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ON SEMI–KAEHLER MANIFOLDS WHOSE TOTALLY REAL BISECTIONAL CURVATURE IS BOUNDED FROM BELOW

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Introduction

R.L. Bishop and S.I. Goldberg [3] introduced the notion of totally real bisectional curvature $B(X,Y)$ on a Kaehler manifold M. It is determined by a totally real plane $[X,Y]$ and its image $[JX,JY]$ by the complex structure J, where $[X,Y]$ denotes the plane spanned by linealy independent vector fields X, and Y. Moreover the above two planes $[X,Y]$ and $[JX,JY]$ are orthogonal to each other. And it is known that two orthonormal vectors X and Y span a totally real plane if and only if X,Y and JY are orthonormal.

C.S. Houh [8] showed that $(n \geq 3)$-dimensional Kaehler manifold with constant totally real bisectional curvature is congruent to a complex space form of constant holomorphic sectional curvature $H(X) = c$, where $H(X)$ is determined by the holomorphic plane $[X,JX]$. Also M.Barros and A.Romero[2] asserted that for a connected indefinite Kaehler manifold M with complex dimension $n \geq 3$ to be an indefinite complex space form with holomorphic sectional curvature c is if and only if it has constant totally real bisectional curvature $\frac{c}{2}$ at any point. Thus in section 2 let us recall the notion of totally real bisectional curvature and calculate the totally real bisectional curvature of the indefinite complex space form $M_s^n(c)$ and the complex quadric Q^n in a complex hyperbolic space $CH^{n+1}(c)$.

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On the other hand, S.I. Goldberg and S. Kobayashi [6] introduced the notion of holomorphic bisectional curvature $H(X,Y)$, which is determined by two holomorphic planes $[X, JX]$ and $[Y, JY]$, and asserted that a complex projective space $CP^n(c)$ is the only compact Kaehler manifold with positive holomorphic bisectional curvature $H(X,Y)$ and constant scalar curvature. If we compare the notion of $B(X,Y)$ with $H(X,Y)$ and $H(X)$, the holomorphic bisectional curvature $H(X,Y)$ turns out to be totally real bisectional curvature $B(X,Y)$ (resp. holomorphic sectional curvature $H(X)$) when two holomorphic planes $[X, JX]$ and $[Y, JY]$ are orthogonal to each other (resp. coincides with each other). From this it follows that the positiveness of $B(X,Y)$ is weaker than the positiveness of $H(X,Y)$, because $H(X,Y) > 0$ implies that both of $B(X,Y)$ and $H(X)$ are positive but we do not know whether $B(X,Y) > 0$ implies $H(X,Y) > 0$ or not.

In section 1 we introduce a local complex exterior derivative formula for semi-Kaehler submanifolds of indefinite complex space forms, which will be used to prove our main result. And in section 2 let us find a relation between the totally real bisectional curvature and the sectional curvature of semi-Kaehler manifolds M. Also the further relation between the totally real bisectional curvature and the holomorphic sectional curvature of M will be treated. Moreover in this section we calculate the totally real bisectional curvature of the complex quadric Q^n immersed in a complex projective space $CP^{n+1}(c)$ with the constant holomorphic sectional curvature c. In section 3 we will prove that a complete Kaehler manifold M with positively lower bounded totally real bisectional curvature $B(X,Y) \geq b > 0$ and constant scalar curvature is congruent to a complex projective space $CP^n(c)$. Before to obtain this result we should verify that a Kaehler manifold M with $B(X,Y) \geq b > 0$ is Einstein. Moreover we also show that the positive constant b in the above estimation is best possible. This means that the condition of a positive lower bound for the totally real bisectional curvature can not be replaced by the non-negativity of this curvature, because we can find that there is a complete Kaehler manifold with non-negative totally real bisectional curvature $B(X,Y) \geq 0$ but not Einstein.

Although S.I. Goldberg and S. Kobayashi [6] showed that a complete Kaehler manifold M with positive holomorphic bisectional curvature

$H(X,Y) > 0$ is Einstein, in order to get this result they should have verified that the Ricci tensor of M is positive definite. In that proof they used the fact that the holomorphic sectional curvature $H(X)$ is positive, which necessary follows from the condition $H(X,Y) > 0$. But the condition of $B(X,Y) > 0$ carries less information than the condition of $H(X,Y) > 0$, it gives us no meaning to use S.I. Goldberg and S. Kobayashi's method to derive the fact that M is Einstein. That is, we can not use the condition of $H(X) > 0$. However, in spite of this weaker condition $B(X,Y) \geq b > 0$ by making use of generalized maximal principal due to H. Omori [13] and S.T. Yau [16] we can also obtain the above result.

It is known that the complete space-like complex submanifold of the indefinite complex space form $M_p^{n+p}(c), c \geq 0$ is totally geodesic. Thus for a case where $c < 0$ we [1] have studied the classification problem of space- like complex submanifolds of indefinite complex hyperbolic space $CH_p^{n+p}(c)$ with bounded scalar curvature. Motivated by this result in section 4 we also study those classification problems with bounded totally real bisectional curvature. Finally in section 5 we study the classification of complex submanifolds M^n of $CP^{n+p}(c), c > 0$ with bounded totally real bisectional curvature.

1. Local formulas

This section is concerned with local formula for indefinite complex submanifolds of semi-Kaehler manifolds. Let M' be an $(n+p)$-dimensional connected semi- Kaehler manifold of index $2(s+t), (n \geq 2, 0 \leq s \leq n,\ 0 \leq t \leq p)$. And let M be an n-dimensional connected semi-Kaehler submanifold of index $2s$ of M'. Then we can choose a local unitary frame field $\{E_A\} = \{E_1, ..., E_{n+p}\}$ on a neighborhood of M' in such a way that, restricted to M, $E_1, ..., E_n$ are tangent to M and the others are normal to M. Here and in the sequel the following convention on the range of indices used throughout this paper, unless otherwise stated:

$$A, B, ... = 1, ..., n, n+1, ..., n+p,$$
$$i, j, ... = 1, ..., n,$$
$$x, y, ... = n+1, ..., n+p.$$

With respect to this frame field, let $\{\omega_A\} = \{\omega_i, \omega_y\}$ be its local dual frame fields. Then the semi-Kaehler metric tensor g' of M' is given by $g' = 2\Sigma_A \epsilon_A \omega_A \otimes \bar{\omega}_A$ and $\{\epsilon_A\} = \{\epsilon_i, \epsilon_x\}$ satisfy

$$\epsilon_i = g'(E_i, \bar{E}_i) = -1 \text{ or } 1 \quad \text{according to} \quad 1 \leq i \leq s \quad \text{or} \quad s+1 \leq i \leq n,$$
$$\epsilon_x = g'(E_x, \bar{E}_x) = -1 \text{ or } 1 \quad \text{according to} \quad n+1 \leq x \leq n+t \quad \text{or} \quad n+t+1 \leq x \leq n+p.$$

The canonical forms ω_A and the connection forms ω_{AB} of the ambient space M' satisfy the structure equations:

$$d\omega_A + \Sigma \epsilon_B \omega_{AB} \wedge \omega_B = 0, \quad \omega_{AB} + \bar{\omega}_{BA} = 0, \tag{1.1}$$

$$d\omega_{AB} + \Sigma \epsilon_C \omega_{AC} \wedge \omega_{CB} = \Omega_{AB}, \qquad \Omega'_{AB} = \Sigma R'_{\bar{A}BC\bar{D}} \omega_C \wedge \bar{\omega}_D, \tag{1.2}$$

where Ω'_{AB} (resp. $R'_{\bar{A}BC\bar{D}}$) denotes the Riemannian curvature form (resp. the components of the Riemannian curvature tensor R) on M.

The second equation of (1.1) means the skew-hermitian symmetry of Ω'_{AB}, which is equivalent to the symmetric conditions

$$R'_{\bar{A}BC\bar{D}} = R'_{\bar{B}AD\bar{C}}.$$

The Bianchi identities $\Sigma_B \epsilon_B \Omega_{AB} \wedge \omega_B = 0$ obtained by the exterior derivative of (1.1) and (1.2) give the further symmetric relations

$$R'_{\bar{A}BC\bar{D}} = R'_{\bar{A}CB\bar{D}} = R'_{\bar{D}BC\bar{A}} = R'_{\bar{D}CB\bar{A}}. \tag{1.3}$$

Now, with respect to the frame chosen above, the Ricci-tensor S' of M' can be expressed as follows;

$$S' = \Sigma \epsilon_C \epsilon_D (S'_{C\bar{D}} \omega_C \otimes \bar{\omega}_D + S'_{\bar{C}D} \bar{\omega}_C \otimes \omega_D),$$

where $S'_{C\bar{D}} = \Sigma_B \epsilon_B R_{\bar{B}BC\bar{D}} = S'_{\bar{D}C} = \bar{S}'_{\bar{C}D}$. The scalar curvature K is also given by

$$K = 2\Sigma_D \epsilon_D S'_{D\bar{D}}.$$

The semi-Kaehler manifold M' is said to be *Einstein* if the Ricci tensor S' is given by

$$S'_{C\bar{D}} = \lambda\epsilon_C\delta_{CD}, \quad \lambda = \frac{K}{2(n+p)},$$

for a constant λ, where λ is called the Ricci curvature of the Einstein manifold.

