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A KANTOROVICH–TYPE CONVERGENCE ANALYSIS FOR THE QUASI–GAUSS–NEWTON METHOD

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1. Introduction

We consider numerical methods for finding a solution to a nonlinear system of algebraic equations

$$f(x) = 0, \tag{1}$$

where the function $f : R^n \to R^n$ is in $x \in R^n$. In [10], a quasi-Gauss-Newton method is proposed and shown the computational efficiency over SQRT algorithm by numerical experiments. The convergence rate of the method has not been proved theoretically. In this paper, we show theoretically that the iterate x_k obtained from the quasi-Gauss-Newton method for the problem (1) actually converges to a root by Kantorovich-type convergence analysis. We also show the rate of convergence of the method is superlinear.

Since the quasi-Gauss-Newton method is a modified Gauss-Newton method, we first discuss Gauss-Newton method briefly. Gauss-Newton method solves

$$J(x)^T J(x)s = -J(x)^T f(x), \tag{2}$$

where $J(x)$ is the Jacobian of f at x, for s at each iteration to have a better approximation to a solution. The equation (2) is usually computed by QR decomposition of $J(x)$. If we use an approximation B for $J(x)$, then (2) becomes

$$B^T Bs = -B^T f(x). \tag{3}$$

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One of the most succesful approximations for $J(x)$ is Broyden's update [1]. Wang showed in [10] that using LDL^T factorization of B^TB leads to superior computational results than the $SQRT$ method for a given set of test problems and using the modified Cholesky factorization in [6] reduces the number of operations from $O(n^3)$ to $O(n^2)+n$.

It is essential to give convergence analysis for the numerical methods that are developed. Such convergence analysis of other methods for systems of nonlinear equations can be found in [2, 3, 8, 7, 5, 9]. In following sections, we first describe Quasi-Gauss-Newton method. And then, we show the method is convergent to a root and its convergence rate is superlinear. It is followed by a Kantorovich-type analysis for the method.

2. Quasi-Gauss-Newton method

In this section, we describe the quasi-Gauss-Newton method in [10] with the modified Cholesky factorization in [6] to solve (3).

The method uses an algorithm in [6] given as Algorithm 2.1 in this section. It modifies a symmetric matrix A by a symmetric matrix of rank one,

$$\overline{A} = A + \alpha z z^T \tag{4}$$

and finds the Cholesky factors of $\overline{A} = \overline{LDL}^T$ from the factors of $A = LDL^T$. If B^TB is modified by a rank-one update as in (4), then the updated matrix $\overline{B}^T\overline{B}$ can be obtained as follows.

$$\overline{B}^T\overline{B} = B^TB + \alpha zz^T = L(D + \alpha pp^T)L^T, \tag{5}$$

where $Lp = z$, and p is obtained from z. If we factor

$$D + \alpha pp^T = \tilde{L}\tilde{D}\tilde{L}^T,$$

the required modified Cholesky factors are of the form,

$$\overline{B}^T\overline{B} = L\tilde{L}\hat{D}\tilde{L}^TL^T.$$

Therefore,

$$\overline{L} = L\hat{L}, \overline{D} = \tilde{D}.$$

Initially, we need the orthogonal factorization of B to have $B^TB = R^TR$. Then, the initial L and D can be obtained from R^TR. The algorithm for updating L and D is:

Algorithm 2.1 [6]

$$\text{Define} \quad \alpha_1 = \alpha, w^{(1)} = z.$$

$$\begin{aligned}
&\text{Do for} j = 1, \ldots n: \\
&\quad p_j = w_j^{(j)}, \\
&\quad \overline{d}_j = d_j + \alpha_j p_j^2, \\
&\quad \beta_j = p_j \alpha_j / \overline{d}_j, \\
&\quad \alpha_{j+1} = d_j \alpha_j / \overline{d}_j \\
&\quad \text{Do for } r = j+1, \ldots, n. \\
&\quad w_r^{(j+1)} = w_r^{(j)} - p_j l_{rj} \\
&\quad \overline{l}_{rj} = l_{rj} + \beta_j w_r^{(j+1)}
\end{aligned}$$

If $\overline{B} = B + \frac{\overline{f}s^T}{s^Ts}$ is used in (5),

$$\overline{B}^T\overline{B} = B^TB + B^T\frac{\overline{f}s^T}{s^Ts} + \frac{s\overline{f}^T}{s^Ts}B + \frac{s\overline{f}^T}{s^Ts}\frac{\overline{f}s^T}{s^Ts}. \tag{6}$$

From the above equation, we can see that B^TB is modified by a rank-2 update and (6) can be rewritten as

$$\overline{B}^T\overline{B} = B^TB + z_1z_1^T - z_2z_2^T,$$

where $z_1 = [\overline{B}^T\overline{f} + (1 - \overline{f}^T\overline{f}/2)\frac{s^T}{s^Ts}]/\sqrt{2}$ and $z_2 = [\overline{B}^T\overline{f} - (1 + \overline{f}^T\overline{f}/2)\frac{s^T}{s^Ts}]/\sqrt{2}$.

