

DIRICHLET FORMS AND DIFFUSION PROCESSES RELATED TO QUANTUM UNBOUNDED SPIN SYSTEMS

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1. Introduction

We study Dirichlet forms and the associated diffusion processes for the Gibbs measures related to the quantum unbounded spin systems (lattice boson systems) interacting via superstable and regular potentials. This work is a continuation of the authors' previous study on the classical systems [LPY] to the quantum cases. In [LPY], we constructed Dirichlet forms and the associated diffusion processes for the Gibbs measures of classical unbounded spin systems. Furthermore, we also showed the essential self-adjointness of the Dirichlet operator and the log-Sobolev inequality for any Gibbs measure under appropriate conditions on the potentials. In this study we try to extend the results of the classical systems to the quantum cases. Because of some technical difficulties, we are only able to construct a Dirichlet form and the associated diffusion process for any Gibbs measure of the quantum systems. We utilize the general scheme of the previous work on the theory in infinite dimensional spaces [AH-K1-2, AKR, AR1-2, Kus, MR, Rö, Sch] and the ideas we employed in our study of the classical systems [LPY].

The Dirichlet forms and the associated diffusion processes have been intensively investigated in connection with their important applications to mathematical physics and theory of random processes (see [AH-KS, Fu, HKPS, MR, Rö, Sil] and references therein). There have been much

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progress in this theory, especially in the theory of Dirichlet forms on locally compact state spaces [Fu, Sil]. There also have been many efforts to extend the general theory to the case where the state spaces are of infinite dimensional, hence non-locally compact topological vector spaces [AHPRS1-2, AH-K1-2, AR1-2, HPS, MR, Sch]. The main problems are to define a Dirichlet form appropriately, to show the closability of the form, and to study the properties of the associated Dirichlet operator [AKR, AK]. This paper can be viewed as a concrete application of the general theory developed in [AR1-2, MR, Rö].

Our motivation to study the theory of Dirichlet forms related to the quantum unbounded spin system comes from the authors' previous investigations on the system [PY1-2]. Characterizing the Gibbs (equilibrium) states of the system there corresponds the Gibbs measures on the infinite dimensional configuration space [PY1]. To speak more precisely, for any bounded $\Lambda \subset \mathbb{Z}^\nu$ we let \mathfrak{A}_Λ be the algebra of bounded operators on $H_\Lambda = \otimes_{i \in \Lambda} H_i$ where H_i is a copy of the Hilbert space $L^2(\mathbb{R}^d)$. The local Hamiltonians are given by

$$H_\Lambda = -\frac{1}{2} \sum_{i \in \Lambda} \Delta_i + V(x_\Lambda),$$

where Δ_i is the Laplacian operator on $L^2(\mathbb{R}^d)$ and $V(x_\Lambda)$ is an interaction function on $(\mathbb{R}^d)^\Lambda$. The local Gibbs states are given by the functional $A \mapsto \text{Tr}_{H_\Lambda}(Ae^{-H_\Lambda})/\text{Tr}_{H_\Lambda}(e^{-H_\Lambda})$, $A \in \mathfrak{A}_\Lambda$. By the Feynman-Kac formula, the density operator $\exp(-H_\Lambda)$ has its integral kernel

$$e^{-H_\Lambda}(x_\Lambda, y_\Lambda) = \int P_{x_\Lambda, y_\Lambda}(d\zeta_\Lambda) e^{-V(\zeta_\Lambda)},$$

where x_Λ and y_Λ are points in $(\mathbb{R}^d)^\Lambda$, $\zeta_\Lambda \in (C([0, 1], \mathbb{R}^d))^\Lambda$, $V(\zeta_\Lambda) = \int_0^1 V(\zeta_\Lambda(t))dt$, and P_{x_Λ, y_Λ} is the conditional Wiener measure on $(C([0, 1], \mathbb{R}^d))^\Lambda$ [Sim]. Then, the normalization factor $\text{Tr}_{H_\Lambda}(e^{-H_\Lambda})$ is given by

$$\text{Tr}_{H_\Lambda}(e^{-H_\Lambda}) = \int dx_\Lambda \int P_{x_\Lambda, x_\Lambda}(d\zeta_\Lambda) e^{-V(\zeta_\Lambda)},$$

where dx_Λ is the Lebesgue measure on $(\mathbb{R}^d)^\Lambda$. Let $E = C_p([0, 1], \mathbb{R}^d)$, the space of \mathbb{R}^d -valued continuous functions on $[0, 1]$ (equipped with

the sup-norm) with the property that $f(0) = f(1)$ for any $f \in E$. Let $\Omega = E^{\mathbb{Z}^v}$ and \mathcal{F} be the σ -algebra on Ω generated by the local σ -algebras (see Section 2.1 for the details). Using the above formula, we were able to define Gibbs specifications [Ge, PY1, Pr] and then define the Gibbs measures in terms of the Gibbs specifications (see Section 2.1).

For given interaction Φ , we denote by $\mathcal{G}^\Phi(\Omega)$ the set of corresponding Gibbs measures on (Ω, \mathcal{F}) . In [PY1], we have shown that $\mathcal{G}^\Phi(\Omega)$ is non-empty, convex, compact in the local convergence topology, and a Choquet simplex (Theorem 2.5). Furthermore, each $\mu \in \mathcal{G}^\Phi(\Omega)$ is regular and satisfies the equilibrium condition. See Definition 2.3 and Definition 2.4. We also defined the Gibbs states through the Gibbs measures and showed that the space of Gibbs states is non-empty, convex, and weak*-compact under suitable conditions on the interactions. See [PY1] for the details.

Let us briefly describe the contents of this paper. We want to define a Dirichlet form on $L^2(\Omega, d\mu)$, where μ is a Gibbs measure, and to construct the associated diffusion process. The main idea is to extend the methods used in [LPY]. In [LPY], the configuration space was $\Omega = (\mathbb{R}^d)^{\mathbb{Z}^v}$ and we introduced three Hilbert spaces $\mathcal{H}_+ \subset \mathcal{H}_0 \subset \mathcal{H}_- (\subset \Omega)$ such that the embeddings were everywhere dense and \mathcal{H}_0 became a rigged Hilbert space rigged by \mathcal{H}_+ and \mathcal{H}_- , and moreover $\mu(\mathcal{H}_-) = 1$ for any Gibbs measure μ . In order to accommodate with the quantum case, we will also introduce a rigged Hilbert space. However, the situation is more complicated because the single site space E is itself of infinite dimensional. We will need therefore another rigging for the single-site space. It turns out that with a suitable Hilbert space $(H_0, (\cdot, \cdot)_0)$, densely embedded in E , and with a Gaussian measure μ^0 on E , (μ^0, H_0, E) becomes an abstract Wiener space [Kuo]. We will show that there exist two other Hilbert spaces $(H_+, (\cdot, \cdot)_+)$ and $(H_-, (\cdot, \cdot)_-)$ such that

$$(1.1) \quad H_+ \subset H_0 \subset H_-$$

is a rigging of H_0 by H_+ and H_- and such that the embeddings are everywhere dense and belong to the Hilbert-Schmidt class and furthermore $H_0 \subset E \subset H_-$. We will represent the sup-norm on E with a notation $|\cdot|_u$ and the norms on H_+ , H_0 , and H_- induced by the inner products $(\cdot, \cdot)_+$, $(\cdot, \cdot)_0$, and $(\cdot, \cdot)_-$, respectively, with the notations

$|\cdot|_+$, $|\cdot|_0$, and $|\cdot|_-$, respectively. The duality between H_+ and H_- given by $(\cdot, \cdot)_0$ will be denoted by $\langle \cdot, \cdot \rangle$. In relevance with the configuration space $\Omega = E^{\mathbb{Z}^\nu}$, we will also introduce three Hilbert spaces $(\mathcal{H}_+, ((\cdot, \cdot))_+)$, $(\mathcal{H}_0, ((\cdot, \cdot))_0)$, and $(\mathcal{H}_-, ((\cdot, \cdot))_-)$ such that

$$(1.2) \quad \mathcal{H}_+ \subset \mathcal{H}_0 \subset \mathcal{H}_-$$

is also a rigging of \mathcal{H}_0 by \mathcal{H}_+ and \mathcal{H}_- . Here, we also have that the embeddings are everywhere dense and belong to the Hilbert-Schmidt class. The induced norms on \mathcal{H}_+ , \mathcal{H}_0 , and \mathcal{H}_- will be denoted by $\|\cdot\|_+$, $\|\cdot\|_0$, and $\|\cdot\|_-$, respectively and the duality between \mathcal{H}_+ and \mathcal{H}_- given by $((\cdot, \cdot))_0$ will be denoted by $\langle\langle \cdot, \cdot \rangle\rangle$. We also topologize Ω via the metric d defined by

$$(1.3) \quad d(\xi, \zeta) = \sum_{i \in \mathbb{Z}^\nu} 2^{-|i|} \frac{|\pi_i(\xi - \zeta)|_u}{1 + |\pi_i(\xi - \zeta)|_u}, \quad \xi, \zeta \in \Omega,$$

where for $i = (i_1, \dots, i_\nu) \in \mathbb{Z}^\nu$, $\pi_i : \Omega \rightarrow E$ is the projection to the i -th site. Then the σ -algebra \mathcal{F} of the local observables becomes the Borel σ -algebra on Ω . Furthermore, in step with the single-site space, we introduce a Banach subspace $\Omega_- \subset \Omega$ such that any Gibbs measure is supported in Ω_- (Lemma 3.4). We will see that for any Gibbs measure $\mu \in \mathcal{G}^\Phi(\Omega)$, μ can be extended to \mathcal{H}_- (*a priori* μ is restricted to Ω_-) and the spaces $L^2(\Omega, d\mu)$ and $L^2(\mathcal{H}_-, d\mu)$ can be considered to be the same spaces (Remark 2.9). Finally, we shall introduce the subspace Ω_{\log} of both \mathcal{H}_- and Ω defined by

$$(1.4) \quad \Omega_{\log} = \{(\xi_i)_{i \in \mathbb{Z}^\nu} \in \Omega : \exists N \in \mathbb{N} \text{ s.t. } |\xi_i|_u \leq N \log(|i| + 1), \forall i \neq 0\}.$$

By the regularity of Gibbs measures, it turns out that (Lemma 2.7)

$$(1.5) \quad \mu(\Omega_{\log}) = 1, \quad \forall \mu \in \mathcal{G}^\Phi(\Omega).$$

Furthermore, by the definitions of subspaces the following inclusions

$$\mathcal{H}_0 \subset \Omega_{\log} \subset \Omega_- \subset \mathcal{H}_-$$

hold (see Section 2.2).