The component $R'_{\bar{A}BC\bar{D}}$ and $R'_{\bar{A}BC\bar{D}\bar{E}}$ (resp. $S'_{A\bar{B}C}$ and $S'_{A\bar{B}\bar{C}}$) of the covariant derivative of the Riemannian curvature tensor R'(resp. the Ricci tensor S') are defined by

$$\Sigma\epsilon_E(R'_{\bar{A}BC\bar{D}E}\omega_E + R'_{\bar{A}BC\bar{D}\bar{E}}\bar{\omega}_E) = dR'_{\bar{A}BC\bar{D}} - \Sigma\epsilon_E(R'_{\bar{E}BC\bar{D}}\omega_{EA}$$

$$+R'_{\bar{A}EC\bar{D}}\omega_{EB} + R'_{\bar{A}BE\bar{D}}\omega_{EC} + R'_{\bar{A}BC\bar{E}}\bar{\omega}_{ED}),$$

$$\Sigma\epsilon_C(S'_{A\bar{B}C}\omega_C + S'_{A\bar{B}\bar{C}}\bar{\omega}_C) = dS'_{A\bar{B}} - \Sigma\epsilon_C(S'_{C\bar{B}}\omega_{CA} + S'_{A\bar{C}}\bar{\omega}_{CB}).$$

The second Bianchi formula is given by

$$(1.4) \qquad R'_{\bar{A}BC\bar{D}E} = R'_{\bar{A}BE\bar{D}C}$$

and hence we have

$$(1.5) \qquad S'_{A\bar{B}C} = S'_{C\bar{B}A} = \Sigma_D\epsilon_D R'_{\bar{B}AC\bar{D}D}, \quad K_A = 2\Sigma_C S_{BC\bar{C}},$$

where $dK = \Sigma_C\epsilon_C(K_C\omega_C + K_{\bar{C}}\bar{\omega}_C)$. The components $S'_{A\bar{B}CD}$ and $S'_{A\bar{B}C\bar{D}}$ of the covariant derivative of $S'_{A\bar{B}C}$ are expressed by

(1.6)

$$\Sigma_D\epsilon_D(S'_{A\bar{B}CD}\omega_D + S'_{A\bar{B}C\bar{D}}\bar{\omega}_D) = dS'_{A\bar{B}C} - \Sigma_D\epsilon_D(S'_{D\bar{B}C}\omega_{DA} + S'_{A\bar{D}C}\bar{\omega}_{DB} + S'_{A\bar{B}D}\omega_{DC}).$$

By the exterior differentiation of the definition of $S'_{A\bar{B}C}$ and by taking account of (1.6) the Ricci formula for the Ricci tensor S' is given as follows:

$$(1.7) \qquad S'_{A\bar{B}C\bar{D}} - S'_{A\bar{B}\bar{D}C} = \Sigma_E\epsilon_E(R'_{\bar{D}CA\bar{E}}S'_{E\bar{B}} - R'_{\bar{D}CE\bar{B}}S'_{A\bar{E}}).$$

Restricting the above canonical forms $\{\omega_A\} = \{\omega_i, \omega_y\}$ to the submanifold M, we have

$$\omega_x = 0 \tag{1.8}$$

and the induced semi-Kaehler metric g of index $2s$ of M is given by $g = 2\Sigma\epsilon_j\omega_j\otimes\bar{\omega}_j$. Then $\{E_j\}$ is a local unitary frame field with respect to this metric and $\{\omega_j\}$ is a local dual field of $\{E_j\}$, which consists of complex-valued 1-forms of type (1,0) on M. Moreover $\omega_1, ..., \omega_n, \bar{\omega}_1, ..., \bar{\omega}_n$ are lineary independent, and they are said to be cannonical 1-forms on M. It follows from (1.8) and the Cartan lemma that the exterior derivatives of (1.8) give rise to

$$\omega_{xi} = \Sigma\epsilon_j h^x_{ij}\omega_j, h^x_{ij} = h^x_{ji}. \tag{1.9}$$

The quadratic form $\Sigma\epsilon_i\epsilon_j h^x_{ij}\omega_i\otimes\omega_j\otimes E_x$ with values in the normal bundle is called the *second fundamental form* of the submanifold M. Similarly, from the structure equation of M' it follows that the structure equations for M are given by

$$d\omega_i + \Sigma\epsilon_j\omega_{ij}\wedge\omega_j = 0, \quad \omega_{ij} + \bar{\omega}_{ji} = 0, \tag{1.10}$$

$$\begin{aligned} d\omega_{ij} + \Sigma\epsilon_k\omega_{ik}\wedge\omega_{jk} &= \Omega_{ij}, \\ \Omega_{ij} &= \Sigma\epsilon_k\epsilon_l R_{\bar{i}jk\bar{l}}\omega_k\wedge\bar{\omega}_l, \end{aligned} \tag{1.11}$$

where Ω_{ij}(resp. $R_{\bar{i}jk\bar{l}}$) denotes the Riemannian curvature form (resp. the components of the Riemannian curvature tensor R) on M. Moreover, the following relationships are defined:

$$d\omega_{xy} + \Sigma\epsilon_x\omega_{xz}\wedge\omega_{zy} = \Omega_{xy}, \quad \Omega_{xy} = \Sigma\epsilon_k\epsilon_l R_{\bar{x}yk\bar{l}}\omega_k\wedge\bar{\omega}_l, \tag{1.12}$$

For the Riemannian curvature tensors R and R' of M and M' respectively from (1.9) and (1.10) the equation of Gauss gives rise to

$$R_{\bar{i}jk\bar{l}} = R'_{\bar{i}jk\bar{l}} - \Sigma\epsilon_x h^x_{jk}\bar{h}^x_{il}, \tag{1.13}$$

The components of the Ricci tensor S and the scalar curvature r of M are given by

$$S_{i\bar{j}} = \Sigma\epsilon_k R'_{\bar{j}ik\bar{k}} - \Sigma\epsilon_r\epsilon_x h^x_{ir}\bar{h}^x_{rj}. \tag{1.14}$$

$$r = 2\Sigma S_{\bar{j}j} = 2\Sigma S_{j\bar{j}} = 2\Sigma\epsilon_j\epsilon_k R'_{\bar{j}jk\bar{k}} - 2h_2, \tag{1.15}$$

where $h^2_{i\bar{j}} = \Sigma\epsilon_k\epsilon_x h^x_{ik}\bar{h}^x_{kj}$ and $h_2 = \Sigma\epsilon_k h^2_{k\bar{k}}$.

Now the components h^x_{ijk} and $h^x_{ij\bar{k}}$ of the covariant derivative of the second fundamental form of M are given by

$$\begin{aligned}\Sigma\epsilon_k(h^x_{ijk}\omega_k + h_{ij\bar{k}}\bar{\omega}_k) = dh^x_{ij} - \Sigma\epsilon_k(h^x_{kj}\omega_{ki} + h^x_{ik}\omega_{kj})\\ + \Sigma\epsilon_y h^y_{ij}\omega_{xy}.\end{aligned}$$

Then substituting dh^x_{ij} into the exterior derivative of (1.4), we have

$$h^x_{ijk} = h^x_{jik} = h^x_{ikj}, \quad h^x_{ij\bar{k}} = -R'_{\bar{x}ij\bar{k}}. \tag{1.16}$$

Similarly the components h^x_{ijkl} and $h^x_{ijk\bar{l}}$ of the covariant derivative of h_{ijk} can be defined by

$$\begin{aligned}\Sigma\epsilon_l(h^x_{ijkl}\omega_l + h^x_{ijk\bar{l}}\bar{\omega}_l) = dh^x_{ijk} - \Sigma\epsilon_l(h^x_{ljk}\omega_{li} + h^x_{ilk}\omega_{lj}\\ + h^x_{ijl}\omega_{lk}) + \Sigma\epsilon_y h^y_{ijk}\omega_{xy},\end{aligned}$$

and the simple calculation give rise to

$$\begin{aligned} h^x_{ijkl} &= h^x_{ijlk},\\ h^x_{ijk\bar{l}} - h^x_{ij\bar{l}k} &= \Sigma\epsilon_r(R_{\bar{l}ki\bar{r}}h^x_{rj} + R_{\bar{l}kj\bar{r}}h^x_{ir})\\ &\quad - \Sigma\epsilon_y R_{\bar{x}yk\bar{l}}h^y_{ij}.\end{aligned} \tag{1.17}$$

A plane section P of the tangent space T_xM' of M' at any point x is said to be *non-degenerate*, provided that $g_x|T_xM'$ is non-degenerate if and only if it has a basis $\{u, v\}$ such that $g(u,u)g(v,v) - g(u,v)^2 \neq 0$, and a holomorphic plane spanned by u and Ju is non-degenerate if and only if it contains some v with $g(v,v)\neq 0$. The sectinal curvature of the non-degenerate holomorphic plane P spanned by u and Ju is called the *holomorphic sectional curvature*, which is denoted by $H(P) = H(u)$. The indefinite Kaehler manifold M' is said to be of *constant holomorphic sectional curvature* if its holomorphic sectional curvature $H(P)$ is constant for all P and for all points of M'. Then M' is called a complex space form, which is denoted by $M^n_s(c)$, provided that it is of constant holomorphic sectional curvature c, of complex dimension n and of index $2s$. The standard models of indefinite complex space forms are the following three kinds which are given by Barros and Romero [2] and Wolf [15] : the indefinite complex Euclidean space C^n_s, the indefinite complex projective space CP^n_s or the indefinite complex hyperbolic space CH^n_s, according as $c = 0, c > 0$ or $c < 0$. For an integer $s (0 < s < n)$ it is seen by [2] and [15] that they are only complete, simply connected and connected indefinite complex space forms of dimension n and of index $2s$.