The algorithm for Quasi-Gauss-Newton method [10] using Broyden's update is:

Algorithm 2.2

$$
\begin{aligned}
&\text{Given} \quad f : R^n \to R^n, x_0 \in R^n, B_0 \in R^{n\times n}.\\
&\quad \text{Get } Q_0R_0 = B_0\\
&\quad L_0 \text{ from } R^T\\
&\quad D_0 = (r_{11}^2, \dots, r_{nn}^2).
\end{aligned}
$$

$$
\begin{aligned}
&\text{Do for} k = 1, \dots :\\
&\text{Solve } L_kD_kL_k^T s_k = -B_k^T f(x_k) \text{ for } s_k,\\
&\quad x_{k+1} := x_k + s_k,\\
&\quad y_k := f(x_{k+1}) - f(x_k),\\
&\quad t_k := -B_k^T f(x_k).\\
&\quad B_{k+1} := B_k + \frac{(y_k - B_k s_k)s_k^T}{s_k^T s_k}.\\
&\quad \text{Get } \tilde{L}\tilde{D}\tilde{L}^T, \overline{LDL^T} \text{ by Algorithm 2.1.}
\end{aligned}
$$

In the next section, we will give a convergence analysis of Algorithm 2.2.

3. Superlinear Convergence of QGN

In this section, we show that the method by (6) is well defined and converges to a solution of (1). We start with the bound deterioration theorem of the method.

THEOREM 3.1. (The Bounded Deterioration Theorem) *Let $D \subseteq R^n$ be an open convex set containing x_k, x_{k+1}, with $x_k \neq x_*$. Let $f : R^n \to R^n$, $B_k \in R^{n\times n}$ and*

$$
(7) \quad B_{k+1}^T B_{k+1} = B_k^T B_k + B_k^T \frac{f_{k+1}s^T}{s^Ts} + \frac{sf_{k+1}^T}{s^Ts}B_k + \frac{sf_{k+1}^T}{s^Ts}\frac{f_{k+1}s^T}{s^Ts}.
$$

If $x_* \in D$ and $J(x)$ obeys the weaker Lipschitz condition,

$$
\|J(x) - J(x_*)\| \le \gamma\|x - x_*\|, \quad \textit{for all } x \in D,
$$

then, for both the Frobenius and l_2 matrix norms,

$$
\begin{aligned}
(8)\qquad & \|(B_{k+1}-J(x_*))^T(B_{k+1}-J(x_*))\| \\
& \le [\|B_k - J(x_*)\| + \frac{\gamma}{2}(\|x_{k+1}-x_*\|_2 + \|x_k - x_*\|_2)]^2.
\end{aligned}
$$

Proof. Let $J_* \equiv J(x_*)$. Adding $-J_*^T B_{k+1} - B_{k+1}^T J_* + J_*^T J_*$ to the both sides of (7), we get

$$
\begin{aligned}
& B_{k+1}^T B_{k+1} - J_*^T B_{k+1} - B_{k+1}^T J_* + J_*^T J_* \\
&= B_k^T B_k - J_*^T B_{k+1} - B_{k+1}^T J_* + J_*^T J_* + B_k^T \frac{(y-B_k s)s^T}{s^T s} \\
&\quad + \frac{s(y-B_k s)^T}{s^T s} B_k + \frac{s(y-B_k s)^T}{s^T s}\frac{(y-B_k s)s^T}{s^T s} \\
(9)\qquad &= [B_k - J_* + \frac{(y-B_k s)s^T}{s^T s}]^T [B_k - J_* + \frac{(y-B_k s)s^T}{s^T s}] \\
&= [(B_k - J_*)[I - \frac{ss^T}{s^T s}] + \frac{(y-J_* s)s^T}{s^T s}]^T [(B_k - J_*)[I - \frac{ss^T}{s^T s}] \\
&\quad + \frac{(y-J_* s)s^T}{s^T s}].
\end{aligned}
$$

Then, it follows that

$$
\|(B_{k+1}-J_*)^T(B_{k+1}-J_*)\| \le [\|(B_k - J_*)[I - \frac{ss^T}{s^T s}]\| + \frac{\|y-J_* s\|_2}{\|s\|_2}]^2.
$$

Using

$$
\|I - \frac{ss^T}{s^T s}\|_2 = 1,
$$

and

$$
\|y - J_* s\|_2 \le \frac{\gamma}{2}(\|x_{k+1}-x_*\|_2 + \|x - x_*\|_2)\|s\|_2
$$

in [4], we have (8).