For each vector $\xi \in E$ and $i \in \mathbb{Z}^\nu$ let us define $\xi^{\{i\}} \in \Omega$ such that $\pi_j(\xi^{\{i\}}) = \delta_{ij}\xi$, $j \in \mathbb{Z}^\nu$. We will consider forms on $L^2(\mathcal{H}_-, d\mu)$ of the following types: for an orthonormal basis $\{e_n\}_{n=1}^\infty$ in H_0 with $e_n \in H_+$, $n = 1, 2, \dots$,

$$\begin{aligned}
 D(\mathcal{E}_\mu) &= \mathcal{F}_{\text{loc}}C_b^\infty \\
 \mathcal{E}_\mu(u, v) &= \sum_{i \in \mathbb{Z}^\nu} \sum_{n=1}^\infty \int \nabla_n^i u \cdot \nabla_n^i v \, d\mu \\
 (1.6) \qquad &= \int ((\nabla u, \nabla v))_0 \, d\mu, \quad u, v \in D(\mathcal{E}_\mu),
 \end{aligned}$$

where $\nabla_n^i u$ is the directional derivative in the direction $e_n^{\{i\}}$, $\nabla u(\cdot) \in \mathcal{H}_+$ is (a coordinate free version of) the gradient of u , and $\mathcal{F}_{\text{loc}}C_b^n$, $n = 0, 1, \dots, \infty$, is the space of locally and finitely based, and n -times boundedly differentiable functions on \mathcal{H}_- . See Section 2.3 for the details.

We will show that, under suitable conditions on the potentials, $\mathcal{F}_{\text{loc}}C_b^\infty$ is dense in $L^2(\mathcal{H}_-, \mu)$, and any vector $k^{\{i\}}$, $k \in H_0$, $i \in \mathbb{Z}^\nu$, is (μ) -admissible in the sense of [AR1]. Furthermore, we will show that there corresponds a Dirichlet operator H_μ with $D(H_\mu) = \mathcal{F}_{\text{loc}}C_b^\infty$ such that

$$\mathcal{E}_\mu(u, v) = (u, H_\mu v)_{L^2(\mathcal{H}_-, d\mu)}, \quad u, v \in D(H_\mu).$$

Thus, by use of the methods of [AR1] or from the general theory [MR, R ], the above form is closable and the closure becomes a Dirichlet form.

We will also construct the associated diffusion process. In [AR2], Albeverio and R ckner gave sufficient conditions for the construction. We will check that all the conditions are fulfilled in our case. See Section 3 for the details.

We organize this paper as follows: In Section 2.1, we introduce notations, definitions and basic assumptions on the potentials, and then describe specific properties of Gibbs measures from [PY1]. In Section 2.2, we give basic spaces with which we can describe Dirichlet forms. In Section 2.3, we introduce Dirichlet forms and Dirichlet operators for Gibbs measures, and give the main results in this paper. In Section 3, we construct Dirichlet forms and the associated diffusion processes employing the methods in [AR1-2, Kus, LPY, R ].

The essential self-adjointness of the Dirichlet operator and the log-Sobolev inequality for any Gibbs measure under appropriate conditions on the potentials are under investigation.

2. Notations, Preliminaries, and Main Results

2.1. Quantum Unbounded Spin Systems; Gibbs Measures

The systems we consider are the quantum unbounded spin systems which were introduced in detail in [PY1-2]. As a preparation, we briefly describe the systems and collect basic results which will be used in the sequel.

Let \mathbb{Z}^ν be the ν -dimensional lattice space and let \mathcal{C} be the class of finite subsets of \mathbb{Z}^ν . At each site $i \in \mathbb{Z}^\nu$ we associate an identical copy of the Hilbert space $L^2(\mathbb{R}^d, dx)$ where dx is the Lebesgue measure on \mathbb{R}^d . For $x = (x^1, \dots, x^d) \in \mathbb{R}^d$ and $i = (i_1, \dots, i_\nu) \in \mathbb{Z}^\nu$ we write

$$(2.1) \quad |x| = \left(\sum_{l=1}^d (x^l)^2 \right)^{1/2}, \quad |i| = \max_{1 \leq k \leq \nu} |i_k|.$$

For each $\Lambda \in \mathcal{C}$, we write

$$(2.2) \quad x_\Lambda = \{x_i : i \in \Lambda\}, \quad dx_\Lambda = \prod_{i \in \Lambda} dx_i.$$

The Hilbert space for quantum unbounded spin systems in $\Lambda \in \mathcal{C}$ is given by

$$(2.3) \quad \begin{aligned} \mathbb{H}_\Lambda &= \otimes_{i \in \Lambda} L^2(\mathbb{R}^d, dx_i) \\ &= L^2((\mathbb{R}^d)^\Lambda, dx_\Lambda). \end{aligned}$$

In this paper, for simplicity, we only consider one-body and two-body interactions, and introduce a Hamiltonian operator on \mathbb{H}_Λ by

$$(2.4) \quad \begin{aligned} H_\Lambda &= -\frac{1}{2} \sum_{i \in \Lambda} \Delta_i + V(x_\Lambda), \\ V(x_\Lambda) &\equiv \sum_{i \in \Lambda} \Phi_{\{i\}}(x_i) + \sum_{\substack{\{i,j\}: \\ i,j \in \Lambda}} \Phi_{\{i,j\}}(x_i, x_j), \end{aligned}$$

where Δ_i is the Laplacian operator for the variable $x_i \in \mathbb{R}^d$ and for any $i, j \in \mathbb{Z}^\nu$, $\Phi_{\{i\}}$ and $\Phi_{\{i,j\}}$ are the interaction potentials which are measurable real valued functions on \mathbb{R}^d and $(\mathbb{R}^d)^2$, respectively. Throughout this paper we impose the following conditions on the interaction:

ASSUMPTION 2.1. The interaction $\Phi = (\Phi_\Delta)_{\Delta \subset \mathbb{Z}^\nu}$ satisfies the following conditions:

(a) There exist a differentiable function $P(x)$ on \mathbb{R}^d and positive constants a and b such that for each $i \in \mathbb{Z}^\nu$, $\Phi_{\{i\}}(x_i) = P(x_i)$ and for some $\gamma > 2$

$$P(x) \geq a|x|^\gamma - b.$$

Moreover, there exists a constant $\alpha > 0$ and $M(\alpha)$ such that the bounds

$$\left| \frac{\partial}{\partial x^l} P(x) \right| \leq M(\alpha) \exp(\alpha|x|), \quad l = 1, 2, \dots, d,$$

hold.

(b) For each $r \in \mathbb{N}$, there exists a differentiable symmetric function $U(\cdot, \cdot; r) : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ such that $\Phi_{\{i,j\}}(x_i, x_j) = U(x_i, x_j; |i - j|) = U(x_j, x_i; |i - j|)$ for any $i, j \in \mathbb{Z}^\nu$. Moreover, there exists a decreasing function Ψ on \mathbb{N} such that $\Psi(r) \leq Kr^{-\nu-\epsilon}$ for some constants K and $\epsilon > 0$ and such that the bounds

$$\begin{aligned} |U(x, y; |i - j|)| &\leq \Psi(|i - j|) \frac{1}{2}(|x|^2 + |y|^2), \\ \left| \frac{\partial}{\partial x^l} U(x, y; |i - j|) \right| &\leq \Psi(|i - j|)(|x| + |y|), \quad l = 1, 2, \dots, d, \end{aligned}$$

hold.

REMARK 2.2. By the above conditions the interaction is invariant under translations of \mathbb{Z}^ν and moreover, superstable and regular [Ru1-2], i.e., there exist $A > 0$ and $c \in \mathbb{R}$ such that for any $x_\Lambda \in (\mathbb{R}^d)^\Lambda$,

$$(2.5) \quad V(x_\Lambda) \geq \sum_{i \in \Lambda} (A|x_i|^2 - c),$$

and if Λ_1, Λ_2 are disjoint finite subsets of \mathbb{Z}^ν and if we write

$$V(x_{\Lambda_1 \cup \Lambda_2}) = V(x_{\Lambda_1}) + V(x_{\Lambda_2}) + W(x_{\Lambda_1}, x_{\Lambda_2}),$$

then the bound

$$(2.6) \quad |W(x_{\Lambda_1}, x_{\Lambda_2})| \leq \sum_{i \in \Lambda_1} \sum_{j \in \Lambda_2} \Psi(|i - j|) \frac{1}{2} (|x_i|^2 + |x_j|^2)$$

holds. Furthermore, by Assumption 2.1 (a) we may take A in (2.5) to be arbitrarily large so that we may assume that

$$\sum_{i \in \mathbb{Z}^\nu} \Psi(|i|) < A,$$

which is needed to obtain a probability estimate, i.e., the regularity of Gibbs measures in Theorem 2.5 [LP, PY1].

For a bounded region $\Lambda \subset \mathbb{Z}^\nu$, the C^* -algebra of local observables is defined by

$$(2.7) \quad \mathfrak{A}_\Lambda = \mathcal{L}(\mathbb{H}_\Lambda),$$

where $\mathcal{L}(\mathbb{H}_\Lambda)$ is the algebra of all bounded operators on \mathbb{H}_Λ . If $\Lambda_1 \cap \Lambda_2 = \emptyset$, then $\mathbb{H}_{\Lambda_1 \cup \Lambda_2} = \mathbb{H}_{\Lambda_1} \otimes \mathbb{H}_{\Lambda_2}$, and \mathfrak{A}_{Λ_1} is isomorphic to the C^* -algebra $\mathfrak{A}_{\Lambda_1} \otimes \mathbf{1}_{\Lambda_2}$, where $\mathbf{1}_{\Lambda_2}$ denotes the identity operator on \mathbb{H}_{Λ_2} . In this way we identify \mathfrak{A}_Λ as a sub-algebra of $\mathfrak{A}_{\Lambda'}$ if $\Lambda \subset \Lambda'$. Let

$$(2.8) \quad \mathfrak{A} = \overline{\bigcup_{\Lambda \in \mathcal{C}} \mathfrak{A}_\Lambda}$$

be the algebra of the quasi-local observables. Notice that \mathfrak{A} has an identity.

The study of the equilibrium statistical mechanics for the quantum unbounded spin systems is to investigate the structure of Gibbs states on \mathfrak{A} [BR]. In [PY1], a characterization of Gibbs states was given via Gibbs measures on a path configuration space. The connection comes through the Feynman-Kac formula mentioned in Introduction.

