J. Korean Math. Soc. **33** (1996), No. 4, pp. 929–954

A QUADRATURE METHOD FOR LOGARITHMIC–KERNEL INTEGRAL EQUATIONS ON CLOSED CURVES

YOUNGMOK JEON AND HA-JINE KIMN

1. Introduction

Consider a Symm's integral equation:

$$\mathcal{L}u(P) = f(P),\ P \in \Gamma \tag{1.1}$$

where

$$\mathcal{L}u(P) := -\frac{1}{\pi}\int_\Gamma \log|P-Q|u(Q)d\Gamma_Q.$$

We assume that Γ is the smooth boundary of a simply connected bounded region Ω, and it is of $Capacity(\Gamma) \neq 1$. Equation (1.1) is derived when we use the single layer potential for a boundary element method of Laplace equation with the Dirichlet boundary data.

Consider a parameterization $P = \gamma(t)$ with $|\gamma'(t)| \neq 0$. We obtain a parameterized equation of equation (1.1):

$$\mathcal{L}u(t) = f(t),\ 0 \le t \le 1, \tag{1.2}$$

where

$$\mathcal{L}u(t) = -2\int_0^1 \log|\gamma(t)-\gamma(s)|u(s)ds$$

with $f(t) := f(\gamma(t))$ and $u(t) := u(\gamma(t))|\gamma'(t)|/(2\pi)$.

Received July 19, 1995.

1991 AMS Subject Classification: 47G30, 65N35, 65R20.

Key words: Symm's integral equation, pseudodifferential operator, qualocation, quadrature.

This research is supported by KOSEF Grant 951-0104-040-2.

Introduce a well-known isometry operator $\mathcal{A} : H^s \to H^{s+1}$, $s \in \mathbb{R}$, where H^s is a periodic Sobolev space (see §2 for H^s):

$$(1.3) \qquad \mathcal{A}u(t) := -2\int_0^1 \log|2e^{-1/2}\sin(\pi(t-s))|u(s)ds.$$

Note that $\mathcal{A} = \mathcal{L}$ when Γ is the circle of radius $e^{-1/2}$. For a general smooth boundary, the operator $\mathcal{L}$ in (1.2) can be expressed as a sum of $\mathcal{A}$ and its compact perturbation.

$$(1.4) \qquad \mathcal{L} = \mathcal{A} + \mathcal{B},$$

where

$$(1.5) \qquad \mathcal{B}u(t) := -2\int_0^1 \log\left|\frac{\gamma(t)-\gamma(s)}{2e^{-1/2}\sin(\pi(t-s))}\right| u(s)ds.$$

The operator $\mathcal{B}$ has a smooth kernel for a smooth boundary Γ, and it has little effect on our numerical analysis (see Theorem 3.1 in §3). Therefore, the operator $\mathcal{A}$ is of our main concern for numerical analysis.

Sloan and Burn [6] considered a numerical method (a fully discrete qualocation method):

$$(1.6) \qquad (\mathcal{L}_h u_h, \chi)_h = (f, \chi)_h, \ \ \chi \in S_h^r,$$

where

$$(1.7) \qquad \mathcal{L}_h u_h(t) := -2h\sum_{k=0}^{N-1} \log|\gamma(t)-\gamma(kh)|u_h(kh), \quad h = \frac{1}{N},$$

and the $J-$ point discrete inner product is defined as

$$(1.8) \qquad (f,g)_h := h\sum_{k=0}^{N-1}\sum_{j=1}^{J} w_j (f\cdot\overline{g})(kh+\xi_j h)$$

with

$$\sum_{j=1}^{J} w_j = 1, \quad w_j > 0.$$

The trial function u_h is a trigonometric function of degree N, and S_h^r is the space of smoothest splines of order r with nodes $\{kh\}_{k=0}^{N-1}$. The discrete inner product $(\cdot,\cdot)_h$ is a quadrature approximation of the usual inner product $(\cdot,\cdot)$ in L_2. Note that smoothest splines of order 1 (S_h^1) consist of piecewise constant functions, and splines of order 2 (S_h^2) have a base $\{v_0, v_1, \cdots, v_{N-1}\}$ consisting of hat functions such that

$$v_k(x) = \begin{cases} 1 - |x - kh|/h, & (k-1)h < x < (k+1)h, \\ 0, & \text{otherwise}. \end{cases} \tag{1.9}$$

Splines of order 4 (S_h^4) are the cubic splines that can be found in an ordinary numerical text book. By using these local basis, the numerical method (1.6) can be chiefly implemented. Moreover, if one uses one point quadrature rule, $J = 1$ in (1.8), the method (1.6) is reduced to a discrete collocation method [6].

Sloan and Burn [6] have proved that at best,

$$\|u - u_h\|_t \leq Ch^3 \|u\|_{t+3}, \quad t \geq -1$$

for $u \in H^{t+3}$ when S_h^2 is used as test functions and when $J = 2$ with special abscissas $\xi_1 = 1/6$ and $\xi_2 = 5/6$ is used for the discrete inner product $(\cdot,\cdot)_h$. Here $\|\cdot\|_s$ is the norm in H^s. Moreover, Saranen and Sloan [5] proposed a modified method of (1.6)

$$(\mathcal{L}_h u_h, \chi)_h = (f, \chi), \quad \chi \in S_h^r, \tag{1.10}$$

and they could show that at best

$$\|u - u_h\|_t \leq Ch^{\min\{r,3\}} \|u\|_{t+\min\{r,3\}}, \quad t \geq -r-1,$$

for $u_h \in S_h^r$ when the same discrete inner product as the above is applied. Then u can be well approximated even when u has poorer regularity (e.g. u is the Dirac delta function). They also noticed that the methods (1.6) and (1.10) can be interpreted as a projection method: that is, equations (1.6) and (1.10) are

$$P_h \mathcal{L}_h u_h = P_h f$$

$$P_h \mathcal{L}_h u_h = \hat{P}_h f,$$

respectively with properly defined projections P_h and $\hat{P}_h$. The use of projections makes more concise analysis possible.