THEOREM 3.2. (Linear Convergence Theorem) *Let all the assumptions of Theorem 3.1 hold. Suppose that* $\|J(x_*)^{-1}\| \leq \beta$ and, *for arbitrary* ϵ *and* $\delta > 0$ *we can choose* x_0 *and* B_0 *such that* $\|x_0 - x_*\| < \epsilon$ *and* $\|B_0 - J(x_*)\| < \delta$. *If* $B_k^T B_k$ *satisfy*

(10)
$$\begin{aligned}&\|(B_{k+1} - J(x_*))^T(B_{k+1} - J(x_*))\|\\ &\quad\leq [\|B_k - J(x_*)\| + \frac{\gamma}{2}(\|x_{k+1} - x_*\| + \|x_k - x_*\|)]^2.\end{aligned}$$

then x_k *converges at least linearly to* x_*.

Proof. We will show that by induction, for $k = 0, 1, 2, \ldots,$

$$\|(B_k - J_*)^T(B_k - J_*)\| \leq [(2 - 2^{-k})\delta]^2 \tag{11}$$

and

$$\|e_{k+1}\| \leq \frac{\|e_k\|}{2}. \tag{12}$$

Choose ϵ and δ such that

$$7\beta\delta \leq 1 \quad \text{and} \quad 3\gamma\epsilon \leq 2\delta. \tag{13}$$

When $k = 0$,

$$\|(B_0 - J_*)^T(B_0 - J_*)\| \leq \|(B_0 - J_*)\|\|(B_0 - J_*)\| \leq \delta^2,$$

by assumption. To show that $\|e_1\| \leq \frac{\|e_0\|}{2}$, we consider the following equation from the iteration,

$$B_0^T B_0(x_1 - x_0) = -B_0^T f_0.$$

Hence,

$$\begin{aligned}e_1 &= e_0 + (B_0^T B_0)^{-1}(-B_0^T f_0)\\ &= e_0 + B_0^{-1}(-f_0 + f_* + J_* e_0) - B_0^{-1} J_* e_0.\end{aligned}$$

In $\|\cdot\|$,

$$\|e_1\| \leq \|B_0^{-1}\|[\|-f_0 + f_* + J_* e_0\| + \|B_0 - J_*\|\|e_0\|].$$

If $\|J(x_*)^{-1}\| \leq \beta$, using Theorem 3.1.4 in [4], it can be shown that

$$\|B_k^{-1}\| \leq \frac{3}{2}\beta.$$

$$\|e_1\| \leq \frac{3}{2}\beta\,[\frac{\gamma}{2}\|e_0\| + \delta]\|e_0\|. \tag{14}$$

Since

$$\frac{\gamma\|e_0\|}{2} \leq \frac{\gamma\epsilon}{2} \leq \frac{\delta}{3},$$

the inequality (14) becomes

$$\begin{aligned} \|e_1\| &\leq \frac{3}{2}\beta[\frac{\delta}{3} + 2\delta]\|e_0\| \\ &\leq 3\beta\delta\|e_0\| \\ &\leq \frac{\|e_0\|}{2}. \end{aligned}$$

For $k = 0, 1, ..., i-1$, let us assume that (11) and (12) hold. Then, for $k = i$, using

$$\|e_i\| \leq \frac{1}{2}\|e_{i-1}\|,$$

$\|e_{i-1}\| \leq 2^{-(i-1)}\epsilon$, (12), and Theroem 3.1,

$$\begin{aligned} \|(B_i - J_*)^T(B_i - J_*)\| &\leq [(2 - 2^{-(i-1)})\delta + \frac{\gamma}{2}\|e_{i-1}\| + \|e_i\|]^2 \\ &\leq [2 - 2^{-(i-1)})\delta + \frac{3}{4}\gamma\|e_{i-1}\|]^2 \\ &\leq [(2 - 2^{-i})\delta]^2. \end{aligned} \tag{15}$$

We omit the proof of (12) since it is identical to the proof of the initial step of the induction.

