special cases. We must show that if λ is small enough, the component of $S(\lambda)$ which reach the level $E_1 = \frac{1}{2}\epsilon$ can not be joined by paths in $S(\lambda)$ to $I_1'(\lambda)$.

So suppose we have a sequence $\lambda_n \to 0$ and points $([A_n], s_n, t_n)$ in $S(\lambda_n)$ which converge to a limit $([A'], s', t')$ with $E_1(A') = \frac{1}{2}\epsilon$ on the complement of a finite set $\{z_1, \cdots, z_n\}$ in $Z_1(r) \amalg Z_2(r)$ where $Z_i(r) = X_i \setminus B(p_i, r)$, $i = 1, 2$.

Since $E_1(A') = \frac{1}{2}\epsilon \neq 0$, A' is a nontrivial, α-twisted, anti-self-dual connection over either component $(X_1, \Sigma_1), (X_2, \Sigma_2)$. And it satisfies the following equation

$$F^+(A') + s' \otimes w + \tau_{t'}(A') + c^*\sigma_x(A') = 0 \tag{4.4.4}$$

where $c^*\sigma_x(A')$ is a "contraction"of a section σ_x by $c \in \mathbb{R}^s$.

REMARK 4.4.5. The equation of (4.4.4) is well defined; Recall that the points $([A_n], s_n, t_n) \in S(\lambda_n)$ satisfy the extended equation $F^+(A_n) + s_n \otimes w + \tau_{t_n}(A_n) + \sigma_x(A_n) = 0$. Since $[A_n]$ converge to $[A']$ on the complement of a finite set $\{z_1, \cdots, z_n\}$ in $(Z_1(r) \amalg Z_2(r))$ $\sigma_x(A_n)$ is supported away from the points in $\{z_1, \cdots, z_n\}$ for large n.

Going to a subsequence we can suppose that;

$$\sigma_x(A_n) \to \Sigma_{i=1}^s c_i x_i h_i(A') p_i(A')$$

where $c_i = 0$ if there is a point z_i in the interior of G_i'.

$c_i \in (0, 1)$ if there is a point z_i on the boundary of G_i'.

$c_i = 1$ if no points z_i lies in the closure of G_i'.

and $x = (x_1, \cdots, x_s) \in B^s(\delta') \subset \mathbb{R}^s$ and $B^s(\delta')$ is a small ball of radius $\delta' > 0$ in $\mathbb{R}^s$.

We define "contraction"of a section σ_x by c to be the section;

$$c^*\sigma_x(A') = \Sigma_{i=1}^s c_i x_i h_i(A') p_i(A'). \tag{4.4.6}$$

Suppose that the m points z_r on where convergence fails, p points lie on at least one of the surfaces Σ_i', $i = 1, \cdots, d$, and q points lie in closure of one of the disjoint neigbourhood G_i' in (X, Σ) used to define σ_x. Then if k' is the Chern class of $[A']$ we have $p + q + k' \leq k$.

Now the size of the set defining the contraction in (4.4.6) is q. The space of solutions $(([A'], s', t'), c) \in \mathcal{C} \times \mathbb{R}^s$ to equation (4.4.4) has dimension $8k' + 4\ell' - 3(1 + b_2^+(X)) - (2g - 2) - 1 + 3 + 2g_1 + q$.

(4.4.7) If $[A'] \in \mathfrak{M}^{\alpha}_{k',\ell',Y} \cap V_{i_1} \cap \cdots \cap V_{ic}$ then $c \geq d - 2p$.