In this paper we consider an approximation operator $\mathcal{L}_h^*$, a modification of $\mathcal{L}_h$ by the subtraction of a singularity:

$$\mathcal{L}_h^* = \mathcal{A}_h^* + \mathcal{B}_h, \tag{1.11}$$

where

(1.12)

$$\mathcal{A}_h^* u_h := -2h \sum_{k=0}^{N-1} \log |2e^{-1/2} \sin(\pi(t - kh))|(u_h(kh) - u_h(t)) + u_h(t),$$

and $\mathcal{B}_h$ is a rectangular rule approximation of $\mathcal{B}$. Clearly, $\mathcal{L}_h^*$ is a better approximation to $\mathcal{L}$ than $\mathcal{L}_h$ is. Our method is: find u_h so that

$$(\mathcal{L}_h^* u_h, \chi)_h = (f, \chi)_h. \tag{1.13}$$

In implementing (1.13), we need to evaluate u_h at other points than node points. It will be evaluated by the Dirichlet kernel interpolation formula,

(1.14)

$$u_h(t) = \begin{cases} h \sum_{m=0}^{N-1} \frac{\sin(N\pi(t-mh))}{\sin(\pi(t-mh))} u_h(mh), & N : \text{odd}, \\ h \sum_{m=0}^{N-1} \cos(t - mh) \frac{\sin(N\pi(t-mh))}{\sin(\pi(t-mh))} u_h(mh), & N : \text{even}. \end{cases}$$

With our modified method, we have an improved convergence

$$\|u - u_h\|_t \le Ch^5 \|u\|_{t+5}, \; t \ge -1, \tag{1.15}$$

when S_h^2 is used as test functions, and when the quadrature method induced by a symmetric two point quadrature rule with the abscissas $\xi_1 = .2308296503$, $\xi_2 = 1 - \xi_1$, is used for the discrete inner product.

This paper is organised in the following way: in §2 we introduce qualocation operators induced by the discrete inner product (1.8); in §3 an abstract numerical analysis for general pseudodifferential operators is given; using the abstract analysis, we give convergence and stability analysis of our method (1.13) in §4. We give results of our numerical experiments in §5.

2. Smooth splines and qualocation operators

Let H^s be the Sobolev space of periodic functions on $[0, 1]$ with norm

$$(2.1) \qquad \|f\|_s^2 := \sum_{m=-\infty, m\neq 0}^{\infty} |m|^{2s}|\hat{f}(m)|^2 + |\hat{f}(0)|^2 < \infty$$

where

$$(2.2) \qquad f(t) = \sum_{m=-\infty}^{\infty} \hat{f}(m)e^{2\pi imt}$$

and

$$(2.3) \qquad \hat{f}(m) = (f, e^{2\pi imt}) = \int_0^1 f(t)e^{-2\pi imt}dt.$$

Here the equality in (2.2) holds in the sense of distribution where the sum is interpreted as the limit of the symmetric partial sum for $f \in H^s$ $(s \le 1/2)$. The integral in (2.3) is understood as a duality paring in case $f \in H^s$ $(s < 0)$. For simplicity of notations, let

$$(2.4) \qquad \phi_k(t) := e^{2\pi ikt}.$$

Let $\Lambda_h = \{k| -N/2 \le k < N/2, N : \text{even}\}$ or $\{k| -(N-1)/2 \le k \le (N-1)/2, N : \text{odd}\}$, and let $\Lambda_h^* = \Lambda_h/\{0\}$. The space of trigonometric polynomials of degree $\le N$, T_h is

$$(2.5) \qquad T_h \equiv span\{\phi_k\}_{k\in\Lambda_h}.$$

Introduce the orthogonal projection P_N in L_2 such that

$$(2.6) \qquad P_N f(t) = \sum_{m\in\Lambda_h} \hat{f}(m)\phi_m(t).$$

Then P_N and T_h have the following useful properties:

1. $\|\phi\|_s \le Ch^{(t-s)}\|\phi\|_t$ for $t \le s$ if $\phi \in T_h$ (the inverse estimate),
2. $\|f - P_N f\|_t \le Ch^{(s-t)}\|f\|_s$ for $t \le s$ and $f \in H^s$ (an approximation property).

We introduce the Fourier series representation for 1-periodic smoothest spline functions of order r on evenly spaced nodes, $\{kh\}_{k=0}^{N-1}$. This is equivalent to the one introduced in §1 (see (1.9)), but it will let us take advantages of the Fourier series analysis.

$$S_h^r := span\{\psi_k\}_{k\in\Lambda_h} \tag{2.7}$$

with

$$\psi_k(t) = \begin{cases} 1 & k=0 \\ \sum_{m\equiv k} \left(\frac{k}{m}\right)^r \phi_m(t) & k \in \Lambda_h^* \end{cases} \tag{2.8}$$

where $m \equiv k$ means that $m-k$ is a multiple of N. Introduce

$$F_r^+(\xi,\eta) = \sum_{l\in\mathbb{Z}^*} \frac{1}{|l+\eta|^r}\phi_l(\xi) = G_r^+(\xi,\eta) - iH_r^+(\xi,\eta) \tag{2.9}$$

$$F_r^-(\xi,\eta) = \sum_{l\in\mathbb{Z}^*} \frac{sign(l)}{|l+\eta|^r}\phi_l(\xi) = G_r^-(\xi,\eta) + iH_r^-(\xi,\eta) \tag{2.10}$$

where $\mathbb{Z}$ is the space of integers and $\mathbb{Z}^* = \mathbb{Z}/\{0\}$. The functions, $G^\pm$ and $H^\pm$ will be explained in Remark 1. Let

$$\Delta_r(\xi,\eta) = \eta^r F_r^\pm(\xi,\eta) \tag{2.11}$$

where $+$ and $-$ sign hold when r is even or odd, respectively. Then we can rewrite $\psi_k(t)$ as

$$\psi_k(t) = \begin{cases} 1 & k=0 \\ \phi_k(t)[1+\Delta_r(Nt,kh)] & k\in\Lambda_h^*. \end{cases} \tag{2.12}$$

Define a *qualocation operator* $P_h : H^s \to T_h$ by

$$P_h(f) = \sum_{k\in\Lambda_h} (f,\psi_k)_h \phi_k. \tag{2.13}$$

REMARK 1. From (2.9) and (2.10),

$$G_r^{\pm}(\xi,\eta)=\sum_{l=1}^{\infty}\left[\frac{1}{(l+\eta)^r}\pm\frac{1}{(l-\eta)^r}\right]\cos 2\pi l\xi \tag{2.14}$$

and

$$H_r^{\pm}(\xi,\eta)=\sum_{l=1}^{\infty}\left[\frac{1}{(l+\eta)^r}\mp\frac{1}{(l-\eta)^r}\right]\sin 2\pi l\xi. \tag{2.15}$$

Then, based on the analysis of [2], [3], [6], we have the following results: for any $r>0$,

$$1+\eta^r G_r^{+}(\xi,\eta)\ge 0,\quad \xi\in(0,1),\quad \eta\in[0,\frac{1}{2}] \tag{2.16}$$

with equality if and only if $(\xi,\eta)=(1/2,1/2)$;

$$1+\eta^r G_r^{-}(\xi,\eta)\ge 0,\quad \xi\in[0,1],\quad \eta\in[0,\frac{1}{2}] \tag{2.17}$$

with equality if and only if $(\xi,\eta)=(0,1/2)$ or $(1,1/2)$, and

$$H_r^{+}(\xi,\eta)\le 0,\quad \xi\in[0,\frac{1}{2}],\quad \eta\in[0,\frac{1}{2}], \tag{2.18}$$

$$H_r^{-}(\xi,\eta)\ge 0,\quad \xi\in[0,\frac{1}{2}],\quad \eta\in[0,\frac{1}{2}]. \tag{2.19}$$

In (2.16), $\xi\in(0,1)$ can be replaced with $\xi\in[0,1]$ if $r>1$, because G_r^{+} is uniformly bounded in ξ for $r>1$.