THEOREM 3.4. (Superlinear Convergence) *Let all the assumptions of Theorem 3.1 hold. Then, the sequence $\{x_k\}$ generated by Algorithm 2.2 is well defined and converges superlinearly to x_*.*

Proof. Define $E_k = B_k - J_*$, and let $\|\cdot\|$ denote the l_2 vector norm. From (9),

$$\begin{aligned}\|E_{k+1}^T E_{k+1}\|_F \leq & \|(I - \frac{s_k s_k^T}{s_k^T s_k}) E_k^T E_k (I - \frac{s_k s_k^T}{s_k^T s_k})\|_F \\ & + 2\|E_k(I - \frac{s_k s_k^T}{s_k^T s_k})\|_F \frac{\|y_k - J_* s_k\|_2}{\|s_k\|_2} + (\frac{\|y_k - J_* s_k\|_2}{\|s_k\|_2})^2.\end{aligned}$$

Using $\frac{\|y - J_* s_k\|_2}{\|s_k\|_2} \leq \frac{\gamma}{2}(\|e_k\|_2 + \|e_{k+1}\|_2)$ in the proof of **Theorem 3.1** and $\|e_{k+1}\| \leq \frac{1}{2}\|e_k\|$, we have

$$\frac{\|y - J_* s_k\|_2}{\|s_k\|_2} \leq \frac{3\gamma}{4}\|e_k\|_2. \tag{17}$$

In view of **Lemma 3.3** and (17),

$$\begin{aligned}\|E_{k+1}^T E_{k+1}\|_F \leq & \|E_k^T E_k\|_F - \frac{1}{2\|E_k^T E_k\|_F} \frac{\|E_k^T E_k s_k\|^2}{\|s_k\|^2} \\ & + \frac{3\gamma\|E_k\|_F}{2}\|e_k\| + \frac{9\gamma^2}{16}\|e_k\|^2.\end{aligned}$$

This can be rewritten as

$$\begin{aligned}\frac{\|E_k^T E_k s_k\|^2}{\|s_k\|^2} \leq & 2\|E_k^T E_k\|_F[\|E_k^T E_k\|_F - \|E_{k+1}^T E_{k+1}\|_F \\ & + \frac{3\gamma}{2}\|E_k\|\|e_k\| + \frac{9\gamma^2}{16}\|e_k\|^2].\end{aligned}$$

From Theorem 3.2, $\|E_k^T E_k\|_F \leq 4\delta^2$ and $\|E_k\| \leq 2\delta$ for all $k \geq 0$, $\sum_{k=0}^{\infty}\|e_k\| \leq 2\epsilon$, and $\sum_{k=0}^{\infty}\|e_k\|^2 \leq \frac{4}{3}\epsilon$,

$$\frac{\|E_k^T E_k s_k\|^2}{\|s_k\|^2} \leq 4\delta^2[\|E_k^T E_k\|_F - \|E_{k+1}^T E_{k+1}\|_F + 3\delta\gamma\|e_k\| + \frac{9\gamma^2}{16}\|e_k\|^2].$$

Summing for $k = 0, 1, \ldots i$,

$$\sum_{k=0}^{i} \frac{\|E_k^T E_k s_k\|^2}{\|s_k\|^2} \leq 4\delta^2[\|E_0^T E_0\|_F - \|E_{i+1}^T E_{i+1}\|_F$$

$$+ 3\delta\gamma \sum_{k=0}^{i} \|e_k\| + \frac{9\gamma^2}{16} \sum_{k=0}^{i} \|e_k\|^2]$$

$$\leq 4\delta^2[\|E_0^T E_0\|_F + 6\delta\gamma\epsilon + \frac{3}{4}\gamma^2\epsilon]$$

$$\leq 4\delta^2[4\delta^2 + 6\delta\gamma\epsilon + \frac{3}{4}\gamma^2\epsilon],$$

which shows that

$$\sum_{k=0}^{i} \frac{\|E_k^T E_k s_k\|^2}{\|s_k\|^2}$$

is finite. This implies (16). Therefore, Quasi-Gauss-Newton method converges superlinearly.