Since the limit $[A']$ is a non trivial, α-twisted, anti-self-dual connection over either component $(X_1, \Sigma_1), (X_2, \Sigma_2)$, $\dim(\mathfrak{M}^{\alpha}_{k',\ell',Y} \cap V_{i_1} \cap \cdots \cap V_{ic}) \geq 0$. Thus we have

(4.4.8) $$\dim(\mathfrak{M}^{\alpha}_{k',\ell',Y}) \geq 2c.$$

By (4.4.7) and (4.4.8)

$$\dim(\mathcal{L}^{\alpha}_{k,\ell,X}(\lambda_n)) = 2d + 3 + 2g_1 \leq 2c + 4p + 3 + 2g_1 \leq \dim(\mathfrak{M}^{\alpha}_{k',\ell',Y}) + 4p + 3 + 2g_1.$$

So we have

(4.4.9) $$2d \leq \dim(\mathfrak{M}^{\alpha}_{k',\ell',Y}) + 4p$$

Since $2d = \dim(\mathfrak{M}^{\alpha}_{k,\ell,X}(g_{\lambda_n})) = 8k + 4\ell - 3(1 + b_2^+(X)) - (2g - 2)$ and by (4.4.9) we get that

(4.4.10) $$8k + 4\ell \leq 8k' + 4\ell' - 1 + 4p.$$

Now decompose the number r of the points $\{z_1, \cdots, z_m\}$ on which convergence fails in $X \setminus \Sigma$ part and s in Σ part $(r + s = m)$. Recall that p is the number of the points z_i which lie on at least one of the surfaces Σ_i', $i = 1, \cdots, d$, and q is the number of the points z_i which lie in the closure of one of the disjoint neighbourhood G_i', $i = 1, \cdots, s$, in (X, Σ) (used to define σ_x). Thus we have

(4.4.11) $$p \leq r \quad \text{and} \quad q \leq m - p.$$

In section 3, we get that $8k + 4\ell = 8k' + 8\Sigma_{i=1}^{r} k_i + 8\Sigma_{j=1}^{s} k_j + 4\ell' + 4\Sigma_{j=1}^{s} \ell_j \geq 8k' + 4\ell' + 8r + 4s$.

Then, by (4.4.10) and (4.4.11), we have

$$8k' + 4\ell' + 8r + 4s \leq 8k + 4\ell \leq 8k' + 4\ell' - 1 + 4p.$$

So $8r + 4s \leq -1 + 4p \leq 4p + q$ $(q \geq 0)$ and by (4.4.11) we have $8r + 4s \leq 4p + q \leq 4r + m - p$. Then $3m + p \leq 0$ and we deduce that $m = p = 0$. Since $0 \leq q \leq m$, $p = q = m = 0$. So we conclude that as $\lambda_n \to 0$ the sequence $([A_n], s_n, t_n) \in S(\lambda_n)$ converges strongly to a limit $([A'], s', t')$ which satisfies $E_1(A') = \frac{1}{2}\epsilon$ and the equation $F^+(A') + s' \otimes w + \tau_{t'}(A') + \sigma_x(A') = 0$.

Specially the Chern number k' of the limit $[A']$ becomes k and the monopole number ℓ' of the limit $[A']$ becomes ℓ where k and ℓ are Chern number and monopole number $[A_n]$ respectively.

In this case we can apply Donaldson's argument (see [4]) and the result is following; let $F : T \to \mathfrak{M}^{\alpha}_{k,\ell,Y}$ be a fiber bundle with fiber $SO(3)$ consisting of isomorphism classes of pairs $([A'], \rho)$ where $\mathfrak{M}^{\alpha}_{k,\ell,Y}$ is a singular, α-twisted moduli space over $Y = (X_1, \Sigma_1) \amalg (X_2, \Sigma_2)$ with the Chern number k and the monopole number ℓ.

Let $\mathcal{O}$ be an open set in $\mathfrak{M}^{\alpha}_{k,\ell,Y}$ with compact closure and $\mathcal{D}$ be a 2η-neighbourhood of $\mathcal{O}$ for a small positive number η. We introduce a notion of strong convergence; for a given λ_n and $\eta > 0$ say that $[A_n] \in \mathfrak{M}^{\alpha}_{k,\ell,X}(g_{\lambda_n})$ is (η, λ_n) close to $[A']$ if there is a bundle map $\varphi; E|_{Z(r)} \to E'|_{Z(r)}$ such that $||A_n - \varphi^*(A')||_{L^p(Z(r))} < \eta$ where $Z(r) = Z_1(r) \amalg Z_2(r)$ for suitably small r and E' is an $SU(2)$-bundle over Y with Chern number k and monopole number ℓ.