For $x, y \in \mathbb{R}^d$, let us denote by $W_{x,y}$ the set of continuous paths $\xi : [0, 1] \rightarrow \mathbb{R}^d$ with $\xi(0) = x, \xi(1) = y$. $W_{x,y}$ is endowed with the standard Borel space structure. By $P_{x,y}$ we mean the conditional Wiener measure on $W_{x,y}$ [Sim]:

$$P_{x,y}(W_{x,y}) = (2\pi)^{-d/2} \exp\left(-\frac{1}{2}|x - y|^2\right).$$

For finite $\Lambda \subset \mathbb{Z}^\nu$ and $x_\Lambda, y_\Lambda \in (\mathbb{R}^d)^\Lambda$, W_{x_Λ, y_Λ} and P_{x_Λ, y_Λ} mean the Cartesian product $\prod_{j \in \Lambda} W_{x_j, y_j}$ and the product measure $\prod_{j \in \Lambda} P_{x_j, y_j}$, respectively. We let

$$(2.9) \quad E = \mathbb{R}^d \times W_{0,0},$$

equipped with the sup-norm $|\cdot|_u$, i.e., for $\xi = (x, \omega) \in E$,

$$(2.10) \quad |\xi|_u = \sup_{0 \leq t \leq 1} |x + \omega(t)|$$

We will also consider the L^2 -norm on E , $\|\xi\|_{L^2[0,1]} = (\int_0^1 |\xi(t)|^2 dt)^{1/2}$, $\xi \in E$, and simply write ξ^2 for $\|\xi\|_{L^2[0,1]}^2$, whenever there is no confusion involved in. We introduce a measure λ on E by

$$(2.11) \quad \lambda(d\xi) = dx P_{x,x}(d\xi), \quad \xi \in E.$$

We will use the same notation λ for any finite product λ^Λ , $\Lambda \in \mathcal{C}$, if there is no confusion involved in. The measure λ is the reference measure with which we characterize the system via measure theory.

Let $\Omega = E^{\mathbb{Z}^\nu}$ and for each $i \in \mathbb{Z}^\nu$, let $\pi_i : \Omega \rightarrow E$ be the projection, $\pi_i(\xi) = \xi_i$, $\xi = (\xi_i)_{i \in \mathbb{Z}^\nu} \in \Omega$. We topologize Ω by the countable seminorms, $\{\rho_i\}_{i \in \mathbb{Z}^\nu} : \rho_i(\xi) \equiv |\pi_i(\xi)|_u$. Notice that this topology is equivalent to the metric topology given by the metric:

$$(2.12) \quad d(\xi, \zeta) = \sum_{i \in \mathbb{Z}^\nu} 2^{-|i|} \frac{\rho_i(\xi - \zeta)}{1 + \rho_i(\xi - \zeta)}, \quad \xi, \zeta \in \Omega.$$

For each subset $\Lambda \subset \mathbb{Z}^\nu$, we have a local σ -algebra \mathcal{F}_Λ , which is the minimal σ -algebra of Borel sets for which $\rho_i, i \in \Lambda$, is continuous. We simply write \mathcal{F} for $\mathcal{F}_{\mathbb{Z}^\nu}$. By $\mathcal{P}(\Omega, \mathcal{F})$ we mean the space of probability measures on (Ω, \mathcal{F}) .

Before introducing Gibbs measures on Ω , we give the notion of regular measures on Ω :

DEFINITION 2.3. A Borel probability measure μ on (Ω, \mathcal{F}) is said to be *regular* if there exist $A^* > 0$ and $\delta > 0$ so that the projection μ_Λ of μ on any $(\Omega, \mathcal{F}_\Lambda)$, being understood as a measure on $(E^\Lambda, \mathcal{B}(E)^\Lambda)$, satisfies

$$g(\zeta_\Lambda | \mu) \leq \exp \left[- \sum_{i \in \Lambda} (A^* \zeta_i^2 - \delta) \right],$$

where $g(\zeta_\Lambda | \mu)$ is such that $\mu_\Lambda(d\zeta_\Lambda) = g(\zeta_\Lambda | \mu) \lambda(d\zeta_\Lambda)$.

For $\Delta \subset \mathbb{Z}^\nu$, ζ_Δ means an element of E^Δ or the projection of $\zeta \in \Omega$ to Δ . We write that

$$\Phi_\Delta(\zeta_\Delta) = \int_0^1 \Phi_\Delta(\zeta_\Delta(t)) dt,$$

$$(2.13) \quad V(\zeta_\Lambda) = \sum_{\Delta \subset \Lambda} \Phi_\Delta(\zeta_\Delta),$$

and for $\xi \in \Omega$ and $\Lambda \subset \mathbb{Z}^\nu$,

$$(2.14) \quad W(\xi_\Lambda, \xi_{\Lambda^c}) = \sum_{i \in \Lambda, j \in \Lambda^c} \Phi_{\{i,j\}}(\xi_i, \xi_j).$$

Let us define

$$(2.15) \quad \mathfrak{S} = \bigcup_{N \in \mathbb{N}} \mathfrak{S}_N,$$

$$\mathfrak{S}_N = \left\{ \xi \in \Omega : \forall l, \sum_{|i| \leq l} \xi_i^2 \leq N^2(2l + 1)^\nu \right\}.$$

This definition is invariant under linear translations of \mathbb{Z}^ν . It can be shown that each regular measure μ on (Ω, \mathcal{F}) satisfies $\mu(\mathfrak{S}) = 1$. We say that such a measure is *tempered*.

The partition function in a finite $\Lambda \subset \mathbb{Z}^\nu$ for the interaction Φ with boundary condition $\xi \in \mathfrak{S}$ is defined by

$$(2.16) \quad Z_\Lambda^\Phi(\xi) \equiv \int \lambda(d\zeta_\Lambda) \exp[-V(\zeta_\Lambda) - W(\zeta_\Lambda, \xi_{\Lambda^c})].$$

We note that the partition function is well defined from the assumptions for Φ . The Gibbs specification $\gamma^\Phi = (\gamma_\Lambda^\Phi)_{\Lambda \in \mathcal{C}}$ with respect to \mathfrak{S} is defined by [Ge, PY1, Pr]

$$(2.17) \quad \gamma_\Lambda^\Phi(A|\xi) = \begin{cases} Z_\Lambda^\Phi(\xi)^{-1} \int \lambda(d\zeta_\Lambda) \exp[-V(\zeta_\Lambda) - W(\zeta_\Lambda, \xi_{\Lambda^c})] \\ \quad \times 1_A(\zeta_\Lambda \xi_{\Lambda^c}), & \text{if } \xi \in \mathfrak{S} \\ 0, & \text{if } \xi \notin \mathfrak{S}, \end{cases}$$

where $A \in \mathcal{F}$ and 1_A is the indicator function of A and $\zeta_\Lambda \xi_{\Lambda^c}$ is the configuration coinciding with ζ_Λ on Λ and with ξ_{Λ^c} on Λ^c , respectively. It is easy to check that the Gibbs specification satisfies the consistent condition [Ge, Pr]: for $\Delta \subset \Lambda$, $\xi \in \mathfrak{S}$,

$$\begin{aligned} \gamma_\Lambda^\Phi \gamma_\Delta^\Phi(A|\xi) &\equiv \int_{\mathfrak{S}} \gamma_\Lambda^\Phi(d\eta|\xi) \gamma_\Delta^\Phi(A|\eta) \\ &= \gamma_\Lambda^\Phi(A|\xi). \end{aligned}$$

We now give a definition of Gibbs measures on (Ω, \mathcal{F}) :

DEFINITION 2.4. A Gibbs measure μ for the potential Φ is a tempered Borel probability measure on (Ω, \mathcal{F}) satisfying the equilibrium equations

$$\mu(A) = \int \mu(d\xi) \gamma_\Lambda^\Phi(A|\xi), \quad \Lambda \in \mathcal{C}, A \in \mathcal{F}.$$

We denote by $\mathcal{G}^\Phi(\Omega)$ the family of all Gibbs measures.

We topologize the space $\mathcal{P}(\Omega, \mathcal{F})$ with the topology of local convergence [Ge]: for each $\mu \in \mathcal{P}(\Omega, \mathcal{F})$ the sets

$$\{ \nu \in \mathcal{P}(\Omega, \mathcal{F}) : \max_{1 \leq k \leq n} |\nu(A_k) - \mu(A_k)| < \varepsilon \}$$

with $A_1, \dots, A_n \in \cup_{\Lambda \in \mathcal{C}} \mathcal{F}_\Lambda$, $\varepsilon > 0$, and $n \geq 1$, form a base of neighborhoods of μ .

The following is one of the main results in [PY1].

THEOREM 2.5 ([PY1], THEOREM 2.7). *Let the hypotheses of Assumption 2.1 hold. Then any Gibbs measure is regular. Furthermore, $\mathcal{G}^\Phi(\Omega)$ is non-empty, convex, compact in the local convergence topology, and a Choquet simplex.*

REMARK 2.6. Under the hypotheses of Assumption 2.1, by the argument mentioned in Remark 2.2, the regularity constant A^* (see Definition 2.3) can be taken to be arbitrarily large. Especially, we will take $A^* > \frac{1}{2}$.

2.2. Basic Spaces

In this subsection we will describe the basic spaces with which the Dirichlet forms are formulated. Recall the definition of E in (2.9). Since E is a real Banach space, there exist a Hilbert space H_0 densely embedded in E , and a Gaussian measure μ^0 on E such that (μ^0, H_0, E) is an abstract Wiener space [Kuo]. In fact, let $C_p^\infty([0, 1], \mathbb{R}^d)$ be the space of \mathbb{R}^d -valued C^∞ -functions on $[0, 1]$, where $[0, 1]$ is understood as a circle by matching the end points 0 and 1. Let Δ_p be the Laplacian operator on $L^2([0, 1], \mathbb{R}^d; dt)$ with the periodic boundary conditions and define

$$(2.18) \quad (f, g)_0 \equiv ((-\Delta_p + 1)^{1/2} f, (-\Delta_p + 1)^{1/2} g)_{L^2[0,1]}, \quad f, g \in C_p^\infty([0, 1], \mathbb{R}^d).$$

Let H_0 be the completion of $C_p^\infty([0, 1], \mathbb{R}^d)$ w.r.t. $(\cdot)_0$ and set the H_0 -norm $(f, f)_0^{1/2}$ by $|f|_0$. Let μ^0 be the measure on E defined by

$$(2.19) \quad \mu^0(d\xi) = \frac{1}{N} \exp\left(-\frac{1}{2} \int_0^1 |\xi(t)|^2 dt\right) \lambda(d\xi), \quad \xi \in E,$$

where N is the normalization constant, which is finite by the Golden-Thompson inequality [Sim]. Then, it can be shown that (μ^0, H_0, E) is an abstract Wiener space. By a direct calculation or from the general theory on the abstract Wiener space [Kuo], it can be shown that there exists a constant c such that

$$(2.20) \quad |\xi|_u \leq c |\xi|_0, \quad \xi \in H_0.$$

The probability measure μ^0 is introduced for the utilization of the abstract Wiener space theory in the sequel. We will also write μ^0 for any finite product $(\mu^0)^\Lambda$, $\Lambda \in \mathcal{C}$.