LEMMA 2.1. *$P_h: T_h\to T_h$ is invertible,*

(1) *unless $\xi=1/2$ with $J=1$ in equation (1.8) when r is even,*

(2) *unless $\xi=0$ or 1 with $J=1$ in equation (1.8) when r is odd,*

where ξ and J are parameters used for $(\cdot,\cdot)_h$.

Proof. By simple calculation, we have

$$P_h(\phi_k)=\sum_{k\in\Lambda_h}(\phi_k,\psi_l)_h\phi_l=(\phi_k,\psi_k)_h\phi_k$$

because

$$(\phi_k, \psi_l)_h = \begin{cases} 0 & k \neq l \\ 1 & k = l = 0 \\ \sum_j w_j[1 + \overline{\Delta_r(\xi_j, lh)}] & k = l \in \Lambda_h^*. \end{cases}$$

Now we need to show

$$\sum_j w_j[1 + \overline{\Delta_r(\xi_j, lh)}] \neq 0$$

Clearly,

$$1 + \overline{\Delta_r(\xi, \eta)} = [1 + \eta^r G_r^{\pm}(\xi, \eta)] - i\eta^r \; H_r^{\pm}(\xi, \eta). \tag{2.20}$$

From Remark 1, we have $Re[1 + \overline{\Delta_r(\xi, \eta)}] > 0$ unless $\xi = 1/2$ if r is even, and $Re[1 + \overline{\Delta_r(\xi, \eta)}] > 0$ unless $\xi = 0$ or 1 when r is odd.

LEMMA 2.2.

$$(P_h f, v) = (f, v)_h, \quad v \in S_h^r.$$

Proof. For any $l \in \Lambda_h$,

$$\begin{aligned} (P_h f, \psi_l) &= (\sum_{k \in \Lambda_h} (f, \psi_k)_h \phi_k, \psi_l) \\ &= \sum_{k \in \Lambda_h} (f, \psi_k)_h (\phi_k, \psi_l) \\ &= (f, \psi_l)_h \end{aligned}$$

because $(\phi_k, \psi_l) = \delta_{kl}$ if $k, \ l \in \Lambda_h$.

COROLLARY 2.3. *If*

$$(f, v)_h = (g, v)_h, \quad v \in S_h^r,$$

then

$$P_h f = P_h g$$

Proof. By Lemma 2.2, $(P_h f, v) = (P_h g, v)$ for $v \in S_h^r$. Then $(P_h f - P_h g, v) = 0$. Write

$$P_h f - P_h g = \sum_{k \in \Lambda_h} c_k \phi_k.$$

Because $(\phi_k, \psi_l) = \delta_{kl}$, $(P_h f - P_h g, \psi_k) = c_k = 0$ for $k \in \Lambda_h$. The corollary follows.

THEOREM 2.4. *For $f \in H^s$ with $s > 1/2$, we have an error estimate*

$$\|f - P_h f\|_t \le C h^{s-t} \|f\|_s$$

if $0 \le t \le s \le t + r$.

Proof. Let $P_h f = \sum_{k \in \Lambda_h} \alpha_k \phi_k$ with $\alpha_k = (f, \psi_k)_h$. Then

$$(2.21) \quad \alpha_k = \begin{cases} \hat{f}(0) + \sum_{l \neq 0} [\sum_j w_j \phi_{lN}(\xi_j h)] \hat{f}(lN), & k = 0 \\ \hat{f}(k)[1 + \sum_j w_j \overline{\Delta_r(\xi_j, kh)}] \\ + \sum_{l \neq 0} \left[\sum_j w_j \phi_{lN}(\xi_j h)[1 + \overline{\Delta_r(\xi_j, kh)}]\right] \hat{f}(k + lN), & k \in \Lambda_h^*. \end{cases}$$

Therefore,

$$|\alpha_0 - \hat{f}(0)|^2 \le C(\sum_{l \neq 0} |\hat{f}(lN)|)^2$$

$$= C \left[\sum_{l \neq 0} \frac{1}{(lN)^{2s}}\right] \left[\sum_{l \neq 0} (lN)^{2s} |\hat{f}(lN)|^2\right]$$

$$\le C h^{2s} \sum_{l \neq 0} (lN)^{2s} |\hat{f}(lN)|^2, \quad s > 1/2$$

Because $\overline{\Delta_r(\xi_j, kh)} \le C|kh|^r$ by (2.11), we have

$$|k|^{2t} |\alpha_k - \hat{f}(k)|^2 \le C|kh|^{2r} |k|^{(2t-2s)} |k|^{2s} |\hat{f}(k)|^2 + T,$$

$$\le C h^{2(s-t)} |kh|^{2(r+t-s)} |k|^{2s} |\hat{f}(k)|^2 + T$$

$$\le C h^{2(s-t)} |k|^{2s} |\hat{f}(k)|^2 + T, \quad r + t - s \ge 0,$$

where

$$T \le C|k|^{2t} (\sum_{l \neq 0} |\hat{f}(lN + k)|)^2$$

$$\le C \sum_{l \neq 0} \frac{|k|^{2t}}{(lN + k)^{2s}} \sum_{l \neq 0} |lN + k|^{2s} |\hat{f}(lN + k)|^2$$

$$= C h^{2(s-t)} \sum_{l \neq 0} \frac{(kh)^{2t}}{|l + kh|^{2s}} \sum_{l \neq 0} |lN + k|^{2s} |\hat{f}(lN + k)|^2$$

$$\le C h^{2(s-t)} \sum_{l \neq 0} |lN + k|^{2s} |\hat{f}(lN + k)|^2,$$

where $t \geq 0,\ s > 1/2,\ |kh| \leq 1/2$. Then

$$\sum_{k\in\Lambda_h} \hat{k}^{2t}|\alpha_k - \hat{f}(k)|^2 \leq Ch^{2(s-t)}\|f\|_s^2 + Ch^{2s}\|f\|_s^2, \quad \hat{k} = \max\{1, |k|\}.$$

Now the theorem is immediate. □

A last comment about P_h is that $P_h P_h \neq P_h$, so P_h is not a projection.