Now we can consider smooth maps τ_λ from the open set $F^{-1}(\mathcal{D})$ in T to the moduli space $\mathfrak{M}^{\alpha}_{k,\ell,X}(g_\lambda)$ such that $\tau_\lambda([A'], \rho) \in \mathfrak{M}^{\alpha}_{k,\ell,X}(g_\lambda)$ is (η, λ) close to $[A']$ and we have two maps from $F^{-1}(\mathcal{D})$ to $(\mathcal{B}^{\alpha}_{k,\ell})^*_{z(r)}$ given by the composites;

$$\begin{array}{ccc} F^{-1}(\mathcal{D}) \subset T & \xrightarrow{\tau_\lambda} & \mathfrak{M}^{\alpha}_{k,\ell,X}(g_\lambda) \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle R_{z(r)}} \\ (\mathfrak{M}^{\alpha}_{k,\ell,Y})^* & \xrightarrow{R_{z(r)}} & (\mathcal{B}^{\alpha}_{k,\ell})^*_{z(r)} \end{array}$$

where $R_{Z(r)}$ is a restriction map to $Z(r) = Z_1(r) \amalg Z_2(r)$. From this we have the following proposition.

PROPOSITION 4.4.12. *Suppose that* $b_2^+(X_i) > 0$, $i = 1, 2$. *Let* $\mathcal{O}$ *be a precompact open set in* $\mathfrak{M}^{\alpha}_{k,\ell,Y}$. *For* $\eta < \eta(\mathcal{O})$ *and* $\lambda < \lambda(\mathcal{O})$ *there is a diffeomorphism* τ_λ *from* $F^{-1}(\mathcal{D}) \subset T$ *to an open set* $S_{\lambda,\eta}$ *in* $\mathfrak{M}^{\alpha}_{k,\ell,X}(g_\lambda)$ *such that*

(1) $\tau_\lambda([A'], \rho)$ *is* (η, λ) *close to* $[A']$
(2) $S_{\lambda,\eta}$ *contains all points in* $\mathfrak{M}^{\alpha}_{k,\ell,X}(g_\lambda)$ *which are* (η, λ) *close to a point of* $\mathcal{O}$
(3) *As* $\lambda \to 0$, *the composites* $R_{z(r)}\tau_\lambda$ *converge in* C^1 *to* $R_{z(r)}F$.

Proof. See proposition 4.6.[4] □

By Proposition 4.4.12 we deduce that for small values of λ we can construct from $([A'], s')$ of the limit $([A'], s', t')$ of the points $([A_n], s_n, t_n)$ a 3-dimensional family $S_{\lambda,\eta}$ parametrized by $SO(3)$.

Let $S'_{\lambda,\eta}$ be the space $\{([A], s, t) \in S(\lambda) = \mathcal{L}^{\alpha}_{k,\ell,X}(\lambda) \cap V_1 \cap \cdots \cap V_d | \ ([A], s) \in S_{\lambda,\eta},\ t \in B^{2g_1}(\delta) \subset \mathbb{R}^{2g_1}$ for small number $\delta\} \subset S(\lambda)$. Then we conclude that $S'_{\lambda,\eta}$ is a complete component of $S(\lambda)$ and for large n the sequence $([A_n], s_n, t_n)$ lie in $S'_{\lambda,\eta}$.