Now, the Riemannian curvature tensor $R_{\bar{A}BC\bar{D}}$ of $M^n_s(c)$ is given by

$$R_{\bar{A}BC\bar{D}} = \frac{c}{2}\epsilon_B\epsilon_C(\delta_{AB}\delta_{CD} + \delta_{AC}\delta_{BD}).$$

In particular, let the ambient space be an indefinite complex space form $M^{n+p}_{s+t}(c')$ of constant holomorphic sectional curvature c'. Then we get

$$R_{\bar{i}jk\bar{l}} = \frac{c'}{2}\epsilon_j\epsilon_k(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}) - \Sigma\epsilon_x h^x_{jk}\bar{h}^x_{il}, \tag{1.18}$$

$$S_{\bar{i}j} = (n+1)\frac{c'}{2}\epsilon_i\delta_{ij} - h^2_{\bar{i}j}, \tag{1.19}$$

$$r = n(n+1)c' - 2h_2, \tag{1.20}$$

$$h^x_{ijk\bar{l}} = \frac{c'}{2}(\epsilon_k h^x_{ij}\delta_{kl} + \epsilon_i h^x_{jk}\delta_{il} + \epsilon_j h^x_{ki}\delta_{jl}) - \Sigma\epsilon_r\epsilon_y(h^x_{ri}h^y_{jk} + h^x_{rj}h^y_{ki} + h^x_{rk}h^y_{ij})\bar{h}^y_{rl}. \tag{1.21}$$

Let us denote by $h_4 = \Sigma\epsilon_i\epsilon_j h^2_{\bar{i}j}h^2_{\bar{j}i}$ and $A_2 = \Sigma\epsilon_x\epsilon_y A_y{}^x A_x{}^y$, where $A_y{}^x = \Sigma\epsilon_i\epsilon_j h^x_{ij}\bar{h}^y_{ij}$. Then, by means of (1.18), the Laplacian $\triangle h_2$ of the function h_2 is given by

$$\triangle h_2 = (n+2)\frac{c'}{2}h_2 - (2h_4 + A_2) + \Sigma\epsilon_x\epsilon_i\epsilon_j\epsilon_k h^x_{ijk}\bar{h}^x_{ijk}. \tag{1.22}$$

First of all, let us introduce a fundamental property for the generalized maximal principal due to H.Omori [13] and S.T.Yau [16].

Theorem 1.1. *Let M be an n-dimensional Riemannian manifold whose Ricci curvature is bounded from below on M. Let F be a C^2-function bounded from below on M, then for any $\epsilon > 0$, there exists a* *point p such that*

$$|\nabla F(p)| < \epsilon, \quad \triangle F(p) > -\epsilon \quad \textit{and} \quad \inf F + \epsilon > F(p).$$

2. Totally real bisectional curvature

Let (M, g) be an n-dimensional semi-Kaehler manifold with almost complex structure J. In this section, we consider a semi-Kaehler manifold with totally real bisectional curvature, which is determined by an non-degenerate anti-holomorphic plane $[u, v]$ and its image $[Ju, Jv]$ by the complex structure J. That is, the totally real bisectional curvature is defined by

$$B(u,v) = g(R(u, Ju)Jv, v)/g(u,u)g(v,v). \tag{2.1}$$

Then for a semi-Kaehler manifold, using the first Bianchi-identity to (2.1), we get

$$\begin{aligned} B(u,v) &= g(R(u, Jv)Jv, u) + g(R(u,v)v, u) \\ &= K(u,v) + K(u, Jv), \end{aligned} \tag{2.2}$$

where $K(u, v)$ means the sectional curvature of the plane spanned by u and v, and $[u, v]$ the totally real plane section such that $g(u, u), g(v, v) = \pm 1$ and $g(u, Ju) = g(v, Jv) = 0$.

Now if we put $u' = \frac{u+v}{\sqrt{2}}$ and $v' = \frac{J(u-v)}{\sqrt{2}}$, then it is easily seen that $g(u', u') = \pm 1, g(v', v') = \pm 1$, and $g(u', Jv') = 0$. Thus $B(u', v') = \frac{g(R(u', Ju')Jv', v')}{g(u', u')g(v', v')}$ implies that

$$\begin{aligned} g(u', u')g(v', v')B(u', v') &= g(R(u', Ju')Jv', v') \\ &= \frac{1}{4} g(u, u)g(v, v)\{H(u) + H(v) + 2B(u, v) - 4K(u, Jv)\}, \end{aligned}$$

where $H(u) = K(u, Ju)$, and $H(v) = K(v, Jv)$ means the holomorphic sectional curvatures of the plane $[u, Ju]$ and $[v, Jv]$ respectively and $K(u, Jv)$ the sectional curvature of the plane $[u, Jv]$. From this together with the fact that

$$g(u', u')g(v', v') = g(u, u)g(v, v) = \pm 1$$

it follows

$$(2.3)\qquad 4B(u',v') - 2B(u,v) = H(u) + H(v) - 4K(u,Jv).$$

If we put $u'' = \frac{u+Jv}{\sqrt{2}}$, and $v'' = \frac{Ju+v}{\sqrt{2}}$, then we get $g(u'',u'') = \pm 1$,$g(v'',v'') = \pm 1$ and $g(u'',v'') = 0$. Using the similar method as in (2.3), we get

$$(2.4)\qquad 4B(u'',v'') - 2B(u,v) = H(u) + H(v) - 4K(u,v).$$

Summing up (2.3) and (2.4), we obtain

$$(2.5)\qquad 2B(u',v') + 2B(u'',v'') = H(u) + H(v).$$

Now we calculate the totally real bisectional curvatures of some manifolds.

EXAMPLE 2.1. Let $M_s^n(c)$ be a complex space form of constant holomorphic sectional curvature c and of index $2s(0\leq s\leq n)$and $[u,v]$ be a totally real plane section. Then

$$\begin{aligned}B(u,v) =& g(R(u,Ju)Jv,v)/g(u,u)g(v,v)\\ =& c\{g(u,v)g(Ju,Jv) - g(u,Jv)g(Ju,v) + g(Ju,v)g(-u,Jv)\\ & - g(Ju,Jv)g(-u,v) - 2g(Ju,Jv)g(-u,v)\}/4g(u,u)g(v,v)\\ =& \frac{c}{2}.\end{aligned}$$

Thus $M_s^n(c)$ is a space of complex space form of constant totally real bisectional curvature $\frac{c}{2}$.

As a Kaehler manifold which is not of constant totally real bisectional curvature we calculate totally real bisectional curvature of the complex quadric Q^n which is a space-like complex Einstein hypersurface of indefinite complex hyperbolic space $CH_1^{n+1}(c'), c' < 0$.

EXAMPLE 2.2. Let Q_s^n be the indefinite complex quadric which is obtained by projecting $N = \{z\in S_{2s}^{2n+3} | -z_1^2 - z_2^2 - \cdots - z_s^2 + z_{s+1}^2 + \ldots + z_{n+2}^2 = 0\}$. Then in a similar way [9] we can see that it is a complex Einstein hypersurface of indefinite complex projective space

$CP_s^{n+1}(c)$ and can be idenfied with the Hermitian symmetric space of non-compact type such that

$$SO^s(n+2)/SO(2)\times SO^s(n).$$

The canonical decomposition of the Lie algebra of the Lie group $SO^s(n+2)$ is given by

$$\mathfrak{G} = \mathfrak{H} + \mathfrak{M},$$

where $\mathfrak{G} = \mathfrak{O}(s, n+2)$, $\mathfrak{H} = \mathfrak{O}(2) + \mathfrak{O}(s, n-s)$ and

$$\mathfrak{M} = \left\{ \left(\begin{array}{cc} 0 & \begin{pmatrix} \xi_1 & \cdots & \xi_s & -\xi_{s+1} & \cdots & -\xi_n \\ \eta_1 & \cdots & \eta_s & -\eta_{s+1} & \cdots & -\eta_n \end{pmatrix} \\ \begin{pmatrix} \xi_1 & \eta_1 \\ \cdot & \cdot \\ \cdot & \cdot \\ \xi_n & \eta_n \end{pmatrix} & 0 \end{array} \right) \middle| \, \xi, \eta \in R_s^n \right\}.$$

The Lie algebra $\mathfrak{O}(s, n-s+2)$ of $SO^s(n+2)$ in the subalgebra of $\mathfrak{GL}(n, R)$ consisting of all S such that

$$S = \begin{pmatrix} a & x \\ {}^t x & b \end{pmatrix}$$

where $a \in \mathfrak{O}(s)$, $b \in \mathfrak{O}(n-s+2)$, $\mathfrak{O}(s)$ is the skew -symmetric matrix and x is an arbitary $s\times(n-s+2)$-matrix.