3. Abstract Numerical Analysis

Consider an integral equation,

$$\mathcal{A}u + \mathcal{B}u = f \tag{3.1}$$

where $\mathcal{A} : H^s \to H^{s+\nu}$ is an invertible pseudodifferential operator of order $-\nu$, and $\mathcal{B}$ is a pseudodifferential operator of order $-\infty$. Then $\mathcal{B}$ is a compact operator from H^s to $H^{s+\nu}$ by Sobolev embedding theorem. Assume (3.1) is uniquely solvable for arbitrary f. Then the system (3.1) is stable because of the Fredholm alternative theorem. We will investigate the unique solvability and convergence of the solution of the following approximation equation,

$$P_h\mathcal{A}_h u_h + P_h\mathcal{B}_h u_h = P_h f \tag{3.2}$$

with $u_h \in T_h$. Here $\mathcal{A}_h$ and $\mathcal{B}_h$ are approximation operators defined by using certain quadrature rules.

Assuming $P_h\mathcal{A}_h : T_h \to T_h$ is invertible, we can introduce a solution operator $R_h : H^s \to T_h$ such that

$$P_h\mathcal{A}_h R_h u = P_h\mathcal{A}u. \tag{3.3}$$

The operator R_h is a solution operator that is obtained from equation (3.2) with $\mathcal{B} = 0$. With aid of the operator R_h, we can write equations (3.1) and (3.2) in standard second kind integral equation forms. Then we have

$$u + \mathcal{M}u = g, \tag{3.4}$$

and

$$u_h + R_h \mathcal{M}_h u_h = R_h g, \tag{3.5}$$

where $\mathcal{M} := \mathcal{A}^{-1}\mathcal{B}$, $\mathcal{M}_h := \mathcal{A}^{-1}\mathcal{B}_h$ and $g = \mathcal{A}^{-1}f$. Note that $\mathcal{M}$ is also a pseudodifferential operator of order $-\infty$.

Now for the stability and error analysis of equation (3.2), we analyse equation (3.5) instead. For the analysis of (3.5), we need some additional assumptions.

(A1) $\|R_h u - u\|_t \le C h^{s-t}\|u\|_s$ for $u \in H^s$, where $-\nu \le t \le s \le t+p$ and $s > 1/2 - \nu$,

(A2) $\|(\mathcal{M}_h - \mathcal{M})u\|_t \le C h^{\lambda}\|u\|_s$ where $u \in T_h$, $\lambda \ge p$ and $t,\ s \in \mathbb{R}$.

In (A1) the condition $t \ge -\nu$, as will be seen in the later sections, is a characteristic of our method. Moreover, $s > 1/2 - \nu$ is needed since $\mathcal{A}u$ is continuous when $u \in H^s$; then $R_h u$ is well-defined. The assumption (A2) is satisfied if $\mathcal{B}$ is an integral operator with smooth kernel (see Corollary 3.3).

THEOREM 3.1. *Then equation (3.5) is uniquely solvable, and we have an error estimate,*

$$\|u - u_h\|_t \le C h^{s-t}\|u\|_s,$$

where $-\nu \le t \le s \le t + p$.

Proof. First, we prove the stability of (3.5):

$$\|(I + R_h \mathcal{M}_h)\psi\|_t \ge C\|\psi\|_t, \ \psi \in T_h \tag{3.6}$$

for an arbitrary $t \in \mathbb{R}$ and a constant C only dependent on t if h is small enough.

Since the operator $(I + \mathcal{M})$ is stable in an arbitrary Sobolev space, we have

$$\|(I + \mathcal{M})\psi\|_t \ge C\|\psi\|_t, \ \psi \in H^t \tag{3.7}$$

for $t \in \mathbb{R}$. The stability estimate (3.6) is proved if we show

$$\|(\mathcal{M} - R_h \mathcal{M}_h)\psi\|_t \le C h^r \|\psi\|_t, \ \psi \in T_h$$

for some $r > 0$.

$$\|(\mathcal{M} - R_h\mathcal{M}_h)\psi\|_t \leq \|(I - R_h)\mathcal{M}\psi\|_t + \|R_h(\mathcal{M} - \mathcal{M}_h)\psi\|_t .$$

Using (A1) and (A2),

$$\begin{aligned}\|(I - R_h)\mathcal{M}\psi\|_t &\leq \|(I - R_h)\mathcal{M}\psi\|_{\max\{t,-\nu\}} \\ &\leq Ch^p\|\mathcal{M}\psi\|_{\max\{t,-\nu\}+p} \\ &\leq Ch^p\|\psi\|_t.\end{aligned}$$

because $\mathcal{M} : H^\lambda \to H^t$ is bounded for arbitrary $\lambda,\ t \in \mathbb{R}$. We also have

$$\begin{aligned}&\|R_h(\mathcal{M} - \mathcal{M}_h)\psi\|_t \\ \leq\ &\|(\mathcal{M} - \mathcal{M}_h)\psi\|_t + Ch^p\,\|(\mathcal{M} - \mathcal{M}_h)\psi\|_{\max\{t,-\nu\}+p} \\ \leq\ &Ch^r\|\psi\|_t\end{aligned} \tag{3.8}$$

where $r \geq p$. Now the stability is proved.

For error analysis, substitute g in (3.5) by using (3.4). Then we have

$$u_h + R_h\mathcal{M}_h u_h = R_h u + R_h\mathcal{M}u.$$

Rewrite the above equation as follows:

$$(I + R_h\mathcal{M}_h)(u_h - P_N u) = (R_h u - P_N u) + (R_h\mathcal{M}u - R_h\mathcal{M}_h P_N u)$$

where P_N is the orthogonal projection in L_2. The stability result yields

$$\|(u_h - P_N u)\|_t = C(\|R_h u - P_N u\|_t + \|R_h\mathcal{M}u - R_h\mathcal{M}_h P_N u\|_t)$$

for $t \in \mathbb{R}$. Since (A2) and the approximation property of the orthogonal projection yield

$$\begin{aligned}\|\mathcal{M}u - \mathcal{M}_h P_N u\|_\lambda &\leq \|\mathcal{M}(u - P_N u)\|_\lambda + \|(\mathcal{M} - \mathcal{M}_h)P_N u\|_\lambda \\ &\leq \|u - P_N u\|_{\lambda-p} + Ch^p\|P_N u\|_\lambda \\ &\leq Ch^p\|u\|_\lambda,\end{aligned}$$

we have, using (A1),

$$\begin{aligned}\|R_h(\mathcal{M}u - \mathcal{M}_h P_N u)\|_t &\\ &\le \|\mathcal{M}u - \mathcal{M}_h P_N u\|_t + Ch^p\|\mathcal{M}u - \mathcal{M}_h P_N u\|_{\max\{t,-\nu\}+p}\\ &\le Ch^p\|u\|_\lambda\end{aligned}$$

for arbitrary $\lambda \in \mathbb{R}$. Now, using (A1) for $-\nu \le t \le s \le t+p$ and $s > -\nu + 1/2$,

$$\begin{aligned}\|R_h u - P_N u\|_t &\le \|R_h u - u\|_t + \|u - P_N u\|_t\\ &\le Ch^{s-t}\|u\|_s + Ch^{s-t}\|u\|_s.\end{aligned}$$

Then

$$\|u_h - P_N u\|_t \le Ch^{s-t}\|u\|_s.$$

The theorem is obtained by applying the triangle inequality. □

Now we give some convergence result for an arbitrary order pseudodifferential operator. This kind of result is also found in [5], and we supply a proof since it has a little bit different flavor.