But we have that for all $[A]$ of $([A], s, t) \in S'_{\lambda,\eta}$, $E_1(A)$ lies in partly above and partly below the level $E_1 = \frac{1}{2}\epsilon$ and as $\lambda \to 0$ the variation of E_1 goes to 0 – so with ϵ fixed and λ approaching zero, this component $S'_{\lambda,\eta}$ eventually lies, say, between the levels $\frac{3}{8}\epsilon$ and $\frac{5}{8}\epsilon$. Such $S'_{\lambda,\eta}$, however, can not contain points in $\mathfrak{M}^{\alpha}_{k,\ell,X}(g_\lambda) \cap U_1(\epsilon)$, because, by Lemma 4.1.1 and Remark 4.4.2, we know that E_1 goes to 0 on $\mathfrak{M}^{\alpha}_{k,\ell,X}(g_\lambda) \cap U_1(\epsilon)$ as $\lambda \to 0$, so E_1 is eventually less than $\frac{1}{4}\epsilon$ for suitably small values of λ. Thus the sequence $([A_n], s_n, t_n) \in S(\lambda_n)$ converging to $([A'], s', t')$ with $E_1(A') = \frac{1}{2}\epsilon$ can not be joined by paths in $S(\lambda)$ to $I'_1(\lambda) \subset E_1^{-1}\ [0, \frac{1}{4}\epsilon]$ for small values of λ. This complete the proof of Lemma 4.4.3 □

Finally we will complete the proof of Theorem 3.11. We can fix λ in accordance with Lemma 4.4.3 and consider the space $\mathcal{L}^{\alpha}_{k,\ell,X}(\lambda) = \{([A], s, t) \in \mathcal{C} | F^+(A) + s \otimes w + \tau_t(A) + \sigma_x(A) = 0\}$. Then we can say that $\mathcal{L}^{\alpha}_{k,\ell,X}(\lambda)$ is a $2d + 3 + 2g_1$-dimensional smooth manifold and $S(\lambda) = \mathcal{L}^{\alpha}_{k,\ell,X}(\lambda) \cap V_1 \cap \cdots V_d$ is a $3 + 2g_1$-dimensional smooth manifold. Then we can show that the space $\{([A], s, t) \in S(\lambda) | E_1(A) \in [0, \frac{1}{2}\epsilon]\} \subset S(\lambda)$ is a compact $3 + 2g_1$-dimensional subset of $S(\lambda)$. By Lemma 4.4.3 the union of path components $S^*(\lambda)$ containing all the points of $I'_1(\lambda)$ is contained in $\{([A],s,t)\in S(\lambda)|E_1(A)\in[0,\frac{1}{2}\epsilon]\}$. Thus $S^*(\lambda)$ is a closed $3+2g_1$-dimensional manifold for suitably small values of λ. By the definition of $I_1'(\lambda)$, $I_1'(\lambda)$ is the intersection of $S(\lambda)$ with the zero section in the bundle $\mathcal{C}\xrightarrow{\pi}U_1(\epsilon)$ (See Diagram 4.4.1). Thus $I_1'(\lambda)$ is the set of zeros of a section of a $3+2g_1$-dimensional bundle over a $3+2g_1$-dimensional compact, oriented manifold $S^*(\lambda)$. So $I_1'(\lambda)$ represents the Euler class of this bundle and hence $\sharp I_1'(\lambda)$ is 0 by the Euler number argument. Because $I_1'(\lambda)$ is obtained from $I_1(\lambda)=(\mathfrak{M}^{\alpha}_{k,\ell,X}(g_\lambda)\cap U_1(\epsilon))\cap V_1\cap\cdots\cap V_d$ by perturbing the term σ_x, we have $\sharp I_1'(\lambda)=\sharp I_1(\lambda)=i_1(\lambda)$ and hence $i_1(\lambda)$ is 0 for all suitably small values of λ.

Similarly for $\sharp I_2(\lambda)=i_2(\lambda)$ we conclude that $i_2(\lambda)$ is 0. Thus we prove that the polynomial invariant $q^{\alpha}_{k,\ell,X}(g_\lambda)$ $([\Sigma_1],\cdots,[\Sigma_d])=i_1(\lambda)+i_2(\lambda)$ is 0 for all suitably small values of λ. □

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