As mentioned in Introduction, we are now going to give a rigging to H_0 by Hilbert spaces H_+ and H_- . For simplicity let us write

$$(2.21) \quad A_p \equiv -\Delta_p + 1.$$

Define the inner products $(\cdot, \cdot)_+$ and $(\cdot, \cdot)_-$ on $C_p^\infty([0, 1], \mathbb{R}^d)$ by

$$(2.22) \quad (f, g)_+ \equiv (A_p f, A_p g)_{L^2[0,1]}, \quad f, g \in C_p^\infty([0, 1], \mathbb{R}^d),$$

and

$$(2.23) \quad (f, g)_- \equiv (f, g)_{L^2[0,1]}, \quad f, g \in C_p^\infty([0, 1], \mathbb{R}^d).$$

The norms on $C_p^\infty([0, 1], \mathbb{R}^d)$ induced by $(\cdot, \cdot)_+$ and $(\cdot, \cdot)_-$ will be denoted by $|\cdot|_+$ and $|\cdot|_-$, respectively. The Hilbert spaces obtained by completing $C_p^\infty([0, 1], \mathbb{R}^d)$ via $|\cdot|_+$ and $|\cdot|_-$ will be denoted by H_+ and H_- , respectively. It is obvious that

$$(2.24) \quad H_+ \subset H_0 \subset H_-$$

is a rigging of H_0 by H_+ and H_- and the embeddings are everywhere dense and belong to the Hilbert-Schmidt class. Furthermore, it follows from (2.20) and the fact that $|\xi|_- \leq |\xi|_u$ that the inclusions

$$(2.25) \quad H_+ \subset E' \subset H_0 \subset E \subset H_-$$

hold, where E' means the topological dual of E . The duality between H_+ and H_- given by $(\cdot, \cdot)_0$ will be denoted by $\langle \cdot, \cdot \rangle$.

In relevance with the configuration space Ω , we introduce the following spaces. Let us fix a $\sigma > 0$. Define a subspace $\Omega_- \subset \Omega$ by

$$(2.26) \quad \Omega_- \equiv \{ \xi \in \Omega : \|\xi\|_u^2 \equiv \sum_{i \in \mathbb{Z}^\nu} e^{-\sigma|i|} |\xi_i|_u^2 < \infty \}.$$

We define Hilbert spaces on the (\mathbb{R} -valued) sequence spaces as follows: for the given $\sigma > 0$ put

$$(2.27) \quad \begin{aligned} l_+ &= \{ (a_i)_{i \in \mathbb{Z}^\nu} : \sum_{i \in \mathbb{Z}^\nu} e^{\sigma|i|} |a_i|^2 < \infty \}, \\ l_0 &= \{ (a_i)_{i \in \mathbb{Z}^\nu} : \sum_{i \in \mathbb{Z}^\nu} |a_i|^2 < \infty \}, \\ l_- &= \{ (a_i)_{i \in \mathbb{Z}^\nu} : \sum_{i \in \mathbb{Z}^\nu} e^{-\sigma|i|} |a_i|^2 < \infty \}. \end{aligned}$$

Obviously the embeddings $l_+ \subset l_0 \subset l_-$ belong to the Hilbert-Schmidt class. Finally, for the configuration space, define Hilbert spaces by

$$(2.28) \quad \mathcal{H}_+ \equiv H_+ \otimes l_+, \quad \mathcal{H}_0 \equiv H_0 \otimes l_0, \quad \text{and} \quad \mathcal{H}_- \equiv H_- \otimes l_-.$$

We denote the inner products on \mathcal{H}_+ , \mathcal{H}_0 , and \mathcal{H}_- by the notations $((\cdot, \cdot))_+$, $((\cdot, \cdot))_0$, and $((\cdot, \cdot))_-$, respectively and the induced norms by the notations $\|\cdot\|_+$, $\|\cdot\|_0$, and $\|\cdot\|_-$, respectively. We see that for $\xi = (\xi_i)_{i \in \mathbb{Z}^\nu}$,

$$(2.29) \quad \begin{aligned} \|\xi\|_+^2 &= \sum_{i \in \mathbb{Z}^\nu} e^{\sigma|i|} |\xi_i|_+^2, \\ \|\xi\|_0^2 &= \sum_{i \in \mathbb{Z}^\nu} |\xi_i|_0^2, \\ \|\xi\|_-^2 &= \sum_{i \in \mathbb{Z}^\nu} e^{-\sigma|i|} |\xi_i|_-^2. \end{aligned}$$

From the definitions,

$$(2.30) \quad \mathcal{H}_+ \subset \mathcal{H}_0 \subset \mathcal{H}_-$$

is a rigging of \mathcal{H}_0 by \mathcal{H}_+ and \mathcal{H}_- and the embeddings are everywhere dense and belong to the Hilbert-Schmidt class. The duality between \mathcal{H}_+ and \mathcal{H}_- given by $((\cdot, \cdot))_0$ will be denoted by $\ll \cdot, \cdot \gg$. Recall the definition of Ω_{\log} in (1.4). It is easy to check that the following inclusions

$$(2.31) \quad \mathcal{H}_0 \subset \Omega_{\log} \subset \Omega_- \subset \mathcal{H}_-$$

hold. Notice also that $\Omega_- \subset \Omega$.

Before closing this subsection we prove the following basic lemma.

LEMMA 2.7. *Under the hypotheses of Assumption 2.1, $\mu(\Omega_{\log}) = 1$ for any $\mu \in \mathcal{G}^\Phi(\Omega)$.*

Proof. Define for each $n \in \mathbb{N}$,

$$\begin{aligned} A_{i,n} &= \{\xi \in \Omega : |\xi_i|_u > n \log(|i| + 1)\}, \quad i \in \mathbb{Z}^\nu, \\ \Omega_{\log,n} &= \{\xi \in \Omega : |\xi_i|_u \leq n \log(|i| + 1), \quad \forall i \neq 0\}. \end{aligned}$$

Then, it follows that $\Omega_{\log}^c = \bigcap_n \Omega_{\log, n}^c$ and so

$$\mu(\Omega_{\log}^c) = \lim_{n \rightarrow \infty} \mu(\Omega_{\log, n}^c).$$

Notice that $\Omega_{\log, n}^c = \bigcup_{i \neq 0} A_{i, n}$. Thus, we have that

$$(2.32) \quad \mu(\Omega_{\log, n}^c) \leq \sum_{\substack{i \in \mathbb{Z}^d \\ i \neq 0}} \mu(A_{i, n}).$$

Due to the regularity of Gibbs measures,

$$(2.33) \quad \mu(A_{i, n}) \leq \int_{\{|\zeta|_u > n \log(|i|+1)\}} \exp(-A^*|\zeta|_-^2 + \delta) P_{x, x}(d\zeta) dx.$$

By the method used in the proof of Proposition 2.10 of [PY1] and using the argument in Remark 2.6, it can be shown that there exist positive constant c such that the bound

$$(2.34) \quad \int \exp(-A^*|\zeta|_-^2) P_{x, x}(d\zeta) \leq \exp(-\frac{1}{2}x^2 + c)$$

holds. Moreover, by Theorem 6.3.8 of [BR] we see that there exist positive constants d_1 and d_2 such that

$$(2.35) \quad \int_{\{|\zeta|_{-x}|_u \geq r\}} P_{x, x}(d\zeta) \leq \exp(-d_1 r^2 + d_2).$$

We divide \mathbb{R}^d into two disjoint regions: $\mathbb{R}^d = D_{n, i} \cup D_{n, i}^c$, where $D_{n, i} = \{x \in \mathbb{R}^d : |x| \leq \frac{1}{2}n \log(|i| + 1)\}$. Then, by (2.34) and (2.35), the integration in (2.33) is bounded by

$$\int_{D_{n, i}} dx \exp\left(-d_1 \left(\frac{1}{2}n \log(|i| + 1)\right)^2 + d_2\right) + \int_{D_{n, i}^c} dx \exp\left(-\frac{1}{2}x^2 + c\right).$$

Substituting the above bound into (2.33) and again into (2.32), we see that there exist positive constants M and b such that

$$\begin{aligned} \mu(\Omega_{\log}^c) &\leq \lim_{n \rightarrow \infty} M \sum_{i \in \mathbb{Z}^d} \exp[-bn \log(|i| + 1)] \\ &= 0. \end{aligned}$$

This completes the proof of the lemma. ■

2.3. Main Results

We introduce a Dirichlet form and the associated diffusion process related to a Gibbs measure $\mu \in \mathcal{G}^\Phi(\Omega)$ and list the main results in this paper. We start with definitions for a (symmetric) Dirichlet form and the associated diffusion process. Let $(\mathcal{H}, (\cdot, \cdot))$ be a real Hilbert space. A pair $(\mathcal{E}, D(\mathcal{E}))$ is a form on \mathcal{H} if $D(\mathcal{E})$ is a linear subspace of \mathcal{H} and $\mathcal{E} : D(\mathcal{E}) \times D(\mathcal{E}) \rightarrow \mathbb{R}$ is a non-negative symmetric bilinear form. Given a form $(\mathcal{E}, D(\mathcal{E}))$ on \mathcal{H} and $\alpha > 0$, we set $\mathcal{E}_\alpha \equiv \mathcal{E} + \alpha(\cdot, \cdot)$, $D(\mathcal{E}_\alpha) = D(\mathcal{E})$. $(\mathcal{E}, D(\mathcal{E}))$ is said to be *closed* if the pre-Hilbert space $(D(\mathcal{E}_1), \mathcal{E}_1)$ is complete and *closable* if it has a closed extension, i.e., there exists a closed form $(\tilde{\mathcal{E}}, D(\tilde{\mathcal{E}}))$ on \mathcal{H} such that $D(\mathcal{E}) \subset D(\tilde{\mathcal{E}})$ and $\mathcal{E} = \tilde{\mathcal{E}}$ on $D(\mathcal{E})$. Clearly, $(\mathcal{E}, D(\mathcal{E}))$ is closable if and only if the following condition is satisfied: if $u_n \in D(\mathcal{E})$, $n \in \mathbb{N}$, such that $u_n \xrightarrow{n \rightarrow \infty} 0$ in \mathcal{H} and $\{u_n\}_{n \in \mathbb{N}}$ is Cauchy (i.e., $\mathcal{E}(u_n - u_m, u_n - u_m) \xrightarrow{n, m \rightarrow \infty} 0$), then $\lim_{n \rightarrow \infty} \mathcal{E}(u_n, u_n) = 0$.