LEMMA 3.2. *Assume $\mathcal{K}$ is a pseudodifferential (integral) operator of order $-k$, i.e., $\|\mathcal{K}\psi\|_{p+k} \le C_p\|\psi\|_p$, for arbitrary p and $\psi \in H^p$. Then, for $\psi \in T_h$,*

$$\|\mathcal{K}\psi - \mathcal{K}_h\psi\|_q \le Ch^{k-q}\|\psi\|_{1/2+\alpha}, \quad \alpha > 0 \tag{3.9}$$

provided $k - q > 1$, where

$$\mathcal{K}(\psi)(t) := \int_0^1 K(t,s)\psi(s)ds,$$

and its approximation operator,

$$\mathcal{K}_h(\psi)(t) := h\sum_{\lambda=0}^{N-1} K(t,\lambda h)\psi(\lambda h).$$

Proof. Let

$$\begin{aligned}\mathcal{K}(\phi_m)(t) &= \int_0^1 K(t,s)\psi(s)ds \\ &= \sum_{l\in\mathbb{Z}} a_{m,l}\phi_l(t),\end{aligned}$$

and

$$\begin{aligned}\mathcal{K}_h(\phi_m)(t) &= h\sum_{\lambda=0}^{N-1} K(t,\lambda h)\phi_m(\lambda h) \\ &= \sum_{l\in\mathbb{Z}} b_{m,l}\phi_l(t).\end{aligned}$$

Note that from our assumption

$$\sum_{l\in\mathbb{Z}} |a_{m,l}|^2 l^{2(p+k)} \le C_p'|m|^{2p}, \quad p\in\mathbb{R}.$$

Introducing an adjoint integral operator $\mathcal{K}^*$ of $\mathcal{K}$ such that

$$\begin{aligned}\mathcal{K}^*(\phi_m)(s) &= \int_0^1 K(t,s)\phi_m(t)dt \\ &= \sum_{l\in\mathbb{Z}} c_{m,l}\phi_l(s),\end{aligned}$$

we can express $a_{m,l}$ and $b_{m,l}$ in terms of $c_{m,l}$. First,

$$\begin{aligned}a_{m,k} &= \int_0^1\int_0^1 K(t,s)\phi_m(s)\phi_{-k}(t)dsdt \\ &= \int_0^1\left[\int_0^1 K(t,s)\phi_{-k}(t)dt\right]\phi_m(s)ds \\ &= \int_0^1\left[\sum_{l\in\mathbb{Z}} c_{-k,l}\phi_l\right]\phi_m ds = c_{-k,-m}.\end{aligned}$$

By the same way,

$$\begin{aligned} b_{m,k} &= \int_0^1 \left[h \sum_{\lambda=1}^{N} K(t, \lambda h)\phi_m(\lambda h) \right] \phi_{-k}(t)dt \\ &= h \sum_{\lambda=1}^{N} \phi_m(\lambda h) \left[\int_0^1 K(t, \lambda h)\phi_{-k}(t)dt \right] \\ &= h \sum_{\lambda=1}^{N} \phi_m(\lambda h) \left[\sum_{l\in\mathbb{Z}} c_{-k,l}\phi_l(\lambda h) \right] \\ &= \sum_{l\in\mathbb{Z}} c_{-k,l} \left[h \sum_{\lambda=1}^{N} \phi_m(\lambda h)\phi_l(\lambda h) \right] \\ &= \sum_{j\in\mathbb{Z}} c_{-k,-m+jN}. \end{aligned}$$

Notice that

$$\sum_{l\in\mathbb{Z}} |c_{-l,-m+jN}|^2 |l|^{2q} = \sum_{l\in\mathbb{Z}} |a_{m-jN,l}|^2 |l|^{2q} \le C_{q-k}^2 |m - jN|^{2(q-k)}.$$

Then for $m \in \Lambda_h$,

$$\begin{aligned} &\|\mathcal{K}(\phi_m) - \mathcal{K}_h(\phi_m)\|_q^2 \\ &= \sum_{l\in\mathbb{Z}} |a_{m,l} - b_{m,l}|^2 |l|^{2q} = \sum_{l\in\mathbb{Z}} \left[\sum_{j\in\mathbb{Z}^*} c_{-l,-m+jN} \right]^2 |l|^{2q} \\ &\le \sum_{l\in\mathbb{Z}} \left[\sum_{j\in\mathbb{Z}^*} |c_{-l,-m+jN}|^2 |-m+jN|^{2r} \right] \cdot \left[\sum_{j\in\mathbb{Z}^*} |-m+jN|^{-2r} \right] |l|^{2q} \\ &\le \left[\sum_{j\in\mathbb{Z}^*} |-m+jN|^{-2r} \right] \cdot \left[\sum_{j\in\mathbb{Z}^*} |-m+jN|^{2r} \cdot \left[\sum_{l\in\mathbb{Z}} |c_{-l,-m+jN}|^2 |l|^{2q} \right] \right] \\ &\le \left[\sum_{j\in\mathbb{Z}^*} |-m+jN|^{-2r} \right] \cdot \left[\sum_{j\in\mathbb{Z}^*} |-m+jN|^{2(q-k+r)} \right] \\ &\le Ch^{2(k-q)}, \ r > 1/2, \ q-k+r < -1/2. \end{aligned}$$

Note in the above proof, r is an intermediate number, and r such that $r > 1/2, \ q - k + r < -1/2$, exists only when $k - q > 1$. Suppose

$$\psi = \sum_{m\in\Lambda_h} \hat{\psi}(m)\phi_m.$$

Then

$$
\begin{aligned}
&\|\mathcal{K}\psi - \mathcal{K}_h\psi\|_q \\
&\leq \sum_{m\in\lambda_h} |\hat{\psi}(m)| \cdot \|\mathcal{K}\phi_m - \mathcal{K}_h\phi_m\|_q \\
&\leq \sqrt{\sum_{m\in\Lambda_h} |\hat{\phi}(m)|^2|m|^{1+2\alpha}} \cdot \sqrt{\sum_{m\in\Lambda_h} \|\mathcal{K}\phi_m - \mathcal{K}^h\phi_m\|_q^2|m|^{-1-2\alpha}}, \quad \alpha > 0 \\
&\leq C\|\psi\|_{1/2+\alpha} \max_{m\in\Lambda_h} \|\mathcal{K}\phi_m - \mathcal{K}_h\phi_m\|_q \\
&\leq Ch^{k-q}\|\psi\|_{1/2+\alpha}.
\end{aligned}
$$

□

From Lemma 3.2 and the inverse estimate on T_h, we have the following consequence.