By changing the metric tensor g of Q_s^n in $CP_s^{n+1}(c)$ to its negative, we can also embedd Q_{n-s}^n into $CH_{n+1-s}^{n+1}(c'), c' = -c < 0$. Before to obtain our results we now calculate the totally real bisectional curvature of $Q_n^n = SO^n(n+2)/SO(2)\times SO^n(n)$ in $CP_n^{n+1}(c)$.

Identifying $(\xi, \eta) \in R_n^n \oplus R_n^n$ with the above matrix in $\mathfrak{M}$ for the case $s = n$, we define an inner product g on $\mathfrak{M} \times \mathfrak{M}$ by

$$g((\xi, \eta), (\xi', \eta')) = \frac{2}{c}\{< \xi, \xi' >_n + < \eta, \eta' >_n\},$$

where $< \xi, \xi' >_n$ is the indefinite inner product in R^n. We also define a complex structure J on $\mathfrak{M}$ by

$$J(\xi, \eta) = (-\eta, \xi).$$

The curvature tensor R at the origin is given by the following

$$R((\xi,\eta),(\xi',\eta')) = ad\begin{pmatrix} 0 & -\lambda & \\ \lambda & 0 & 0 \\ 0 & & B \end{pmatrix}, \quad B\in O(n),$$

where $\lambda = {}^t\eta'\xi - {}^t\eta\xi'$ and $B = \frac{c}{4}\{\xi\wedge\xi' + \eta\wedge\eta'\}$, in which $\wedge$ is defined by $(\xi\wedge\xi')\eta = \frac{4}{c}\{\xi'{}^t\xi'\eta - \xi'{}^t\xi\eta\}$. Thus for unit time-like elements $u = (\xi,\eta), v = (\xi',\eta')$ in $\mathfrak{M}$, the holomorphic bisectional curvature is given by

(2.6)
$$\begin{aligned} H(u,v) &= g(R(u,Ju)Jv,v) \\ &= \frac{2}{c}\{< -B\eta',\xi' >_n + < B\xi',\eta' >_n\} - \frac{c}{2}g(v,v) \\ &= \frac{8}{c}\{< \xi,\xi' >_n< \eta,\eta' >_n - < \xi,\eta' >_n< \xi',\eta >_n\} + \frac{c}{2}. \end{aligned}$$

And the holomorphic sectional curvature $H(u)$ is given by

$$(2.7)\qquad H(u) = g(R(u,Ju)Ju,u) = \frac{8}{c}(|\xi|^2|\eta|^2 - < \xi,\eta >_n^2) + \frac{c}{2} \geq \frac{c}{2},$$

where $|\xi|^2 = < \xi,\xi >_n^2$.

Now we consider the totally real bisectional curvature of the indefinite complex quadric Q_n^n in $CP_n^{n+1}(c)$. Let [u,v] be a totally real plane section such that $u = (\xi,\eta), v = (\xi',\eta')$, and $Jv = (-\eta',\xi')$. Then u, v, Ju and Jv become orthonormal unit elements in $\mathfrak{M}$. That is

$$\begin{aligned} g(u,v) &= \frac{2}{c}\{< \xi,\xi' > + < \eta,\eta' >\} = 0, \\ g(u,Jv) &= \frac{2}{c}\{< \xi,-\eta' > + < \eta,\xi' >\} = 0. \end{aligned}$$

From these together with (2.6) the totally real bisectional curvature is given by

$$(2.8)\qquad B(u,v) = -\frac{8}{c}\{< \xi,\xi' >_n^2 + < \xi,\eta' >_n^2\} + \frac{c}{2}.$$

As we have already seen, if we change the metric tensor g of Q^n_n in $CP^{n+1}_n(c)$ to its negative, we can embedd the complex quadric Q^n into $CH^{n+1}_1(c'), c' = -c$. Thus a metric tensor g' of Q^n in $CH^{n+1}_1(c')$ can be given by

$$g'((\xi,\eta),(\xi',\eta') = \frac{2}{c'}\{<\xi,\xi'>_n + <\eta,\eta>_n\},$$

for a $u = (\xi,\eta)$, $v = (\xi',\eta')$ in $\mathfrak{M}$ for the case $s = n$. Thus by changing c into c' of the equations (2.6),(2.7) and (2.8) we can obtain the holomorphic bisectional curvature, holomorphic sectional curvature, and the totally real bisectional curvature of Q^n embedded in $CH^{n+1}_1(c')$ respectively as follows:

(2.9)

$$H'(u,v) = \frac{8}{c'}\{<\xi,\xi'>_n<\eta,\eta'>_n - <\xi,\eta'>_n<\eta,\xi'>_n\} + \frac{c'}{2},$$

$$H'(u) = \frac{8}{c'}\{<\xi,\xi>_n<\eta,\eta>_n - <\xi,\eta>_n^2\} + \frac{c'}{2}, \tag{2.10}$$

$$B'(u,v) = -\frac{8}{c'}\{<\xi,\xi'>_n^2 + <\xi,\eta'>_n^2\} + \frac{c'}{2}. \tag{2.11}$$

Now we set $\xi = (x_j)$, $\xi' = (y_j)$, and $\eta' = (z_j)\in R^n_n$. To get an upper bound of $B'(u,v)$ by using the *Lagrange multiplier rule* let us calculate the maximal value of the following function

$$f = f(\xi,\xi',\eta') = <\xi,\xi'>_n^2 + <\xi,\eta'>_n^2 = (-\Sigma x_j y_j)^2 + (-\Sigma x_j z_j)^2$$

under the condition such that $g_1 = \frac{c}{2} - \Sigma x_j^2 \geq 0$, and $g_2 = \Sigma y_j^2 + \Sigma z_j^2 - \frac{c}{2} = 0$. The multiplier λ_1 and λ_2 is yet to be determined. From the multiplier rule we get three equations

$$\begin{aligned} f_{x_k} &= 2y_k\Sigma x_j y_j + 2z_k \Sigma x_j z_j = -2\lambda_1 x_k, \\ f_{y_k} &= 2x_k \Sigma x_j y_j = 2\lambda_2 y_k, \\ f_{z_k} &= 2z_k \Sigma x_j z_j = 2\lambda_2 z_k, \end{aligned}$$

for $k = 1, 2, ..., n$, where f_{x_k}, f_{y_k} and f_{z_k} means the partial derivative of f with respect to x_k,y_k and z_k respectively. Thus the above equation can be represented by the following vector notation

$$\lambda_1\xi - <\xi, \xi'>_n \xi' - <\xi, \eta'>_n \eta' = 0, \tag{2.12}$$

$$- <\xi, \xi'>_n \xi = \lambda_2\xi', \tag{2.13}$$

$$- <\xi, \eta'>_n \xi = \lambda_2\eta'. \tag{2.14}$$

From (2.13) and (2.14) it follows that $<\xi, \xi'>_n^2 - \lambda_2|\xi|^2 = 0$ and $<\xi, \eta'>_n^2 - \lambda_2|\eta|^2 = 0$. Thus

$$f = \lambda_2(|\xi'|^2 + |\eta'|^2) = \frac{c}{2}\lambda_2 \tag{2.15}$$

where $|\xi'|^2 = <\xi', \xi'>_n^2$, and $|\eta'|^2 = <\eta', \eta'>_n^2$. Taking the inner product (2.12) with ξ, then we get

$$f = <\xi, \xi'>_n^2 + <\xi, \eta'>_n^2 = -\lambda_1|\xi|^2. \tag{2.16}$$

Multiplying λ_2 to (2.12) and using (2.13) and (2.14), we have that

$$(f + \lambda_1\lambda_2)\xi = 0. \tag{2.17}$$

Thus for a case of $\xi = 0$, by (2.16) $f = 0$ that is, minimum value of f. For a case of $\xi \neq 0$, by (2.17) $f = -\lambda_1\lambda_2$. From this and (2.16) and (2.17) it follows that $f = \frac{c}{2}\lambda_2 = -\lambda_1\lambda_2 = -\lambda_1|\xi|^2 > 0$. Since $\lambda_1\lambda_2 \neq 0, \lambda_1 = -\frac{c}{2}, \lambda_2 = |\xi|^2$. Also $\lambda_1 g_1 = 0$ gives that $\lambda_2 = |\xi|^2 = \frac{c}{2}$ because of the fact $\lambda_1 \neq 0$. Hence the maximal value of f is $(\frac{c}{2})^2$, where $c = -c'$. Thus $\frac{c'}{2} \leq B'(u, v) \leq -\frac{3}{2}c'$.