4. A Kantorovich-type Analysis of QGN

THEOREM 4.1. *Let all the assumptions of Theorem 3.1 hold. Then,*

$$\|B(x_k) - J(x_k)\|_F \leq \|B(x_0) - J(x_0)\|_F + \frac{3}{2}\gamma \sum_{j=1}^{k} \|x_j - x_{j-1}\|_2.$$

Proof. Using Lemma 8.2.1 in [4],

$$\|B_k - J_k\|_F \leq \|B_{k-1} - J_{k-1}\|_F + \frac{3}{2}\gamma\|x_k - x_{k-1}\|$$

$$\leq \|B_0 - J_0\|_F + \frac{3}{2}\gamma \sum_{j=1}^{k} \|x_j - x_{j-1}\|.$$

We now present the Kantorovich-type analysis for the Quasi-Gauss-Newton method. Let us define $S(y, \delta) = \{x \in R^n | \|x - y\| < \delta, y \in R^n\}$ and $\overline{S}(y, \delta)$ the closure of $S(y, \delta)$.

THEOREM 4.2. *Let all the assumptions of Theorem 3.1 hold and B_k be nonsingular. Assume that δ, β and η are positive scalars such that $\|B_0 - J_0\|_F \le \delta$, $\|B_0^{-1}\|_F \le \beta$, $\|B_0^{-1} f(x_0)\| \le \eta$,*

$$h = \frac{\beta\gamma\eta}{(1-3\beta\delta)^2} \le \frac{1}{9},$$

and $\beta\delta < 1/3$. If $\overline{S}(x_0, t_*) \subset D$, where

$$t_* = \frac{2}{9\beta\gamma}(1-3\beta\delta)(1-\sqrt{1-9h}),$$

then x_k by algorithm 2.2 is well-defined, remains in $\overline{S}(x_0, t_*)$ and converges to x_*.

Proof. Let

$$f(t) = \frac{9}{4}\gamma t^2 - (\frac{1-3\beta\delta}{\beta})t + \frac{\eta}{\beta}. \tag{18}$$

Consider the iteration,

$$t_{k+1} - t_k = \beta f(t_k), \tag{19}$$

where $t_0 = 0, k = 0, 1, 2....$ Then, $f(t_{k-1}) = 1/\beta(t_k - t_{k-1})$. Taylor series expansion of f yields

$$f(t_k) = f(t_{k-1}) + f'(t_{k-1})(t_k - t_{k-1}) + \frac{1}{2}f''(t_{k-1})(t_k - t_{k-1})^2. \tag{20}$$

In view of (18) and (20),

$$f(t_k) = 3[\frac{3}{4}\gamma(t_k - t_{k-1}) + \frac{3}{2}\gamma t_{k-1} + \delta](t_k - t_{k-1}). \tag{21}$$

Substituting (21) into (19),

(22)

$$\begin{aligned} t_{k+1} - t_k &= 3\beta[\frac{3}{4}\gamma(t_k - t_{k-1}) + \frac{3}{2}\gamma t_{k-1} + \delta](t_k - t_{k-1}) \\ &= 3\beta[\frac{3}{4}\gamma(t_k + t_{k-1}) + \delta](t_k - t_{k-1}). \end{aligned}$$

$k = 1, 2, \ldots$. By $t_0 = 0$ and (19), we have $t_1 = \eta$. It can be shown by induction that t_k is a strictly increasing sequence. Next, we show that $t_k \le t_*$. Using the fact that t_* is a root of $f(t)$,

$$\begin{aligned} t_* - t_{k+1} &= t_* - t_k - \beta f(t_k) \\ &= \beta[\frac{1}{\beta}(t_* - t_k) - f(t_k)] \\ &= \beta\{[f(t_*) - f(t_k) - f'(t_k)(t_* - t_k)] \\ &\quad + [f'(t_k) + \frac{1}{\beta}](t_* - t_k)\}. \end{aligned}$$

By (19) and (20),

$$\begin{aligned} t_* - t_{k+1} &= \beta\{\frac{1}{2}f''(t_k)(t_* - t_k)^2 + [f'(t_k) + \frac{1}{\beta}](t_* - t_k)\} \\ &= \beta\{\frac{9}{4}\gamma(t_* - t_k) + \frac{9}{2}\gamma t_k + 3\delta\}(t_* - t_k) \\ &= \beta\{\frac{9}{4}\gamma(t_* + t_k) + 3\delta\}(t_* - t_k). \end{aligned}$$

Since $t_0 = 0$ and $t_* > 0$, we have

$$t_k \le t_*, \qquad k = 0, 1, 2, \ldots \tag{23}$$

Therefore, there exists a $\tilde{t} \le t_*$ such that

$$\lim_{k \to \infty} t_k = \tilde{t}. \tag{24}$$

Using (19) and (24), we have $f(\tilde{t}) = 0$, hence, $\tilde{t} = t_*$ since t_* is the smallest root of f. We have shown that $lim_{t \to \infty} = t_*$.