COROLLARY 3.3. *Assume $\mathcal{K}$ is an integral operator with an smooth kernel. For $\psi \in T_h$,*

$$
\|\mathcal{K}\psi - \mathcal{K}_h\psi\|_t \leq Ch^\lambda\|\psi\|_s \tag{3.10}
$$

where $s,\ t \in \mathbb{R}$, $\lambda > 0$.

4. First Kind Integral Equations on Smooth Curves

In this section, we analyse the numerical method (1.13):

$$
P_h\mathcal{L}_h^* u_h = P_h\mathcal{L}u, \tag{4.1}
$$

which is obtained by using the qualocation operator P_h and by substituting f with $\mathcal{L}u$. Rewrite (4.1) as

$$
P_h(\mathcal{A}_h^* + \mathcal{B}_h)u_h = P_h(\mathcal{A} + \mathcal{B})u. \tag{4.2}
$$

Remember that

$$
\mathcal{A}u(x) = -2\int_0^1 \log|2e^{-1/2}\sin\pi(x-y)|u(y)dy, \tag{4.3}
$$

and

$$\mathcal{B}u(x) = -2\int_0^1 \log\left|\frac{\gamma(x)-\gamma(y)}{2e^{-1/2}\sin\pi(x-y)}\right| u(y)dy. \tag{4.4}$$

The operator $\mathcal{A}$ is an isometry operator from H^s to H^{s+1} for any $s \in \mathbb{R}$ such that

$$\mathcal{A}u(x) = \sum_{m\in\mathbb{Z}} \frac{1}{\hat{m}}\hat{u}(m)\phi_k(x), \quad \hat{m} = \max\{1, |m|\}, \tag{4.5}$$

where

$$u(x) = \sum_{m\in\mathbb{Z}} \hat{u}(m)\phi_m(x).$$

The operator $\mathcal{B}$ has a smooth kernel if γ is a smooth parametrization.

In view of (1.7) and (1.12),

$$\mathcal{A}_h u(x) = -2h\sum_{m=0}^{N-1} \log|2e^{-1/2}\sin\pi(x-mh)|u(mh), \quad x \neq mh, \tag{4.6}$$

$$\begin{aligned} \mathcal{A}_h^* u(x) &= -2h\sum_{m=0}^{N-1} \log|2e^{-1/2}\sin\pi(x-mh)|(u(mh)-u(x)) + u(x) \\ &= (\mathcal{A}_h u)(x) + u(x)[1-\mathcal{A}_h(1)], \end{aligned} \tag{4.7}$$

and

$$\mathcal{B}_h\phi(x) = -2h\sum_{m=0}^{N-1} \log\left|\frac{P(x)-P(mh)}{2e^{-1/2}\sin\pi(x-mh)}\right| \phi(mh). \tag{4.8}$$

Taken from analysis in [6],

$$\mathcal{A}_h\phi_k(x) = \sum_{m\equiv k} \frac{1}{\hat{m}}\phi_m(x), \tag{4.9}$$

where the notation $m \equiv k$ represents $m \equiv k\ (mod\ N)$. Define

$$G(x) := \sum_{l\in\mathbb{Z}^*} \frac{1}{|l|}\phi_l(x), \quad x \notin \mathbb{Z}, \tag{4.10}$$

and

$$\Gamma(x,\eta) := \sum_{l\in\mathbb{Z}^*} \frac{|\eta|}{|l+\eta|}\phi_l(x), \quad x \notin \mathbb{Z}. \tag{4.11}$$

Then

$$\mathcal{A}_h\phi_k(x) = \begin{cases} 1 + hG(Nx), & k = 0, \\ \frac{1}{|k|}\phi_k(x)[1+\Gamma(Nx,kh)], & k \in \Lambda_h^*, \end{cases} \tag{4.12}$$

and

$$\mathcal{A}_h^*\phi_k(x) = \begin{cases} 1, & k = 0 \\ \frac{1}{|k|}\phi_k(x)[1+\Gamma^*(Nx,kh)], & k \in \Lambda_h^*, \end{cases} \tag{4.13}$$

where

$$\Gamma^*(x,\eta) = \Gamma(x,\eta) - |\eta|G(x). \tag{4.14}$$

By simple calculation,

$$\begin{aligned} \Gamma^*(x,\eta) = & \sum_{l=1}^{\infty}\left[\frac{|\eta|}{l+\eta} + \frac{|\eta|}{l-\eta} - \frac{2|\eta|}{l}\right]\cos 2\pi l x \\ & + i\sum_{l=1}^{\infty}\left[\frac{|\eta|}{l+\eta} - \frac{|\eta|}{l-\eta}\right]\sin 2\pi l x. \end{aligned} \tag{4.15}$$

By substituting f with $\mathcal{A}_h^*\phi_k$ in (2.21), and followed by simple calculation,

$$P_h\mathcal{A}_h^*\phi_k(x) = \alpha_k^*\phi_k(x), \tag{4.16}$$

where

$$\alpha_k^* = \begin{cases} 1, & k = 0, \\ \frac{1}{|k|}\sum_j w_j[1+\Gamma^*(\xi_j,kh)][1+\overline{\Delta_r(\xi_j,kh)}], & k \in \Lambda_h^*. \end{cases} \tag{4.17}$$

By the same way, we have

$$P_h \mathcal{A} \phi_k(x) = \beta_k \phi_k(x), \tag{4.18}$$

where

$$\beta_k = \begin{cases} 1, & k = 0, \\ \frac{1}{|k|} \sum_j w_j [1 + \overline{\Delta_r(\xi_j, kh)}], & k \in \Lambda_h^* \end{cases}. \tag{4.19}$$

Let us define

$$\begin{aligned} E^*(\eta) &:= |k|(\alpha_k^* - \beta_k) \\ &= \sum_j \Gamma^*(\xi_j, \eta)[1 + \overline{\Delta_r(\xi_j, \eta)}]. \end{aligned} \tag{4.20}$$

$E^*(\eta)$ is an important quantity in determining the order of convergence for our method, and it is shown in the next theorem.

Consider a solution operator $R_h := (P_h \mathcal{A}_h^*)^{-1} P_h \mathcal{A}$. R_h is well-defined if $P_h \mathcal{A}_h^*$ is invertible (see Lemma 4.2).

LEMMA 4.1. *Assume $P_h \mathcal{A}_h^* : T_h \to T_h$ is invertible. Suppose $|E^*(\eta)| \le C|\eta|^p$. For $f \in H^s$ with $s > -1/2$, we have*

$$\|R_h f - f\|_t \le C h^{(s-t)} \|f\|_s,$$

where $-1 \le t \le s \le t + p$

Proof. See the proof of Theorem 6.1 in [6].

Now we examine the invertibility of $P_h \mathcal{A}_h^* : T_h \to T_h$.