On the other hand, from (2.5) and (2.10) it follows that

$$2B'(u', v') + c' \leq 2B'(u', v') + 2B'(u'', v'') = H'(u) + H'(v) \leq c'.$$

Thus $B'(u', v') \leq 0$. Together with this fact, consequently we get

$$\frac{c'}{2} \leq B'(u, v) \leq 0.$$

3. Complete Kaehler manifolds with positive totally real bisectional curvature

Let M be an n-dimensional Kaehler manifold with the complex structure J. We can choose a local field of orthonormal frames $u_1, \ldots, u_n, u_{1^*} = Ju_1, \ldots, u_{n^*} = Ju_n$ on a neighborhood on M. With respect to this frame field, let $\theta_1, \ldots, \theta_n, \theta_{1^*}, \ldots, \theta_{n^*}$ be the field of dual frames.

Let us denote by $\theta = (\theta_{AB}, \theta_{A^*B}, \theta_{AB^*}, \theta_{A^*B^*}), A, B = 1, \ldots, n$ the connection form of M. Then we have
(3.1)
$$\theta_{AB} = \theta_{A^*B^*}, \theta_{AB^*} = -\theta_{A^*B}, \theta_{AB} = -\theta_{BA}, and \quad \theta_{AB^*} = \theta_{BA^*}.$$

Now we set $e_A = \frac{1}{\sqrt{2}}(u_A - iu_A)$, $e_{\bar{A}} = \frac{1}{\sqrt{2}}(u_A + iu_A)$. Then $\{e_A, e_{\bar{A}}\}$ constitute a local field of unitary frames. And let us denote by $\omega_A = \theta_A + i\theta_{A^*}$ and $\bar{\omega}_A = \theta_A - i\theta_{A^*}$ its dual frame fields respectively. Then the components of Kaehler metric $g = 2\Sigma_A \omega_A \otimes \bar{\omega}_A$ and the metric components of the Riemannian curvature tensor are given by the following respectively

$$g_{B\bar{C}} = g_{BC} + ig_{BC^*}, \tag{3.2}$$

$$R_{\bar{A}BC\bar{D}} = -\{K_{ABCD} + K_{A^*BC^*D} + i(-K_{ABC^*D} + K_{A^*BCD})\}, \tag{3.3}$$

where $R_{\bar{A}BC\bar{D}} = g_{\bar{A}E}R^E{}_{BC\bar{D}}$. Thus for the case of $A = B$, $C = D$, $B \neq C$ in (3.3), the totally real bisectional curvature is given by

$$R_{\bar{B}BC\bar{C}} = -K_{B^*BC^*C} = K_{BB^*C^*C} = B(u_B, u_C). \tag{3.4}$$

For the case of $A = B = C = D$ in (3.3), the holomorphic sectional curvature is given by

$$R_{\bar{B}BB\bar{B}} = g(R(u_B, Ju_B)Ju_B, u_B) = H(u_B). \tag{3.5}$$

REMARK 3.1. From (1.8) and (3.4) we know that for any totally real plane section $[u, v]$ the totally real bisectional curvature $B(u, v)$ of a complex space form $M_n(c)$ is $\frac{c}{2}$ which is the same as in Example 2.1.

On the other hand, S.I. Goldberg and S. Kobayashi [6] showed that a Kaehler manifold with positive holomorphic bisectional curvature and constant scalar curvature is Einstein. It is well known that the Ricci 2-form is harmonic if and only if the scalar curvature is constant. In order to prove that the second Betti number of a compact connected Kaehler manifold M with positive holomorphic bisectional curvature $H(X,Y)>0$ is one they have used the fact that $H(X)>0$. Thus the Ricci 2-form is propotional to the Kaehler 2-form , so that M becomes to an Einstein manifold.

But from the condition $B(X,Y)>0$ we do not know whether $H(X)$ is positive or not, because the condition $B(X,Y)>0$ is weaker than that of $H(X,Y)>0$. Thus in order to get the above result it is impossible for us to use $H(X)>0$ with the condition of $B(X,Y)>0$. From this point of view due to H.Omori [13] and S.T. Yau's [16] maximal principal we can obtain the following.

THEOREM 3.1. *Let M be a complete n-dimensional Kaehler manifold with constant scalar curvature. Assume that the totally real bisectional curvature is lower bounded for some positive constant b. Then M is Einstein.*

Proof. Since $(S_{B\bar{C}})$ is a Hermitian matrix, it can be diagonalizable. Thus $S_{B\bar{C}}=\lambda_B\delta_{BC}$, where λ_B is a real valued function. From this it follows that $r=2\Sigma_B S_{B\bar{B}}=2\Sigma_B\lambda_B$. Now we put $S_2=\Sigma_{B,\bar{C}}S_{B\bar{C}}S_{C\bar{B}}$. Then it yields easily that

$$S_2-\frac{r^2}{4n}=\Sigma\lambda_B^2-\frac{(\Sigma\lambda_B)^2}{n}=\frac{1}{2n}\Sigma_{B,C}(\lambda_B-\lambda_C)^2. \tag{3.6}$$

Since we have assumed that the scalar curvature r of M is constant, from (1.5) it follows $\Sigma_B S_{B\bar{B}C}=\Sigma_B S_{C\bar{B}B}=0$. Together with this fact using (1.5) and the Ricci formula (1.7) we have that

$$\begin{aligned}\triangle S_{B\bar{C}}&=\Sigma_D S_{B\bar{C}D\bar{D}}=\Sigma_D S_{D\bar{C}B\bar{D}}\\&=\Sigma_{E,D}(R_{\bar{D}BD\bar{E}}S_{E\bar{C}}-R_{\bar{D}BE\bar{C}}S_{D\bar{E}}),\end{aligned}$$

from which, if we use the first Bianchi-identity (1.3) to the final term, we have

$$\begin{aligned}\triangle S_{B\bar{C}}&=\Sigma_E(S_{B\bar{E}}S_{E\bar{C}}-\Sigma_D R_{\bar{D}EB\bar{C}}S_{D\bar{E}})\\&=\lambda_B S_{B\bar{C}}-\Sigma_A\lambda_A R_{\bar{A}AB\bar{C}}.\end{aligned}$$

Thus we get

$$(3.7)\qquad \frac{1}{2}\triangle S_2 = \frac{1}{2}|\nabla S|^2 + \Sigma_{B,C} S_{\bar{C}B}(\lambda_B S_{B\bar{C}} - \Sigma_A \lambda_A R_{\bar{A}AB\bar{C}}),$$

where $|\nabla S|^2 = 2\Sigma S_{A\bar{B}C}\bar{S}_{A\bar{B}C}$. Since the second term of the right hand side is reduced to

$$\Sigma_{A,B}(\lambda_B^2 R_{\bar{A}AB\bar{B}} - \lambda_A\lambda_B R_{\bar{A}AB\bar{B}}) = \frac{1}{2}\Sigma_{A,B}(\lambda_A - \lambda_B)^2 R_{\bar{A}AB\bar{B}},$$

we get the following inequality by (3.7)

$$(3.8)\qquad \triangle S_2 \geq \Sigma(\lambda_A - \lambda_B)^2 R_{\bar{A}AB\bar{B}},$$

where the above equality holds if and only if the Ricci tensor S is parallel on M.

Now let us consider a non-negative function $f = S_2 - \frac{r^2}{4n}$. Then from (3.6),(3.8) and the assumption it follows that

$$(3.9)\qquad \triangle f \geq 2nbf,$$

where the above equality holds if and only if the Ricci tensor S is parallel on M. In order to prove this theorem, we need the following lemma.

LEMMA 3.2. *Under the same assumption as stated in Theorem 3.1 the Ricci-curvature is bounded from below.*

Proof. From the assumption and (2.5) it follows that

$$H(u) + H(v) \geq 4b.$$

Using (3.5) to the above equation for $u = u_A, v = u_B, A \neq B$, then we can rewritten the above inequality as the following

$$R_{\bar{A}AA\bar{A}} + R_{\bar{B}BB\bar{B}} \geq 4b.$$

If we put $R_A = R_{\bar{A}AA\bar{A}}$, then

$$(3.10)\qquad R_A + R_B \geq 4b \qquad (A \neq B).$$

Thus $\Sigma_{A<B}(R_A + R_B) \geq 2n(n-1)b$ implies that

$$\Sigma_A R_A \geq 2nb, \tag{3.11}$$

where the equality holds if and only if $R_A = 2b$ for any A.