Let us now consider the case $\mathcal{H} \equiv L^2(\Omega, d\mu)$ where Ω is a Hausdorff locally convex topological vector space over \mathbb{R} equipped with its Borel σ -algebra $\mathcal{B}(\Omega)$ and μ is a (probability) measure on $(\Omega, \mathcal{B}(\Omega))$. A form $(\mathcal{E}, D(\mathcal{E}))$ is said to be *Markovian* if for each $\varepsilon > 0$, there exists a real function $\phi_\varepsilon(t)$, $t \in \mathbb{R}$, such that $\phi_\varepsilon(t) = t$ for $0 < t < 1$, $-\varepsilon \leq \phi_\varepsilon(t) \leq 1 + \varepsilon$ for any $t \in \mathbb{R}$, $0 \leq \phi_\varepsilon(t) - \phi_\varepsilon(s) \leq t - s$ whenever $s < t$, and for any $u \in D(\mathcal{E})$, it holds that $\phi_\varepsilon(u) \in D(\mathcal{E})$ and $\mathcal{E}(\phi_\varepsilon(u), \phi_\varepsilon(u)) \leq \mathcal{E}(u, u)$.

DEFINITION 2.8. A Dirichlet form on a real Hilbert space $L^2(\Omega, d\mu)$ is a closed Markovian form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(\Omega, d\mu)$.

We focus on the space $L^2(\Omega, d\mu)$ where $\Omega = E^{\mathbb{Z}^d}$ and $\mu \in \mathcal{G}^\Phi(\Omega)$. Before continuing further, we check that $L^2(\Omega, d\mu)$ would be considered as $L^2(\mathcal{H}_-, d\mu)$ by the following argument.

REMARK 2.9. In Lemma 3.4, we will show that any Gibbs measure $\mu \in \mathcal{G}^\Phi(\Omega)$ is supported on the Banach space Ω_- (see (2.26) for the definition). We notice that Ω_- is dense in both Ω and \mathcal{H}_- . Furthermore, it can be shown that Ω_- is a Borel subset of both Ω and \mathcal{H}_- , i.e., $\Omega_- \in \mathcal{B}(\Omega)$ and $\Omega_- \in \mathcal{B}(\mathcal{H}_-)$. We know that $\Omega_{\text{log}} \subset \Omega_-$ and

$\mu(\Omega_{\log}) = 1$ by Lemma 2.7. Since for any $A \in \mathcal{B}(\mathcal{H}_-)$, $A \cap \Omega_- \in \mathcal{B}(\Omega)$, with the same notation we define for any $A \in \mathcal{B}(\mathcal{H}_-)$,

$$\mu(A) = \mu(A \cap \Omega_-).$$

In this way μ can be considered as a probability measure on $(\mathcal{H}_-, \mathcal{B}(\mathcal{H}_-))$. Furthermore, since $\mu(\Omega_-) = 1$, we may identify $L^2(\Omega, d\mu)$ with $L^2(\mathcal{H}_-, d\mu)$.

From now on we fix a Gibbs measure $\mu \in \mathcal{G}^\Phi(\Omega)$ and we will take $L^2(\mathcal{H}_-, d\mu)$ as the underling Hilbert space on which Dirichlet forms will be constructed.

For $n = 0, 1, \dots, \infty$, denote by $C^n(\mathbb{R}^m)$ (resp. $C_b^n(\mathbb{R}^m)$) the spaces of n -times continuously differentiable functions on \mathbb{R}^m (resp. n -times continuously differentiable bounded functions on \mathbb{R}^m with bounded derivatives). Recall that (μ^0, H_0, E) is an abstract Wiener space. Let \mathcal{P}_{fin} be the class of finite rank projection operators P on H_0 with images in H_+ . If $\{l_1, \dots, l_n\} \subset H_+$ is a (orthonormal w.r.t. H_0) base for the range space of P , it can be written that

$$P(k) = \sum_{i=1}^n \langle k, l_i \rangle l_i, \quad k \in H_0.$$

Therefore, P extends continuously to H_- and we write the continuous extension of P to H_- with the same notation P . Let $\mathcal{F}_{\text{loc}}C^n$ (resp. $\mathcal{F}_{\text{loc}}C_b^n$) be the spaces of functions u on \mathcal{H}_- of the following type: there exist $m \in \mathbb{N}$, $\Lambda \in \mathcal{C}$, $P_i \in \mathcal{P}_{\text{fin}}$ for $i \in \Lambda$, with $\sum_{i \in \Lambda} \dim P_i H_0 = m$, and $f \in C^n(\mathbb{R}^m)$ (resp. $f \in C_b^n(\mathbb{R}^m)$) such that

$$\begin{aligned} u(\xi) &= f(P_i \circ \pi_i(\xi), P_j \circ \pi_j(\xi), \dots), \quad i, j, \dots \in \Lambda \\ &\equiv f(l_{i_1}(\xi), l_{i_2}(\xi), \dots, l_{i_{m(i)}}(\xi), \dots), \end{aligned}$$

where $\{l_{i_1}, l_{i_2}, \dots, l_{i_{m(i)}}\} \subset \mathcal{H}_+$ is a base of the range space of $P_i \circ \pi_i$, $i \in \Lambda$.

We will show in Lemma 3.4 that $\mathcal{F}_{\text{loc}}C_b^\infty$ is dense in $L^2(\mathcal{H}_-, d\mu)$. Now, let us define for $u \in \mathcal{F}_{\text{loc}}C_b^1$, the following Gâteaux-type derivative (in direction k) by

$$(2.36) \quad \frac{\partial}{\partial k} u(\xi) \equiv \left. \frac{d}{ds} u(\xi + sk) \right|_{s=0}, \quad \xi, k \in \mathcal{H}_-.$$

Since any Gibbs measure $\mu \in \mathcal{G}^\Phi(\Omega)$ is supported on \mathcal{H}_- by Lemma 3.4, $\frac{\partial}{\partial k}$ of (2.36) defines an operator with a dense domain in $L^2(\mathcal{H}_-, d\mu)$. Let us define a form on $L^2(\mathcal{H}_-, d\mu)$ by

$$(2.37) \quad \mathcal{E}_{\mu,k}(u, v) = \int \frac{\partial}{\partial k} u \cdot \frac{\partial}{\partial k} v \, d\mu, \quad u, v \in \mathcal{F}_{\text{loc}} C_b^\infty.$$

As in [AR1], we say that a vector $k \in \mathcal{H}_- \setminus \{0\}$ is (μ) -admissible if the form (2.37) is closable. Remember that for $\xi \in E$ and $i \in \mathbb{Z}^\nu$, $\xi^{(i)} \in \Omega$ is such that $\pi_j(\xi^{(i)}) = \delta_{ij}\xi$, $j \in \mathbb{Z}^\nu$. Let $\{e_n\}_{n \in \mathbb{N}}$ be an orthonormal basis of $H_0 \subset E$ with $e_n \in H_+$, $n = 1, 2, \dots$. Define a form on $L^2(\mathcal{H}_-, d\mu)$ by

$$(2.38) \quad \begin{aligned} D(\mathcal{E}_\mu) &= \mathcal{F}_{\text{loc}} C_b^\infty, \\ \mathcal{E}_\mu(u, v) &= \sum_{i \in \mathbb{Z}^\nu} \sum_{n=1}^\infty \int \nabla_n^i u \cdot \nabla_n^i v \, d\mu, \quad u, v \in D(\mathcal{E}_\mu), \end{aligned}$$

where $\nabla_n^i u = \frac{\partial}{\partial e_n^{(i)}} u$. We notice that for each $u \in \mathcal{F}_{\text{loc}} C_b^\infty$,

$$\mathcal{E}_\mu(u, u) = \sum_{i \in \mathbb{Z}^\nu} \sum_{n=1}^\infty \int \nabla_n^i u \cdot \nabla_n^i u \, d\mu < \infty.$$

Therefore, the form (2.38) is well defined. In fact, since the summation w.r.t. $i \in \mathbb{Z}^\nu$ in the above is a finite sum, it suffices to show the finiteness of $\sum_{n=1}^\infty \int \nabla_n^i u \cdot \nabla_n^i u \, d\mu$. Suppose that $u \in \mathcal{F}_{\text{loc}} C_b^\infty$ is of the form: $u(\xi) = f(\dots, P_i \circ \pi_i(\xi), \dots) = f(\dots, l_1(\pi_i(\xi)), \dots, l_m(\pi_i(\xi)), \dots)$, where f is a smooth function (on a finite dimensional space) and $\{l_1, \dots, l_m\} \subset H_+$ is a basis of the range space of P_i . We notice that $\sum_{n=1}^\infty (l_k(e_n))^2 < \infty$, $k = 1, \dots, m$. Then,

$$\begin{aligned} &\sum_{n=1}^\infty \int \nabla_n^i u \cdot \nabla_n^i u \, d\mu \\ &\leq \sum_{k=1}^m \int \left(\partial_k^i f(\dots, l_1(\pi_i(\xi)), \dots, l_m(\pi_i(\xi)), \dots) \right)^2 d\mu(\xi) \\ &\quad \cdot \sum_{k=1}^m \sum_{n=1}^\infty (l_k(e_n))^2 \\ &< \infty, \end{aligned}$$

where $\partial_k^i f$ is the partial derivative of f w.r.t. the k -th variable in the range space of P_i . This proves the assertion. Obviously the form (2.38) does not depend on the choice of orthonormal bases in H_0 [Rö].

Let us consider a coordinate free version of the form (2.38) [Rö]. If $u \in \mathcal{F}_{\text{loc}} C_b^\infty$ is of the form $u(\xi) = f(l_{i_1}(\xi_i), l_{i_2}(\xi_i), \dots, l_{i_{m(i)}}(\xi_i), \dots)$, then

$$\frac{\partial u}{\partial k}(\xi) = \sum_i \sum_{m=1}^{m(i)} \partial_m^i f(l_{i_1}(\xi_i), l_{i_2}(\xi_i), \dots, l_{i_{m(i)}}(\xi_i), \dots) \langle l_{i_m}, k_i \rangle,$$

where the i -summation is a finite sum and $\partial_m^i f$ is the derivative of f with respect to the i_m -th variable. Therefore, for $u \in \mathcal{F}_{\text{loc}} C_b^\infty$ and $\xi \in \mathcal{H}_-$ fixed, $k \rightarrow \frac{\partial u(\xi)}{\partial k}$ is a continuous linear functional on \mathcal{H}_- . Define $\nabla u(\xi) \in \mathcal{H}_+$ by

$$(2.39) \quad \ll \nabla u(\xi), k \gg = \frac{\partial u(\xi)}{\partial k}, \quad k \in \mathcal{H}_-$$

It follows from (2.37) - (2.39) that

$$(2.40) \quad \begin{aligned} D(\mathcal{E}_\mu) &= \mathcal{F}_{\text{loc}} C_b^\infty \\ \mathcal{E}_\mu(u, v) &= \int_{\mathcal{H}_-} ((\nabla u(\xi), \nabla v(\xi)))_0 d\mu(\xi). \end{aligned}$$

The form $(\mathcal{E}_\mu, \mathcal{F}_{\text{loc}} C_b^\infty)$ is a densely defined positive definite symmetric bilinear form on $L^2(\mathcal{H}_-, d\mu)$.