LEMMA 4.2. *Use splines of even order as test functions. $P_h \mathcal{A}_h^* : T_h \to T_h$ is invertible unless the quadrature method $J = 1$ with $\xi = 1/2$ (in (1.8)) is used.*

Proof. In view of equation (4.16), we need to show $\alpha_k^* \neq 0$. We prove that

$$Re\left[[1 + \Gamma^*(\xi, \eta)][1 + \overline{\Delta_r(\xi, \eta)}]\right] > 0,$$

if $|\eta| \leq \frac{1}{2}$ and $|\xi| \leq 1$. Using the notations defined in Remark 1,

$$1 + \Gamma^*(\xi, \eta) = Re[1 + \Gamma^*(\xi, \eta)] + i|\eta| H_1^+(\xi, \eta),$$

and

$$1 + \overline{\Delta_r(\xi, \eta)} = 1 + \eta^r G_r^+(\xi, \eta) - i\eta^r H_r^+(\xi, \eta).$$

Then

$$\begin{aligned} Re\left[[1 + \Gamma^*(\xi, \eta)][1 + \overline{\Delta_r(\xi, \eta)}]\right] =& Re[1 + \Gamma^*(\xi, \eta)] \cdot [1 + \eta^r G_r^+(\xi, \eta)] \\ &+ |\eta| H_1^+(\xi, \eta) \cdot \eta^r H_r^+(\xi, \eta). \end{aligned}$$

Using (2.18) with $H_r^+(\xi, \eta) = -H_r^+(1 - \xi, \eta)$, we have

$$|\eta| H_1^+(\xi, \eta) \cdot \eta^r H_r^+(\xi, \eta) \geq 0, \ \ 0 \leq \xi \leq 1.$$

Now

$$\begin{aligned} Re[1 + \Gamma^*(\xi, \eta)] >& 1 - \frac{1}{6} - \frac{1}{8} \sum_{l=2}^{\infty} \frac{1}{l(l - \eta)(l + \eta)} \\ >& 1 - \frac{1}{6} - \frac{1}{8} \sum_{l=2}^{\infty} \frac{1}{l(l + 1)} \\ >& 1 - \frac{1}{6} - \frac{1}{16} > 0. \end{aligned}$$

Since $[1 + \eta^r G_r^+(\xi, \eta)] > 0$ unless $\xi = 1/2$ by (2.16), we have the lemma.

Like the method of Sloan and Burn [6], the stability of this method is not known for splines of odd order either.

LEMMA 4.3. *Use splines of even order $r \geq 2$ as test functions.*

(1) $|E^*(\eta)| \leq C|\eta|^2$ *at least,*

(2) $|E^*(\eta)| \leq C|\eta|^3$ *if the quadrature method $J = 1$ with $\xi = 0$ or 1 is used for the discrete inner product (1,8).*

(3) $|E^*(\eta)| \leq C|\eta|^5$ *if the two-point symmetric quadrature, J=2 with $\xi_1 = \xi$ and $\xi_2 = 1 - \xi$, is used where $\xi = .2308296503$ and $1 - \xi$ are zeros of*

$$G^*(x) = 2 \sum_{l=1}^{\infty} \frac{1}{l^3} \cos(2\pi l x).$$

Proof.

$$E^*(\eta) = \sum_{j=1}^{J} \Gamma^*(\xi_j,\eta) + \Gamma^*(\xi_j,\eta)\overline{\Delta_r(\xi_j,\eta)}$$

where

$$\overline{\Delta_r(\xi,\eta)} = \eta^r[G_r^+(\xi,\eta) - iH_r^+(\xi,\eta)]$$

with G_r^+ and H_r^+ in Remark 1, and

$$\Gamma^*(x,\eta) = \sum_{l=1}^{\infty}\left[\frac{|\eta|}{l+\eta} + \frac{|\eta|}{l-\eta} - \frac{2|\eta|}{l}\right]\cos(2\pi lx) + i\sum_{l=1}^{\infty}\left[\frac{|\eta|}{l+\eta} - \frac{|\eta|}{l-\eta}\right]\sin(2\pi lx).$$

Simple calculation yields

$$\Gamma^*(x,\eta) = 2|\eta|^3G^*(x) + 2|\eta|^5H^*(x) + 2|\eta|^7\overline{\Gamma^*}_1(x,\eta) + i2\eta|\eta|\overline{\Gamma^*}_2(x,\eta)$$

where

$$G^*(x) = \sum_{l=1}^{\infty}\frac{1}{l^3}\cos(2\pi lx), \qquad H^*(x) = \sum_{l=1}^{\infty}\frac{1}{l^5}\cos(2\pi lx),$$

$$\overline{\Gamma^*}_1(x,\eta) = \sum_{l=1}^{\infty}\frac{1}{l^5(l^2-\eta^2)}\cos(2\pi lx),$$

$$\overline{\Gamma^*}_2(x,\eta) = -\sum_{l=1}^{\infty}\frac{1}{l^2-\eta^2}\sin(2\pi lx).$$

Then $E^*(\eta) = C\eta|\eta| + O(\eta)^3$ in general. When $\xi = 0$, $1/2$ or 1 with $J = 1$, $\sin(2\pi\xi) = 0$. Then

$$E^*(\eta) = C|\eta|^3 + O(\eta^5).$$

When the quadrature is symmetric with $J = 2$, (that is, $\xi_2 = 1-\xi_1$), we have

$$E^*(\eta) = \sum_{j=1}^{J} w_j Re[\Gamma^*(\xi_j,\eta)]\cdot[1+\eta^rG_r^+(\xi_j,\eta)] + |\eta|H_1^+(\xi_j,\eta)\cdot\eta^rH_r^+(\xi_j,\eta).$$

Simple calculation gives us

$$H_r^+(\xi_j,\eta) = -2\eta \sum_{l=1}^{\infty} \frac{r}{l^{r+1}} \sin(2\pi l x) + O(\eta^3)$$

for any r. If $G^*(\xi_1) = G^*(\xi_2) = 0$, then

$$Re[\Gamma^*(\xi_j,\eta)] \cdot [1 + \eta^r G_r^+(\xi_j,\eta)] = C\eta^5 + O(\eta^7)$$

and

$$|\eta| H_1^+(\xi_j,\eta) \cdot \eta^r H_r^+(\xi_j,\eta) = C|\eta|\eta^{2+r} + O(\eta^{5+r}).$$

Since $r \geq 2$, and the lemma is immediate.

By Lemmas 4.1, 4.2 and 4.3 and smoothness of the kenel of the operator $\mathcal{B}$, our method satisfies the assumptions (A1), (A2) with $\nu = -1$ with $p = 2,\ 3$ or $p = 5$. Therefore, we have the following theorem immediately by applying Theorem 3.1.