On the other hand, from the fact that

$$\begin{aligned} r = 2\Sigma_A S_{A\bar{A}} = 2\Sigma_{A,B} R_{\bar{A}AB\bar{B}} &= 2(\Sigma_A R_A + \Sigma_{A\neq B} R_{\bar{A}AB\bar{B}}) \\ &\geq 2\Sigma_A R_A + 2n(n-1)b \end{aligned}$$

it follows

$$\Sigma_A R_A \leq \frac{r}{2} - n(n-1)b, \tag{3.12}$$

where the equality holds if and only if $R_{\bar{A}AB\bar{B}} = b$ for any i, j $(i \neq j)$. In this case due to C.S.Houh [8] M is congruent to $M_n(2b)$. From (3.11) and (3.12) we know that $r \geq 2n(n+1)b$. Thus from the assumption the scalar curvature r is positive constant. Also (3.10) gives $\Sigma_{B=2}^{n}(R_1 + R_B) \geq 4(n-1)b$, so that

$$(n-2)R_1 + \Sigma_B R_B \geq 4(n-1)b. \tag{3.13}$$

From this and (3.12) it follows

$$(n-2)R_1 \geq 4(n-1)b - \Sigma_B R_B \geq 4(n-1)b - \{\frac{r}{2} - n(n-1)b\}.$$

Thus if we use the similar method to the other index, we can assert the following

$$(n-2)R_B \geq (n-1)(n+4)b - \frac{r}{2}$$

for any index B, so that R_B is bounded from below for $n \geq 3$. Moreover the above equality holds for some index B if and only if M is congruent to $M^n(2b)$. Accordingly the Ricci-curvature is given by

$$\begin{aligned} \lambda_A = S_{A\bar{A}} = \Sigma_B R_{\bar{A}AB\bar{B}} &= R_A + \Sigma_{A\neq B} R_{\bar{A}AB\bar{B}} \\ &> R_A + (n-1)b. \end{aligned} \tag{3.14}$$

Thus the Ricci-curvature is bounded from below. Now Lemma 3.2 is proved. □

Now we will complete the proof of Theorem 3.1. For a constant $a > 0$, we consider a smooth positive function $F = (f + a)^{-\frac{1}{2}}$. Thus, from Lemma 3.2 we can apply Theorem 1.1(H.Omori [13] and S.T.Yau [16]) to the function $F = (f + a)^{-\frac{1}{2}}$ for the given f. Given any positive number $\epsilon > 0$, there exists a point p such that

$$(3.15) \qquad |\nabla F|(p) < \epsilon, \quad \triangle F(p) > -\epsilon, \quad F(p) < inf F + \epsilon.$$

It follows from these properties that we have

$$(3.16) \qquad \epsilon(3\epsilon + 2F(p)) > F(p)^4 \triangle f(p) \geq 0.$$

Thus for a convergent sequence $\{\epsilon_m\}$ such that $\epsilon_m > 0$ and $\epsilon_m \to 0$ as $m \to \infty$, there is a point sequence $\{p_m\}$ so that the sequence $\{F(p_m)\}$ satisfies (3.15) and converges to F_0, by taking a subsequence, if necessary, because the sequence $\{F(p_m)\}$ is bounded. From the definition of the infimum and (3.15) we have $F_0 = inf F$ and hence $f(p_m) \to f_0 = sup f$. It follows from (3.16) that we have

$$\epsilon_m\{3\epsilon_m + 2F(p_m)\} > F(p_m)^4 \triangle f(p_m)$$

and the left hand side converges to 0 because the function F is bounded. Thus we get

$$F(p_m)^4 \triangle f(p_m) \to 0 \quad (m \to \infty).$$

As is already seen, the Ricci-curvature is bounded from below i.e., so is any λ_B. Since $r = 2\Sigma_B \lambda_B$ is constant, λ_B is bounded from above. Hence $F = (f + a)^{-\frac{1}{2}}$ is bounded from below by a positive constant. From (3.17) it follows that $\triangle f(p_m) \to 0$ as $m \to \infty$. Taking $b > 0$, by (3.9) we have that

$$\triangle f(p_m) \geq 2nbf(p_m) \geq 0.$$

Thus we have $f(p_m) \to 0 = inf f$. Since $f(p_m) \to sup f$, $sup f = inf f = 0$. Hence $f = 0$ on M. That is, M is Einstein or $b \leq 0$. This completes the above proof of Theorem 3.1. □

REMARK 3.2. The positive constant $b > 0$ in Theorem 3.1 is best possible. This means that the condition of a positive lower bound for the totally real bisectional curvature can not be replaced by the non-negativity of this curvature, because there is a complete Kaehler manifold with non-negative totally real bisectional curvature $B(u,v)\geq 0$ but not Einstein as follows: Consider a product manifold $M = CP^{n^1}(c_1) \times CP^{n^2}(c_2)$. Then from (3.8) we know that its totally real bisectional curvature is given by

$$R_{\bar{A}AB\bar{B}} = \begin{cases} R_{\bar{a}ab\bar{b}} = \frac{c_1}{2} & \text{if } A = a, B = b, \\ 0 & \text{if } A = a, B = s, \\ R_{\bar{r}rs\bar{s}} = \frac{c_2}{2} & \text{if } A = r, B = s, \end{cases}$$

where indices $A, B(A\neq B), ...; 1, ..., n_1 . n_1+1, ..., n_2$, and $a, b, ..; 1, ..., n_1$, $r, s, ..; n_1 + 1, ..., n_2$.

And its Ricci-tensor is given by the following

$$S_{A\bar{B}} = \Sigma_C R_{\bar{B}AC\bar{C}} = \Sigma_a R_{\bar{B}Aa\bar{a}} + \Sigma_r R_{\bar{B}Ar\bar{r}}$$
$$= \begin{cases} \frac{n_1+1}{2} c_1 \delta_{bc} & \text{if } B = c . A = b, \\ 0 & \text{if } B = s, A = b, \\ \frac{n_2+1}{2} c_2 \delta_{ts} & \text{if } B = s, A = t. \end{cases}$$

Thus for case where $(n_1 + 1)c_1 \neq (n_2 + 1)c_2$, $M = CP^{n^1}(c_1) \times CP^{n^2}(c_2)$ is not Einstein.

Since a complete Kaehler manifold M with the assumption in Theorem 3.1 is known to be Einstein and its scalar curvature r is positive constant, its Ricci-tensor is positive definite. Thus by using a theorem of Myers we can assert that M is compact [9]. Now let us introduce a theorem of S.I. Goldberg and S. Kobayashi [6], which is slight different from the original one.

THEOREM A. *An n-dimensional compact connected Kaehler manifold with an Einstein metric of positive totally real bisectional curvature is globally isometric to CP^n with the Fubini-Study metric.*

Though the original theorem in [6] are assumed with positive holomorphic bisectional curvature, it can be easily cheked that the result in Theorem A also holds if we assume with positive totally real bisectional curvature. Thus combining Theorem A and Theorem 3.1 we can assert the following.

THEOREM 3.3. *Let M be a complete $n(\geq 3)$-dimensional Kaehler manifold with constant scalar curvature. Assume that the totally real bisectional curvature is lower bounded for some positive constant b. Then M is globally isometric to CP^n with the Fubini-Study metric.*

4. Space-like complex submanifolds

Let $M' = CH_p^{n+p}(c)$ be an $(n+p)$-dimensional indefinite complex hyperbolic space of index $2p(>0)$, and M be an $n(\geq 3)$-dimensional space-like complex submanifold of $CH_p^{n+p}(c)$, $(c<0)$. Then by the equation of Gauss

$$R_{\bar{i}ij\bar{j}} = \frac{c}{2} - \Sigma_x \epsilon_x h_{ij}^x \bar{h}_{ij}^x \geq \frac{c}{2}, \tag{4.1}$$

where we have used the fact that $\epsilon_x = -1$, because the normal space of M is time-like. Thus from (4.1) we know that there is a totally real bisectional plane section [u,v] such that $B(u,v)\geq\frac{c}{2}$.

Now we will give here some remarks of the totally real bisectional curvature of semi-Kaehler submanifolds of indefinite complex space forms.

REMARK 4.1. For the complex submanifold M of a complex space form $M' = M^{n+p}(c)$ we have

$$R_{\bar{i}ij\bar{j}} = \frac{c}{2} - \Sigma_x h_{ij}^x \bar{h}_{ij}^x \leq \frac{c}{2}.$$

Thus its totally real bisectional curvature is upper bounded such that $B(u,v)\leq\frac{c}{2}$. For this example let M be a complex quadric Q_n embedded in $CP^{n+1}(c)$. Since Q_n is known to be Hermitian symmetric space of compact type, its sectional curvature is non-negative(cf.[9]). Thus from (2.2) and the above inequality we know that the totally real bisectional curvature $B(u,v)$ is given by $0\leq B(u,v)\leq\frac{c}{2}$. Moreover, in the paper [12] the holomorphic sectional curvature $H(u)$ of Q_n is holomorphically pinched as $\frac{c}{2}\leq H(u)\leq c$.

REMARK 4.2. ([1]) Let M be a complete space-like complex submanifold of an indefinite complex space form $M_p^{n+p}(c)$ with $c \geq 0$. Then M is totally geodesic. Thus $B(u,v) = \frac{c}{2}$.

REMARK 4.3. ([1]) Let $M = M_s^n(c)$ be an n-dimensional indefinite complex space form immersed in $M' = M_{s+t}^{n+p}(c'), c' = 0$, and $t = p$.

If $c' \neq 0$, then $c' = kc$ and $n + p \geq \binom{n+k}{k} - 1$ for some positive integer k.

If $c' = 0$ if and only if $c = 0$.