We are now in a position to say one of the main results in this paper.

THEOREM 2.10. *Let the hypotheses of Assumption 2.1 be satisfied and $\mu \in \mathcal{G}^\Phi(\Omega)$. Then the form (2.38) is closable and the closure is a Dirichlet form.*

Next, we consider the associated diffusion process. Let $(\mathcal{E}, D(\mathcal{E}))$ be an arbitrary Dirichlet form on $L^2(\Omega, d\mu)$. Here, we do not restrict Ω and μ to any special case. Define

$$D(L) = \{u \in D(\mathcal{E}) : v \mapsto \mathcal{E}(u, v) \text{ is continuous w.r.t. } (\cdot, \cdot)_{L^2}^{1/2} \text{ on } D(\mathcal{E})\},$$

and let $(L, D(L))$ be the linear operator defined by $(-Lu, v) = \mathcal{E}(u, v)$. Then, L is the generator of a strongly continuous Markovian semigroup $(T_t)_{t \geq 0}$, i.e., $T_t = e^{-tL}$, $t \geq 0$ and for all $u \in L^2(\Omega, d\mu)$, $0 \leq u \leq 1$ implies $0 \leq T_t u \leq 1$, $t \geq 0$. See [Fu, Rö] for details.

DEFINITION 2.11. Let $(\mathcal{E}, D(\mathcal{E}))$ be a Dirichlet form on $L^2(\Omega, d\mu)$. A Markov process $(\tilde{\Omega}, \tilde{\mathcal{F}}, (X_t)_{t \geq 0}, (P_\xi)_{\xi \in \Omega})$ with state space Ω is said to be *associated with* $(\mathcal{E}, D(\mathcal{E}))$ if for any $u : \Omega \rightarrow \mathbb{R}$, \mathcal{F} -measurable, bounded, and every $t \geq 0$,

$$(T_t u)(\xi) = \int_{\tilde{\Omega}} u(X_t) dP_\xi \quad \text{for } \mu - \text{a.e. } \xi \in \Omega.$$

A Markov process $(\tilde{\Omega}, \tilde{\mathcal{F}}, (X_t)_{t \geq 0}, (P_\xi)_{\xi \in \Omega})$ associated with $(\mathcal{E}, D(\mathcal{E}))$ is called a *diffusion* if it is a Hunt process having continuous sample paths P_ξ -almost surely for each $\xi \in \Omega$.

We have also the following result.

THEOREM 2.12. *Let the hypotheses of Assumption 2.1 be satisfied and $\mu \in \mathcal{G}^\Phi(\Omega)$. Then there exists a diffusion process with state space \mathcal{H}_- associated with the closure of $(\mathcal{E}_\mu, D(\mathcal{E}_\mu))$ in (2.38).*

The proofs of Theorem 2.10 and Theorem 2.12 will be produced in the next section.

We introduce the Dirichlet operator associated to a Gibbs measure. Let $P(x)$ and $U(x, y; |i - j|)$ be the one-body and two-body potentials given in Assumption 2.1. For a later use, we define

$$(2.41) \quad \tilde{P}(x) = P(x) - \frac{1}{2}|x|^2, \quad x \in \mathbb{R}^d.$$

Recall the definition of the operator A_p in (2.21). For $\xi \in \Omega_{\log}$, let $\beta(\xi)$ be an element of \mathcal{H}_- given by

$$(2.42) \quad \beta(\xi) = (\beta_i(\xi))_{i \in \mathbb{Z}^d}, \quad \xi \in \Omega_{\log},$$

$$\beta_i(\xi) = - \left[A_p^{-1}(\nabla^i \tilde{P}(\xi_i)) + \xi_i + \sum_{\substack{j \in \mathbb{Z}^d \\ j \neq i}} A_p^{-1}(\nabla^i U(\xi_i, \xi_j; |i - j|)) \right],$$

where $\nabla^i \tilde{P}$, e.g., is the gradient of the function \tilde{P} on \mathbb{R}^d w.r.t. $x_i = (x_i^1, \dots, x_i^d) \in \mathbb{R}^d$ variable and for $\xi_i \in E$, $\nabla^i \tilde{P}(\xi_i)$ is an element in E defined by

$$\nabla^i \tilde{P}(\xi_i)(t) \equiv \nabla^i \tilde{P}(\xi_i(t)), \quad t \in [0, 1].$$

(There may occur some confusion comparing with the definition in (2.13). However, we will stick on this convention because it will be obvious what is meant from the context.) It turns out that when $k \in \mathbb{H}\text{-}\Omega_{\text{loc}}$ (see Section 3 for the notation), $\beta_k(\xi) \equiv \ll \beta(\xi), k \gg$ is the osmotic velocity (in the direction k). In Section 3, we will show that $\beta(\xi) \in \mathcal{H}_-$ if $\xi \in \Omega_{\text{log}}$. Denote by $C^k(\mathcal{H}_-, \mathcal{B})$ the set of mappings from \mathcal{H}_- into a Banach space \mathcal{B} that are k -times continuously differentiable in the sense of Fréchet. See, e.g., [AK]. Define $C_b^k(\mathcal{H}_-, \mathcal{B})$ as the subset of $C^k(\mathcal{H}_-, \mathcal{B})$ which is characterized by the boundedness in usual operator norms of the derivatives

$$f^{(l)} : \mathcal{H}_- \rightarrow \mathcal{L}(\mathcal{H}_-, \mathcal{L}(\mathcal{H}_-, \dots, \mathcal{L}(\mathcal{H}_-, \mathcal{B}) \dots)), \quad l = 0, 1, 2, \dots, k.$$

For $f : \mathcal{H}_- \rightarrow \mathbb{R}$, identify $f'(\cdot) \in \mathcal{L}(\mathcal{H}_-, \mathbb{R})$ with the vector $\widehat{f}'(\cdot) \in \mathcal{H}_+$ and $f''(\cdot)$ with the operator $\widehat{f}''(\cdot) \in \mathcal{L}(\mathcal{H}_-, \mathcal{H}_+)$ by the formulae

$$(2.43) \quad \begin{aligned} f'(\xi)\zeta &= \ll \widehat{f}'(\xi), \zeta \gg, \\ (f''(\xi)\zeta)\phi &= \ll \widehat{f}''(\xi)\zeta, \phi \gg, \quad \zeta, \phi, \xi \in \mathcal{H}_-. \end{aligned}$$

For the function $f \in C_b^2 \equiv C_b^2(\mathcal{H}_-, \mathbb{R})$ we use the symbol $\nabla f = f'$ and $\Delta f = \text{Tr}_{\mathcal{H}_0}(f'')$. We notice that for $u \in \mathcal{F}_{\text{loc}}C_b^\infty$, the quantities ∇u , defined here and in (2.39), are the same thing. We introduce in the space C_b^2 the norm

$$\|f\|_{C_b^2} \equiv \sup_{\xi \in \mathcal{H}_-} \{ |f(\xi)| + \|f'(\xi)\|_+ + \|f''(\xi)\|_{\mathcal{L}(\mathcal{H}_-, \mathcal{H}_+)} \}.$$

Define a differential operator H_μ on the domain $D(H_\mu) = \mathcal{F}_{\text{loc}}C_b^\infty$ by the formula.

$$(2.44) \quad H_\mu u(\xi) = -\Delta u(\xi) - \ll \beta(\xi), \nabla u(\xi) \gg, \quad u \in \mathcal{F}_{\text{loc}}C_b^\infty, \quad \xi \in \mathcal{H}_-.$$

In the next section, we will show that H_μ is a well-defined symmetric operator in $L^2(\mathcal{H}_-, d\mu)$ and that the relation

$$(2.45) \quad \mathcal{E}_\mu(u, v) = (u, H_\mu v)_{L^2}, \quad u, v \in \mathcal{F}_{\text{loc}}C_b^\infty,$$

holds. Since \mathcal{E}_μ is associated to the symmetric operator H_μ , \mathcal{E}_μ is closable. We call the operator H_μ in $L^2(\mathcal{H}_-, d\mu)$ with $D(H_\mu) = \mathcal{F}_{\text{loc}}C_b^\infty$ the *Dirichlet operator* associated to μ .

3. Dirichlet Forms and Associated Diffusion Processes

In this section we produce the proofs of Theorem 2.10 and Theorem 2.12. The methods of the proofs turn out to be similar to those used in [LPY]. In the rest of this section we assume that the properties in Assumption 2.1 hold and a Gibbs measure $\mu \in \mathcal{G}^\Phi(\Omega)$ is given. We begin with the following result:

LEMMA 3.1. *Let β be defined as in (2.42). Then $\|\beta(\xi)\|_-$ is finite for any $\xi \in \Omega_{\log}$ and $\|\beta\|_- \in L^2(\mathcal{H}_-, d\mu)$.*

Proof. Notice that

$$(3.1) \quad \|\beta(\xi)\|_-^2 = \sum_{i \in \mathbb{Z}^d} e^{-\sigma|i|} |\beta_i(\xi)|_-^2.$$

We recall that as an operator on $L^2([0, 1], \mathbb{R}^d; dt)$, A_ρ^{-1} is a bounded linear operator with a norm less than 1. We use this fact, the Schwarz inequality, and the conditions in Assumption 2.1 to conclude that

$$(3.2) \quad |\beta_i(\xi)|_-^2 \leq M \left(e^{2\alpha|\xi_i|_u} + |\xi_i|_u^2 + \sum_{\substack{j \in \mathbb{Z}^d \\ j \neq i}} \Psi(|i-j|) (|\xi_i|_u^2 + |\xi_j|_u^2) \right)$$

for some constant $M > 0$. It follows then from Assumption 2.1 (a) and the definition of Ω_{\log} that $\|\beta(\xi)\|_- < \infty$ for any $\xi \in \Omega_{\log}$. Since $\mu(\Omega_{\log}) = 1$, $|\beta|_- : \mathcal{H}_- \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$ is finite μ -a.e. Due to the regularity of μ (Definition 2.3 and Theorem 2.5) and Remark 2.6, we see that e.g.,

$$(3.3) \quad \int e^{2\alpha|\xi_i|_u} d\mu(\xi) \leq N \int_E e^{2\alpha|\zeta|_u} \mu^0(d\zeta).$$

where μ^0 is a Gaussian measure on E and N is a constant independent of ξ . Now we recall the Fernique’s theorem in the abstract Wiener space (see e.g., [Kuo], Chapter III, Theorem 3.1): there exists $a > 0$ such that

$$(3.4) \quad \int_E e^{a|\zeta|_u^2} \mu^0(d\zeta) < \infty.$$

Therefore, the right hand side of (3.3) is finite. By similar calculations we see that $\int |\beta_i(\xi)|_-^2 d\mu(\xi) < \infty$, and hence $\int \|\beta(\xi)\|_-^2 d\mu(\xi) < \infty$. This completes the proof of the lemma. ■