THEOREM 4.4. *Let $u_h \in T_h$ be a solution of* (1.13) *with $f \in H^{s+1}$, $s > -1/2$, and assume $-1 \leq t \leq s \leq t+p$ where $|E^*(\eta)| \leq C|\eta|^p$. Then* (1.13) *is uniquely solvable, and we have error estimate*

$$\|u - u_h\|_t \leq C h^{s-t} \|u\|_s \tag{4.21}$$

where $u \in H^s$ is a solution of (1.2).

5. Numerical Examples

Let u be a solution of (1.2) and u_h be a solution of (1.13). By the nature of our method, the solution we get will be $\{u_h(kh)\}_{k=1}^N$ By considering u and u_h as distributions; we can consider $|(u,g) - (u_h,g)|$ for a smooth function g, where $(\cdot,\cdot)$ is considered as a duality paring. Since

$$|(u,g) - (u_h,g)| \leq C\|u - u_h\|_t \|g\|_{-t},\ t \in R,$$

therefore we can see the convergence of $\|u - u_h\|_t$ through $|(u,g) - (u_h,g)|$.

Let

$$f_P(u) = -2\int_0^1 \log|P-\gamma(x)|u(x)dx, \quad P \in \Omega, \tag{5.1}$$

where Ω is a simply connected region and its boundary Γ is parametrized by $\gamma(x)$. Introduce

$$f_P(u_h) = -2h\int_0^1 \log|P-\gamma(x)|u_h(x)dx \tag{5.2}$$

and

$$f_{P,h}(u_h) = -2\sum_{k=0}^{N-1} \log|P-\gamma(kh)|u_h(kh).$$

Then $f_P(u_h)$, $f_{P,h}(u_h)$ approximate $f_P(u)$ at $P \in \Omega$. Now

$$|f_P(u) - f_{P,h}(u_h)| \leq |f_P(u) - f_P(u_h)| + |f_P(u_h) - f_{P,h}(u_h)|.$$

Because $g_P(x) := \log|P-\gamma(x)|$ is smooth for P not on the boundary, we have

$$|f_P(u) - f_P(u_h)| \leq \|g_P\|_1\|u-u_h\|_{-1}.$$

Because f_P is a pseudodifferential operator of order $-\infty$, Corollary 3.3 gives us

$$|f_P(u_h) - f_{P,h}(u_h)| \leq C_P h^\lambda \|u_h\|_s$$

with arbitrary λ, $s \in \mathbb{R}$. Then

$$|f_P(u) - f_{P,h}(u_h)| \leq C_P\|u-u_h\|_{-1}. \tag{5.3}$$

Consider harmonic functions:

$$f(x,y) = Re[\{(x+1)+iy\}^q],$$

where $q = .5,\ 1.5,\ 3.5$ on the ellipse:

$$\Gamma := \{(\cos(t), 2\sin(t)) : 0 \leq t \leq 2\pi\}.$$

	$_P(u) - f_{P,h}(u_h)\|$	r_h	$\|f_P(u) - f_{P,h}(u_h)\|$	r_h
1/16	.235e-1		.901e-3	
1/32	.812e-2	1.53	.577e-3	.642
1/64	.287e-2	1.50	.205e-3	1.50
1/128	.102e-2	1.50	.724e-4	1.50
1/256	.359e-3	1.50	.256e-4	1.50

Table 1 : Numerical results for the one point method of order 3 and two point method of order 5; $P = (0, .4)$, $f(x, y) = Re[(x + iy)^{.5}]$.

h	$\|f_P(u) - f_{P,h}(u_h)\|$	r_h	$\|f_P(u) - f_{P,h}(u_h)\|$	r_h
1/16	.236e-2		.246e-2	
1/32	.721e-4	5.03	.959e-5	8.00
1/64	.274e-4	1.39	.248e-5	1.95
1/128	.695e-5	1.98	.422e-6	2.55
1/256	.150e-5	2.22	.742e-7	2.51

Table 2 : Numerical results for the one point method of order 3 and two point method of order 5; $P = (0, .4)$, $f(x, y) = Re[(x + iy)^{1.5}]$

h	$\|f_P(u) - f_{P,h}(u_h)\|$	r_h	$\|f_P(u) - f_{P,h}(u_h)\|$	r_h
1/16	.192e-1		.476e-1	
1/32	.920e-2	5.79	.103e-3	8.86
1/64	.117e-2	1.06	.785e-5	3.71
1/128	.146e-3	2.98	.244e-6	5.01
1/256	.182e-4	3.00	.759e-8	5.01

Table 3 : Numerical results for the one point method of order 3 and two point method of order 5; $P = (0, .4)$, $f(x, y) = Re[(x + iy)^{3.5}]$

Using the single layer potential representation, we have the boundary integral equation (1.1), where the single layer density u is sought. Then the single layer density function will have regularity: $u \in H^{q-1/2-\epsilon}$ for $\epsilon > 0$. Equation (1.1) is solved by the qualocation method (1.13) for u_h. Then we expect the convergence of order $\min\{p, q+1/2-\epsilon\}$ in the norm $\|\cdot\|_{-1}$. The tables represent the numerical results with the method of the order 3 and those of the order 5, respectively. As expected, the convergence of order 5 method is not better than that of order 3 method asymptotically when u has low regularity. When u is sufficiently smooth, the maximal orders of convergence are achieved. In our numerical experiments the maximal convergence occurs for less regular u than that expected by Formula (5.3) with Theorem 4.4. This is because of common error cancellation phenomena that happen when we evaluate approximate potential (5.2).

In the table, the rate of convergence r_h is evaluated by the formula:

$$r_h = \frac{\log\left[|f_P(u) - f_{P,h/2}(u_{h/2})|/|f_P(u) - f_{P,h}(u_h)|\right]}{\log 2}.$$

References

1. K. E. Atkinson, *A Survey of Numerical Methods for the Solution of Fredholm Inte ral Equations of the Second Kind*, SIAM Pub., 1976.
2. G. Brown, G. A. Chandler, I. H. Sloan and D. C. Wilson, *Properties of certain trigonometric series arising in numerical analysis*, J. Math. Anal Appl. **162** (1991), 371-380.
3. G. A. Chandler and I. H. Sloan, *Spline qualocation methods for boundary integral equations*, Numer. Math. **58** (1990), 537-569.
4. R. Kress, *A Nyström method for boundary integral equations in domains with corners*, Numer. Math. **58** (1990), 445-461.
5. J. Saranen and I. H. Sloan, *Quadrature methods for logarithmic-kernel equations on closed curv s*, IMA J. Numer. Anal. **12** (1992), 537-567.
6. I. H. Sloan and B. J. Burn, *An unconventional quadrature method for logarithm c-kernel equations on closed curves*, J. Integral Eqns. and appl. **4** (1992), 117-151.

Youngmok Jeon
Department of Mathematics
Ajou University
Suwon 441-749, Korea

Ha-Jine Kimn
Department of Computer Science
Ajou University
Suwon 441-749, Korea