In particular for the case $t = p$, $c' \neq 0$,

If $c' > 0$, then $c' = c$. Thus M is totally geodesic and $B(u,v) = \frac{c}{2}$.

If $c' < 0$, then $c' = c$ or $2c$, the first case arising only when M is totally geodesic and the other arising only when $s = 0$ and $B(u,v) = \frac{c}{4}$.

REMARK 4.4. Let Q^n be a space-like complex quadric of a complex hyperbolic space $CH_1^{n+1}(c')$ of index 2, which is defined by $-z_1^2 + \Sigma_{j=2}^{n+2} z_j^2 = 0$ in the homogeneous coordinate system of $CH_1^{n+1}(c'), c' < 0$. Then Q^n is Einstein, and it satisfies $\frac{c}{2} \leq B(u,v) \leq 0$ for any totally real bisectional plane [u,v].

From the above Remark 4.2 we know that a complete space-like complex submanifold of $M' = M_p^{n+p}(c), c \geq 0$, is totally geodesic. It gives us no meaning to consider the complete space-like submanifold of $M_p^{n+p}(c), c \geq 0$, with lower bounded totally real bisectional curvature. Thus in this section we consider the classification problem of the complete space-like submanifold of $CH_p^{n+p}(c), c < 0$, with lower bounded totally real bisectional curvature.

Now suppose that there exist a lower bound $b \in R$ such that

$$R_{\bar{i}ij\bar{j}} \geq b \quad for \quad any \quad i,j \quad (i \neq j). \tag{4.2}$$

From this and together with (4.1) it follows that

$$2\Sigma_x \epsilon_x h_{ij}^x \bar{h}_{ij}^x \leq c - 2b \quad for \quad any \quad i,j \quad (i \neq j). \tag{4.3}$$

By (1.20),(3.11),(3.12) and (3.5) we have

$$2nb \leq \Sigma_j R_j \leq n(n+1)c/2 - h_2 - n(n-1)b.$$

Thus we have the following

$$(4.4) \qquad 2h_2 \leq n(n+1)(c-2b),$$

where the above equality holds if and only if $R_j = 2b$ for any j. That is, $M = M^n(2b)$.

On the other hand, by (3.14) and (1.20) we have that

$$(4.5) \qquad (n-2)R_j \geq (n-1)(n+4)b - n(n+1)c/2 + h_2.$$

Using (1.18), the holomorphic sectional curvature is given by $R_j = R_{\bar{j}j\bar{j}j} = c - \Sigma_x \epsilon_x h^x_{jj} \bar{h}_{jj}{}^x$, from which it follows that

$$(4.6) \quad \Sigma_x \epsilon_x h^x_{jj} \bar{h}_{jj}{}^x = c - R_j \leq \{(n-1)(n+4)(c-2b) - 2h_2\}/2(n-2).$$

With these estimations of the above inequalities we prove here the following.

THEOREM 4.1. *Let M be an $n(\geq 3)$-dimensional complete complex submanifold of $CH^{n+p}_p(c)$, $p > 0$, with totally real bisectional curvature $\geq b$. Then the following holds*

(1) *b is smaller than or equal to $\frac{c}{4}$.*

(2) *If $b = \frac{c}{4}$, then M is a complex space form $CH^n(\frac{c}{2})$, $p \geq \frac{n(n+1)}{2}$.*

(3) *If $b = \frac{n(n+p+1)c}{2(n+2p)(n+1)}$, then M is a complex space form $CH^n(\frac{c}{2})$, $p = \frac{n(n+1)}{2}$.*

Proof. Since M is space-like, the normal space of M can be regarded as a time-like space. Thus the matrix $(h^2_{j\bar{k}})$ given in section 1 is a negative semi-definite Hermitian one, whose eigenvalue $\mu_j{}'$ are non-positive real valued function on M. The matrix (A^x_y) is also by the definition positive semi-definite Hermitian one and its eigenvalues $\mu_x{}'$ are non-negative real valued functions on M. Then it is easily [1] seen that

$$(4.7) \qquad \Sigma_x \epsilon_x \mu_x = TrA = h_2,$$

$$h_2^2 \geq h_4 = \Sigma_j \mu_j^2 \geq \frac{h_2^2}{n},$$

$$h_2^2 \geq A_2 = \Sigma_x \mu_x^2 \geq \frac{h_2^2}{p}.$$

Also from the estimating of the norm of $\Sigma_x\{\epsilon_x h^x_{jk}\bar{h}^x_{il} - \frac{h_2}{n(n+1)}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl})\}$ it follows that

$$A_2 \geq \frac{2}{n(n+1)} h_2^2, \tag{4.8}$$

where the above equality holds if and only if M is a space of constant holomorphic sectional curvature.

By (1.22) and (4.7) we have

$$\triangle h_2 \leq \frac{n+2}{2} ch_2 - 2h_4 - A_2 \leq \frac{n+2}{2} ch_2 - \frac{2}{n} h_2^2 - A_2. \tag{4.9}$$

From this and (4.8) it follows that

$$\triangle h_2 \leq \frac{n+2}{2n(n+1)} h_2 \{n(n+1)c - 4h_2\}, \tag{4.10}$$

where the above equality holds if and only if M is a space of constant curvature.

On the other hand, by the hypothesis of the Theorem and using (1.20) and $r \geq 2n(n+1)b$ we have

$$n(n+1)c - 4h_2 \geq n(n+1)(4b - c), \tag{4.11}$$

from this and (4.10) it follows that

$$\triangle h_2 \leq \frac{n+2}{2}(4b - c)h_2. \tag{4.12}$$

Now we are in a position to prove the first assertion. In fact let us suppose that $b > \frac{c}{4}$. Set $f = -h_2$. Then for given any positive number a, a function F which is defined by $(f + a)^{-\frac{1}{2}}$ is smooth bounded function. Since μ_j is known to be non-positive, the Ricci-curvature $S_{jj} = \frac{n+1}{2}c - \mu_j$ is lower bounded. The function $f = -h_2$ is also bounded by (4.11). By using the similar method to that of Theorem 3.1 we can prove that $f = 0$, that is, M is totally geodesic. From this fact and (4.11) it follows that

$$0 > n(n+1)c \leq n(n+1)(4b - c) > 0.$$

Thus this makes a contradiction. Hence $b \le \frac{c}{4}$. We have proved the first assertion.

For the second assertion we put $b = \frac{c}{4}$. Noticing $h_2 \le 0$, by (4.11) and (4.12) we get

$$\triangle h_2 \le \frac{2(n+2)}{n(n+1)} h_2 \{ \frac{n(n+1)}{4} c - h_2 \} \le 0.$$

From this, taking a smooth no-negative function F such that $F = \frac{n(n+1)}{4} c - h_2$, we have

$$\triangle(-F) \le \frac{2(n+2)}{n(n+1)} \{ \frac{n(n+1)}{4} c - F \} F \le - \frac{2(n+2)}{n(n+1)} F^2.$$

Thus we get $\triangle F \ge \frac{2(n+2)}{n(n+1)} F^2$. Since the Ricci-curvature is bounded from below, we can apply a theorem due to Nishikawa [11] to the function F. Then we get $F = 0$ on M. That is, $h_2 = \frac{n(n+1)}{4} c$. Thus by (4.10) M is a space of constant holomorphic sectional curvature. Moreover by (4.4) its holomorphic sectional curvature is $R_j = 2b$ for any j. That is M is congruent to $M^n(2b) = CH^n(\frac{c}{2})$. Thus the second assertion is now verified.

Now we will prove the last assertion. By (1.20) and (4.4),(4.7) we get

(4.13)

$$\begin{aligned}\triangle h_2 &\le \{np(n+2)ch_2 - 2(n+2p)h_2^2\}/2np \\ &\le \frac{h_2}{2np} \{np(n+2)c - (n+2p)n(n+1)(c-2p)\} \\ &\le \frac{h_2}{2p} \{2(n+1)(n+2p)b - n(n+p+1)c\}.\end{aligned}$$

From this and the assumption it follows that

$$\triangle h_2 \le 0,$$

where the above equality holds if and only if $h_2 = 0$ or $h_2 = \frac{n(n+1)}{2}(c - 2b)$ by virtue of (4.4). That is, $R_j = c$ for any j and $R_{\bar{i}ij\bar{j}} = \frac{c}{2}$ for any $i, j (i \ne j)$ or $R_j = 2b$ for any j and $R_{\bar{i}ij\bar{j}} = b$ for any $i, j (i \ne j)$.