Define for $k \in \mathcal{H}_-$, $\tau_k : \mathcal{H}_- \rightarrow \mathcal{H}_-$ by $\tau_k(\xi) = \xi + k$, and let $\tau_k(\mu)$ be the image measure of μ under τ_k . μ is said to be *k-quasi-invariant* if $\tau_{sk}(\mu)$ is absolutely continuous w.r.t. μ for all $s \in \mathbb{R}$. In this case we set

$$(3.5) \quad a_{sk}(\xi) \equiv \frac{d\tau_{sk}(\mu)}{d\mu}(\xi), \quad \xi \in \mathcal{H}_-$$

The following are adapted from [AR1] and the similar notions appear in, e.g., [AK, KT, Kus]. Let us fix a $k \in \mathcal{H}_- \setminus \{0\}$ such that μ is *k-quasi-invariant*. We also fix $l \in \mathcal{H}_+$ such that $l(k) = \ll l, k \gg = 1$. We define a measure σ_k on $(\mathcal{H}_-, \mathcal{B}(\mathcal{H}_-))$ by

$$(3.6) \quad \sigma_k(A) = \int_{\mathbb{R}} \tau_{sk}(\mu)(A) ds, \quad A \in \mathcal{B}(\mathcal{H}_-).$$

σ_k is a σ -finite measure on $(\mathcal{H}_-, \mathcal{B}(\mathcal{H}_-))$ which is finite on compacts. Furthermore, we have that

$$(3.7) \quad \sigma_k(A + sk) = \sigma_k(A), \quad A \in \mathcal{B}(\mathcal{H}_-), s \in \mathbb{R}.$$

It is easy to prove the following result [AR1, Kus]:

PROPOSITION 3.2 ([AR1], PROPOSITION 4.2). μ and σ_k are absolutely continuous w.r.t. each other and for any $\mathcal{B}(\mathcal{H}_-)$ -measurable function $u : \mathcal{H}_- \rightarrow \mathbb{R}_+$ we have that

$$(3.8) \quad \begin{aligned} \int u(\xi) \mu(d\xi) &= \int_{\mathcal{H}_-} \int_{\mathbb{R}} u(\xi + sk) \frac{d\mu}{d\sigma_k}(\xi + sk) ds \mu(d\xi) \\ &= \int_{\mathcal{H}_1} \int_{\mathbb{R}} u(\zeta + sk) \frac{d\mu}{d\sigma_k}(\zeta + sk) ds \nu_k(d\zeta), \end{aligned}$$

where $\mathcal{H}_1 = p_k(\mathcal{H}_-)$ and $\nu_k = p_k(\mu)$, in which $p_k : \mathcal{H}_- \rightarrow \mathcal{H}_-$ is defined by $p_k(\xi) = \xi - l(\xi)k$, $\xi \in \mathcal{H}_-$.

Now, we introduce a family $V(sk)$, $s \in \mathbb{R}$, of operators on $L^2_{\mathbb{C}}(\mathcal{H}_-, d\mu)$ (i.e., the canonical complexification of $L^2(\mathcal{H}_-, d\mu)$) by

$$(3.9) \quad (V(sk)u)(\xi) \equiv \sqrt{a_{-sk}(\xi)}(u \circ \tau_{sk})(\xi), \quad \xi \in \mathcal{H}_-, u \in L^2_{\mathbb{C}}(\mathcal{H}_-, d\mu).$$

Note that since μ is k -quasi-invariant, the map $u \mapsto u \circ \tau_{sk}$ respects μ -classes of functions. Hence, $V(sk)$ is well defined by (3.9) as an operator on $L^2_{\mathbb{C}}(\mathcal{H}_-, d\mu)$ for each $s \in \mathbb{R}$. Clearly, the map $s \mapsto V(sk)$ is a strongly continuous unitary representation of the abelian group \mathbb{R} . By Stone’s theorem there exists a unique self-adjoint operator $\pi(k)$ with domain $D(\pi(k))$ such that

$$(3.10) \quad V(sk) = \exp(is\pi(k)), \quad s \in \mathbb{R}.$$

We introduce the “smoothness condition”

$$(3.11) \quad 1 \in D(\pi(k)).$$

If (3.11) holds we set

$$(3.12) \quad \beta_k \equiv 2i\pi(k)1.$$

β_k is called the *drift coefficient* or *osmotic velocity* [AH-K1, AR1], or the *logarithmic derivative of μ in the direction k* [AKR].

We say that a vector $k \in \mathcal{H}_-$ is *local* if there exists a $\Delta \in \mathcal{C}$ such that $\pi_i(k) = 0$ if $i \notin \Delta$, and by Ω_{loc} we denote the local vectors in Ω . Furthermore, by $H\text{-}\Omega_{\text{loc}}$ we denote the vectors in Ω_{loc} with components in the Hilbert space H_0 . The following Proposition plays a key role for showing the closability of the form (2.38) (cf. [AR1], Proposition 4.5).

PROPOSITION 3.3. *Let $k \in H\text{-}\Omega_{\text{loc}}$. Then, the following properties hold:*

- (i) μ is k -quasi-invariant and $1 \in D(\pi(k))$.
- (ii) $\mathcal{F}_{\text{loc}}C_b^1 \subset D(\pi(k))$ and for any $u \in \mathcal{F}_{\text{loc}}C_b^1$,

$$i\pi(k)u = \frac{\partial}{\partial k}u + \frac{1}{2}\beta_k u.$$

- (iii) If $(\frac{\partial}{\partial k})^*$ denotes the adjoint of $\frac{\partial}{\partial k}$, then $\mathcal{F}_{\text{loc}}C_b^1 \subset D((\frac{\partial}{\partial k})^*)$ and for every $u \in \mathcal{F}_{\text{loc}}C_b^1$,

$$\left(\frac{\partial}{\partial k}\right)^* u = -\frac{\partial}{\partial k}u - \beta_k u.$$

Proof. If (i) is true, then (ii) is obvious [AR1] and (iii) follows from (ii) and the fact that $(i\pi(k))^* = -i\pi(k)$. So, it remains only to show (i). Let k be such that for a $\Delta \in \mathcal{C}$, $\pi_i(k) = 0$ if $i \notin \Delta$. Recall the Gibbs condition in Definition 2.4 and the Hilbert space H_0 of (2.18), which is densely embedded in E . Using the Gibbs condition and Assumption 2.1 we have for any $A \in \mathcal{F}$ that

$$\begin{aligned}
 \tau_{sk}(\mu)(A) &= \mu(\tau_{-sk}(A)) \\
 &= \int_{\mathfrak{E}} \mu(d\xi) Z_{\Delta}(\xi)^{-1} \int \lambda(d\zeta_{\Delta}) \\
 &\quad \exp[-V(\zeta_{\Delta}) - W(\zeta_{\Delta}, \xi_{\Delta^c})] 1_{\tau_{-sk}(A)}(\zeta_{\Delta} \xi_{\Delta^c}) \\
 &= \int_{\mathfrak{E}} \mu(d\xi) Z_{\Delta}(\xi)^{-1} N^{|\Delta|} \int \mu^0(d\zeta_{\Delta}) \\
 (3.13) \quad &\quad \exp[-\tilde{V}(\zeta_{\Delta}) - W(\zeta_{\Delta}, \xi_{\Delta^c})] 1_A(\tau_{sk}(\zeta_{\Delta}) \xi_{\Delta^c}).
 \end{aligned}$$

Here, we have used (2.19) to obtain the third equality and \tilde{V} has been defined by

$$\tilde{V}(\zeta_{\Delta}) = V(\zeta_{\Delta}) - \frac{1}{2} \sum_{i \in \Delta} |\zeta_i|^2.$$

We use the the change of variables $\tau_{sk}(\zeta_{\Delta}) \rightarrow \zeta_{\Delta}$ in (3.13) and apply the translation of the Gaussian measures in abstract Wiener spaces (see e.g., [Kuo], Chapter II, Theorem 1.2). After this, we use once again the Gibbs condition to obtain the following result:

$$\begin{aligned}
 \tau_{sk}(\mu)(A) &= \int_A \mu(d\xi) \exp \left[- \sum_{i \in \Delta} [\tilde{P}(\xi_i - sk_i) - \tilde{P}(\xi_i)] \right. \\
 &\quad - \sum_{\substack{\{i,j\} \subset \Delta \\ i \neq j}} [U(\xi_i - sk_i, \xi_j - sk_j; |i - j|) - U(\xi_i, \xi_j; |i - j|)] \\
 &\quad \left. - \sum_{i \in \Delta} \sum_{j \in \Delta^c} [U(\xi_i - sk_i, \xi_j; |i - j|) - U(\xi_i, \xi_j; |i - j|)] \right. \\
 (3.14) \quad &\quad \left. + s \sum_{i \in \Delta} \langle \xi_i, k_i \rangle - \frac{1}{2} s^2 \sum_{i \in \Delta} |k_i|_0^2 \right],
 \end{aligned}$$

where $\tilde{P}(\xi_i) = P(\xi_i) - \frac{1}{2}|\xi_i|_-^2$ and the last two factors come from the translation of the Gaussian measures. Here, $\langle \cdot, k_i \rangle$ is understood as a random variable on E (and hence on \mathcal{H}_-) defined $\mu^{\otimes 3}$ -a.e. (see [Kuo], Chapter I, Lemma 4.7). In fact, if $k_i \in H_+$, then $\langle \xi_i, k_i \rangle$ is just the dual pairing between vectors of H_- and H_+ . Notice that when $A = \mathcal{H}_-$, the integration in (3.14) is equal to 1. Therefore, equation (3.14) says that for all $s \in \mathbb{R}$, $\tau_{s,k}(\mu)$ is absolutely continuous w.r.t. μ with a Radon-Nikodym derivative $a_{s,k}(\xi)$, the factor $\exp[\dots]$ in (3.14). From this we can calculate that