We prove the followings by induction on k:

$$\{x_k\} \subset \overline{S}(x_0, t_*), \tag{25}$$

$$\|B_k^{-1}\|_F \le 3\beta, \quad k = 1, 2, \ldots, \tag{26}$$

and

$$\|x_{k+1} - x_k\| \le t_{k+1} - t_k, \; k = 1, 2, \ldots. \tag{27}$$

When $k = 0$,

$$\|x_1 - x_0\| = \eta = t_1 - t_0 \leq t_*,$$

from $t_0 = 0$ and (23). Suppose that (25) hold for $k = 0, 1, ..., i - 1$. Then,

$$\|x_k - x_0\| \leq \sum_{i=0}^{k-1} \|x_{i+1} - x_i\| \leq \sum_{i=0}^{k-1} (t_{i+1} - t_i) = t_k \leq t_*.$$

This implies $x_k \in \overline{S}(x_0, t_*)$. To show (26), we use Theorem 4.1,

$$\|B_k - J_k\|_F \leq \|B_0 - J_0\|_F + \frac{3}{2}\gamma \sum_{i=1}^{k} \|x_i - x_{i-1}\|,$$

for all k, and

$$\frac{5}{2}\gamma t_* + 2\delta \leq \frac{2}{3}\beta.$$

Now, we can proceed as follows:

$$\begin{aligned}
&\|B_0^{-1}(B_k - B_0)\|_F \\
&\quad\leq \|B_0^{-1}\|_F(\|B_k - J_k\|_F + \|J_k - J_0\|_F + \|J_0 - B_0\|_F) \\
&\quad\leq \beta(2\delta + \frac{3}{2}\gamma \sum_{i=1}^{k} \|x_i - x_{i-1}\| + \gamma\|x_k - x_0\|) \\
&\quad\leq \beta(2\delta + \frac{5}{2}\gamma \sum_{i=1}^{k} \|x_i - x_{i-1}\|) \\
&\quad\leq \beta(2\delta + \frac{5}{2}\gamma t_k) \\
&\quad\leq \beta(2\delta + \frac{5}{2}\gamma t_*) \\
&\quad\leq \frac{2}{3}.
\end{aligned}$$

Therefore,

$$\|B_k^{-1}\|_F \leq \frac{\beta}{1 - 2/3} = 3\beta,$$

by Lemma 2.3.2 in [9]. It remains to show (27). Since

$$B_{k-1}^T B_{k-1}(x_k - x_{k-1}) + B_{k-1}^T f_{k-1} = 0$$

is equivalent to

$$B_{k-1}(x_k - x_{k-1}) + f_{k-1} = 0,$$

$$\begin{aligned}
\|x_{k+1} - x_k\| &\le \|B_k^{-1}\|_F \|f_k\| \\
&= \|B_k^{-1}\|_F \|f_k - f_{k-1} - B_{k-1}(x_k - x_{k-1})\| \\
&\le \|B_k^{-1}\|_F (\|f_k - f_{k-1} - J_{k-1}(x_k - x_{k-1})\| \\
&\quad + \|(J_{k-1} - B_{k-1})(x_k - x_{k-1})\|).
\end{aligned}$$

By (26), Lemma 4.1.12 in [4], (27), and (22),

$$\begin{aligned}
&\|x_{k+1} - x_k\| \\
&\le 3\beta\{\frac{\gamma}{2}\|x_k - x_{k-1}\| + \frac{3}{2}\gamma \sum_{i=0}^{k-2} \|x_{i+1} - x_i\| + \delta\}\|x_k - x_{k-1}\| \\
&\le 3\beta\{\frac{\gamma}{2}(t_k - t_{k-1}) + \frac{3}{2}\gamma t_{k-1} + \delta\}(t_k - t_{k-1}) \\
&\le 3\beta\{\frac{3}{4}\gamma(t_k - t_{k-1}) + \frac{3}{2}\gamma t_{k-1} + \delta\}(t_k - t_{k-1}) \\
&\le 3\beta\{\frac{3}{4}\gamma(t_k + t_{k-1}) + \delta\}(t_k - t_{k-1}) \\
&= t_{k+1} - t_k.
\end{aligned}$$

This shows that there is an $x_* \in D$ such that $\lim_{k\to\infty} x_k = x_*$.

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