Now we put $F = -h_2 + \frac{np(n+2)}{2(n+2p)}c = a - h_2$, $a < 0$. By (4.11) and the assumption (3) we have $np(n+2)c - 2(n+2p)h_2 \geq 0$. From this we know that the function F is non-negative. Thus by (4.13) we have

$$\text{(4.14)} \quad \triangle(-F) \leq \frac{n+2p}{np} h_2(a - h_2) = \frac{n+2p}{np}(a - F)F \leq -\frac{n+2p}{np}F^2.$$

That is, $\triangle F \geq \frac{n+2p}{np}F^2$. From this we can apply a theorem of Nishikawa [11]. Thus we have $F \equiv 0$ on M. That is, $h_2 = a = \frac{np(n+2)}{2(n+2p)}c$. Thus $R_j = 2b$ for any j and $R_{\bar{i}ij\bar{j}} = b$ for any $i, j (i \neq j)$. Hence M is congruent to $CH^n(2b)$ and $p \geq \frac{n(n+1)}{2}$. By Remark 4.3, $2b = c$ or $\frac{c}{2}$. Thus we conclude that $b = \frac{c}{4}$. From this and together with the assumption (3) we have that $p = \frac{n(n+1)}{2}$. Thus the proof of Theorem 4.1 is completely verified. □

5. Complex submanifold

In this section we study an n-dimensional complex submanifold M of $(n+p)$ -dimensional complex projective space $CP^{n+p}(c), c > 0$, with bounded totally real bisectional curvature. In this case both the tangent space and the normal space of M in $CP^n(c)$ are space-like. Thus the signs ϵ_i and ϵ_x given in section 1 will be denoted by 1.

For a complex submanifold M of $CP^{n+p}(c)$ let us denote the function h_2 by $h_2 = \Sigma_{i,j,x} h^x_{ij} \bar{h}^x_{ij}$. Thus by using (1.21) and the fact that $h^x_{ij\bar{k}} = 0$ we have

(5.1)

$$\begin{aligned}(h_2)_{k\bar{l}} =& \Sigma h^x_{ijk} \bar{h}^x_{ijl} + \Sigma\{\frac{c}{2}(h^x_{ij}\delta_{kl} + h^x_{jk}\delta_{il} + h^x_{ki}\delta_{jl})\bar{h}^x_{ij} \\ &- (h^x_{ri}h^y_{jk} + h^x_{rj}h^y_{ki} + h^x_{rk}h^y_{ij})\bar{h}^y_{rl}\bar{h}^x_{ij}\}.\end{aligned}$$

Also the function h_4 is given by $h_4 = \Sigma h^2_{i\bar{j}} h^2_{j\bar{i}} = \Sigma h^x_{ij}\bar{h}^x_{jk}h^y_{kl}\bar{h}^y_{il}$. Thus from this and also (1.21) it follows that

(5.2)

$$\begin{aligned}\triangle h_4 =& 2\Sigma[\{\frac{n+2}{2}ch^x_{ij} - (h^x_{ir}h^2_{j\bar{r}} + h^x_{jr}h^2_{i\bar{r}} + A_z{}^x h^z_{ij})\}\bar{h}^x_{jk}h^y_{kl}\bar{h}^y_{li} \\ &+ h^x_{ijm}\bar{h}^x_{jkm}h^2_{ki} + h^x_{ijm}h^x_{jk}h^y_{kl}\bar{h}^y_{lim}].\end{aligned}$$

By using these formulas we have the following Theorem.

THEOREM 5.1. *Let M be an $n(\geq 3)$-dimensional complex submanifold of a complex projective space $CP^{n+p}(c)$. If there exist a positive constant b such that $b > \frac{n^3+2n^2+2n-2}{2n(n^2+2n+3)}c$ and the totally real bisectional curvature of M is greater than or equal to b, then M is congruent to a complex projective space $CP^n(c)$.*

Proof. Since in this case the matrix $(h^2_{i\bar{j}})$ and (A^x_y) defined in section 1 are positive semi-definite Hermitian ones, their eigenvalues, say μ_j and μ_y, are all real valued non-negative function on M. Now we choose a local field $\{e_A\} = \{e_j, e_y\}$ of unitary frames such that $h^2_{i\bar{j}} = \mu_i\delta_{ij}$, $A_y{}^x = \mu_x\delta_{xy}$. Then by using this frame to (5.2) and noticing that the second and the third term of the right hand side are non-negative we have

$$(5.3)\qquad \triangle h_4 \geq (n+2)ch_4 - 2h_6 - 2\Sigma\mu_i\mu_j h^x_{ij}\bar{h}^x_{ij} - 2\Sigma\mu_i\mu_x h^x_{ij}\bar{h}^x_{ij}.$$

On the other hand, by using the equation of Gauss (4.1) to the assumption and (4.6) we have the following inequality

(5.4)

$$\begin{aligned}\Sigma\mu_i\mu_j h^x_{ij}\bar{h}^x_{ij} =& \Sigma_{x,i}\mu_i^2 h^x_{ii}\bar{h}^x_{ii} + \Sigma_{x,i\neq j}\mu_i\mu_j h^x_{ij}\bar{h}^x_{ij}\\ \leq& \{(n-1)(n+4)(c-2b) - 2h_2\}\Sigma\mu_i^2/2(n-2)\\ &+ \frac{1}{2}(c-2b)\Sigma\mu_i(h_2-\mu_i)\\ =& \{(n-1)(n+4)-(n-2)\}(c-2b)h_4/2(n-2)\\ &- \frac{1}{n-2}h_2h_4 + \frac{c-2b}{2}h_2^2,\end{aligned}$$

where we have used $h_2 = \Sigma_i\mu_i$ and $h_4 = \Sigma\mu_i^2$. Moreover, we know that the above inequality holds if and only if $M\equiv CP^n(2b)$ or $M\equiv CP^n(c)$.

Since $\mu_x\geq 0$, it follows

$$\mu_x \leq \Sigma\mu_x = \Sigma h^x_{ij}\bar{h}^x_{ij} = h_2,$$

where the equality holds if and only if $\mu_y = 0$ for any $y\neq x$. Using this fact and also (4.1),(4.3) and (4.6), we have the following inequality

(5.5)

$$\begin{aligned}\Sigma\mu_i\mu_x h^x_{ij}h^x_{ij} &\le h_2\Sigma\mu_i h^x_{ij}h^x_{ij}\\ &= h_2\{\Sigma\mu_i h^x_{ii}h^x_{ii} + \Sigma_{x,i\neq j}\mu_i h^x_{ij}h^x_{ij}\}\\ &\le h_2\Big[\frac{(n-1)(n+4)(c-2b)-2h_2}{2(n-2)}\Sigma\mu_i\\ &\quad + \frac{c-2b}{2}\Sigma_{i\neq j}i\mu_i\Big]\\ &= \frac{h_2^2}{2(n-2)}\big[\{(n^2+3n-4)+(n^2-3n+2)\}(c-2b)-2h_2\big]\\ &= \frac{(n^2-1)(c-2b)-h_2^2}{n-2}h_2^2,\end{aligned}$$

where we have used

$$\Sigma_{i\neq j}i\mu_i = (n-1)\Sigma\mu_i = (n-1)h_2.$$

Moreover the above equality of (5.5) holds if and only if $A_y{}^x = 0$, that is, $h_2 = 0$. Thus $M \equiv CP^n(c)$.

Substituting (5.4) and (5.5) into (5.3), we have

$$\begin{aligned}\triangle h_4 &\ge (n+2)ch_4 - 2h_6 - \frac{n^2+2n-2}{n-2}(c-2b)h_4\\ &\quad + \frac{2}{n-2}h_2h_4 - \frac{2n^2+n-4}{n-2}(c-2b)h_2^2 + \frac{2}{n-2}h_2^3\\ &\ge (n+2)ch_4 - \frac{2(n-3)}{n-2}h_2h_4 + \frac{2}{n-2}h_2^3\\ &\quad - \frac{c-2b}{n-2}\{(n^2+2n-2)h_4 + (2n^2+n-4)h_2^2\}\\ &\ge \frac{n+2}{n}ch_2^2 - \frac{2(n-4)}{n-2}h_2^3 - \frac{3n^2+3n-6}{n-2}(c-2b)h_2^2\\ &= \frac{h_2^2}{n(n-2)}\big[(n^2-4)c - 2n(n-4)h_2 - 3n(n^2+n-2)(c-2b)\big],\end{aligned}$$

where we have used $h_6 \le h_2h_4$ to the second inequality and $h_2^2 \ge h_4 \ge \frac{h_2^2}{n}$ to the third inequality respectively. From this, using (4.4), it follows that

$$\begin{aligned}\triangle h_4 \geq & \frac{h_2^2}{n(n-2)}\{(n^2-4)c - n(n^3-n-6)(c-2b)\} \\ = & \frac{h_2^2}{n}\{2n(n^2+2n+3)b - (n^3+2n^2+2n-2)c\}.\end{aligned}$$

Thus $\triangle h_4 \geq Bh_4$ for a positive constant $B = \{2(n^2+2n+3)b - (n^2+2n+2-\frac{2}{n})c\}/n$. By (4.4) the function h_2 is bounded from above and $h_4 \leq h_2^2$. Hence h_4 is also bounded from above. For a constant $a > 0$ let us take a function F such that $F = (f+a)^{-\frac{1}{2}}$, where we have put $f = h_4$. Then by using a similar method as the proof of Theorem 3.1 we get $Supf = Inff = 0$. Thus $f \equiv 0$, i.e., $h_2 = 0$. Hence M is totally geodesic and congruent to a complex projective space $CP^n(c)$. Thus we completed the proof of Theorem 5.1. □

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