$$(3.15) \quad \frac{d}{ds} a_{-sk}(\xi) \Big|_{s=0} = - \sum_{i \in \Delta} \left[(\nabla^i \tilde{P}(\xi_i), k_i)_- + \langle \xi_i, k_i \rangle + \sum_{\substack{j \in \mathbb{Z}^d \\ j \neq i}} (\nabla^i U(\xi_i, \xi_j; |i-j|), k_i)_- \right],$$

where the function $\nabla^i \tilde{P}(\xi_i)$ for $\xi_i \in E$, as explained below (2.42), is defined to be an element of E given by $\nabla^i \tilde{P}(\xi_i)(t) = \nabla^i \tilde{P}(\xi_i(t))$, $t \in [0, 1]$, in which $\nabla^i \tilde{P}(x_i)$ is the gradient of the function $\tilde{P}(x_i)$. If $k \in \mathcal{H}_+ \cap \mathbb{H}\text{-}\Omega_{loc}$, then the right hand side of (3.15) is just the dual pairing $\ll \beta(\xi), k \gg$ (for $\xi \in \Omega_{log}$), where $\beta(\xi)$ was defined in (2.42). By Lemma 2.7 and Lemma 3.1, we see that $|\ll \beta(\xi), k \gg| \leq \|\beta(\xi)\|_- \|k\|_+ \in L^2(\mathcal{H}_-, d\mu)$. Therefore, the right hand side of (3.15) is the logarithmic derivative β_k . Under the weak condition that $k \in \mathbb{H}\text{-}\Omega_{loc}$, however, we should directly show that the right hand side of (3.15) belongs to $L^2(\mathcal{H}_-, d\mu)$. But, since the method is much more the same as in the proof of Lemma 3.1, we omit it. The proof of the proposition is now completed. ■

We now establish the relation (2.45). First we prove the following Lemma. Recall the definition of the Banach space Ω_- in (2.26).

LEMMA 3.4. *Let the hypotheses of Assumption 2.1 be satisfied and $\mu \in \mathcal{G}^\Phi(\Omega)$. Then, μ is supported in the Banach space $\Omega_- \subset \mathcal{H}_-$ (i.e., $\mu(U) > 0$ for any open $U \subset \Omega_-$) and $\mathcal{F}_{loc} C_b^\infty$ is dense in $L^p(\mathcal{H}_-, d\mu)$, $1 \leq p < \infty$.*

Proof. First, we will show that $\text{supp } \mu = \Omega_-$. Suppose that there exists an open set $U \subset \Omega_-$ such that $\mu(U) = 0$. Take a countable set

$Z \subset \mathbb{H}\text{-}\Omega_{\text{loc}}$ which is dense in Ω_- . Then

$$\Omega_- = \cup_{k \in Z} \tau_{-k}(U),$$

where $\tau_{-k}(U) = \{\xi - k : \xi \in U\}$. By Proposition 3.3 (i), μ is k -quasi-invariant for each $k \in Z$, i.e.,

$$\tau_k \ll \mu.$$

Therefore, by assumption, $\mu(\tau_{-k}(U)) = 0$ and hence $\mu(\Omega_-) = 0$. This is a contradiction because $\Omega_{\text{log}} \subset \Omega_-$ and $\mu(\Omega_{\text{log}}) = 1$. So, we conclude that $\text{supp } \mu = \Omega_-$.

Now let us prove the denseness of $\mathcal{F}_{\text{loc}}C_b^\infty$ in $L^p(\mathcal{H}_-, d\mu)$. We follow the ideas used in [MR, Sch]. First, we show that for any $l \in \mathcal{H}_+$, there exists a sequence $\{l_n\}_{n \in \mathbb{N}} \subset \mathcal{F}_{\text{loc}}C^\infty$ such that $\lim_{n \rightarrow \infty} l_n(\xi) = \langle \cdot, l, \xi \rangle, \xi \in \mathcal{H}_-$. For each $\Lambda \in \mathcal{C}$, define a projection operator $\hat{\pi}_\Lambda : \mathcal{H}_- \rightarrow \mathcal{H}_-$ given by

$$(\hat{\pi}_\Lambda(\xi))_i = \begin{cases} \xi_i, & \text{if } i \in \Lambda \\ 0, & \text{if } i \in \Lambda^c. \end{cases}$$

Recall the finite rank projections \mathcal{P}_{fin} introduced below Remark 2.9. For each $P \in \mathcal{P}_{\text{fin}}$ define $\hat{P} : \mathcal{H}_- \rightarrow \mathcal{H}_-$ by

$$(3.16) \quad (\hat{P}\xi)_i = P\xi_i, \quad \xi = (\xi_i)_{i \in \mathbb{Z}^d} \in \mathcal{H}_-.$$

We notice that for each $k \in H_+, \zeta \in H_-$, and $P \in \mathcal{P}_{\text{fin}}$,

$$(3.17) \quad |\langle k, P\zeta \rangle| \leq |k|_+ |\zeta|_-$$

and

$$(3.18) \quad \langle k, P_n \zeta \rangle \rightarrow \langle k, \zeta \rangle$$

if $P_n \rightarrow \mathbf{1}$ strongly in H_0 . From (3.16) – (3.17) we see that for each $l \in \mathcal{H}_+, \xi \in \mathcal{H}_-$, and $P \in \mathcal{P}_{\text{fin}}$,

$$(3.19) \quad |\langle \cdot, \hat{P}\xi \rangle| \leq \|l\|_+ \|\xi\|_-.$$

Now let us choose $\{\Lambda_n\}$, $\Lambda_n \in \mathcal{C}$, and $\{P_n\}$, $P_n \in \mathcal{P}_{\text{fin}}$ such that $\Lambda_n \uparrow \mathbb{Z}^p$ and $P_n \rightarrow \mathbf{1}$ strongly in H_0 and define $l_n(\xi) \equiv \ll l, \widehat{P}_n \circ \widehat{\pi}_{\Lambda_n}(\xi) \gg$. Then, clearly, $l_n \in \mathcal{F}_{\text{loc}}C^\infty$, and using (3.16) – (3.19) and a dominated convergence theorem for a series, we see that

$$(3.20) \quad \lim_{n \rightarrow \infty} l_n(\xi) \equiv \ll l, \xi \gg, \quad l \in \mathcal{H}_+, \quad \xi \in \mathcal{H}_-$$

We now show that there exists a countable sequence in $\mathcal{F}_{\text{loc}}C_b^\infty$ which separates points in \mathcal{H}_- . Let $\{\xi_j\}_{j \in \mathbb{N}}$ be a countable dense set in \mathcal{H}_- , and for each $j \in \mathbb{N}$, by the Hahn-Banach theorem we can choose $l_j \in \mathcal{H}_+$ with $\|l_j\|_+ = 1$ and $l_j(\xi_j) = \|\xi_j\|_-$. It follows that $\|\xi\|_- = \sup_j l_j(\xi)$ for every $\xi \in \mathcal{H}_-$. Now from above we see that for each $j \in \mathbb{N}$, there exist $\{l_j^m\}_{m \in \mathbb{N}} \subset \mathcal{F}_{\text{loc}}C^\infty$ so that $l_j^m \rightarrow l_j$ (weakly) as $m \rightarrow \infty$. Let $f_{n,m,j}(\xi) = n \sin(l_j^m(\xi)/n)$, so that $f_{n,m,j} \in \mathcal{F}_{\text{loc}}C_b^\infty$. Since $\lim_{n \rightarrow \infty} n \sin(t/n) = t$ for each $t \in \mathbb{R}$, it is not hard to see that the collection $\{f_{n,m,j}\}_{n,m,j \in \mathbb{N}}$ separates points in \mathcal{H}_- .

Since \mathcal{H}_- is a Polish space, a countable collection of measurable functions that separates points must generate the entire Borel σ -algebra ([Co], Corollary 8.6.8). Therefore, the σ -algebra generated by $\{f_{n,m,j}\}_{n,m,j \in \mathbb{N}}$, and hence the σ -algebra generated by $\mathcal{F}_{\text{loc}}C_b^\infty$ is $\mathcal{B}(\mathcal{H}_-)$. Now define

$$\mathcal{B} = \left\{ u \in \mathcal{B}_b(\mathcal{H}_-) : \text{There exists } \{u_n\}_{n \in \mathbb{N}} \in \mathcal{F}_{\text{loc}}C_b^\infty \right. \\ \left. \text{so } \int |u_n - u|^p d\mu \rightarrow 0 \text{ as } n \rightarrow \infty \right\}.$$

Then, from a Monotone Class Theorem ([Sh], A0.6), we see that $\mathcal{B} = \mathcal{B}_b(\mathcal{H}_-)$. Since $\mathcal{B}_b(\mathcal{H}_-)$ is dense in $L^p(\mathcal{H}_-, d\mu)$, we conclude that $\mathcal{F}_{\text{loc}}C_b^\infty$ is dense in $L^p(\mathcal{H}_-, d\mu)$, $1 \leq p < \infty$. ■

We now have the following result:

PROPOSITION 3.5. *The Dirichlet operator*

$$H_\mu u(\xi) = -\Delta u(\xi) - \ll \beta(\xi), \nabla u(\xi) \gg, \quad u \in C_b^2,$$

is a positive-definite symmetric operator in $L^2(\mathcal{H}_-, d\mu)$. Furthermore, H_μ is associated with the form $\mathcal{E}_\mu(u, v)$ of (2.38) by the equality $\mathcal{E}_\mu(u, v) = (u, H_\mu v)_{L^2}$.

Before proving the proposition we give some comments. Since the embeddings $\mathcal{H}_+ \subset \mathcal{H}_0$ and $\mathcal{H}_0 \subset \mathcal{H}_-$ are Hilbert Schmidt operators,

$\Delta u = \text{Tr}_{\mathcal{H}_0} u''$ is well-defined for any $u \in C_b^2$. For $u \in C_b^2$, the function $\ll \beta(\cdot), \nabla u(\cdot) \gg$ is finite μ -a.e. and belongs to $L^2(\mathcal{H}_-, d\mu)$ by Lemma 3.1, and so H_μ is well-defined on C_b^2 .

Proof of Proposition 3.5. The proposition follows from the definition of \mathcal{E}_μ in (2.38) and Proposition 3.3. We notice that for any orthonormal basis $\{e_n\}_{n \in \mathbb{N}} \subset H_0$ with $e_n \in H_+$, $n \in \mathbb{N}$, the vectors $e_n^{\{i\}}$ belong to $H\text{-}\Omega_{\text{loc}}$. Therefore, from (2.38) and Proposition 3.3, it follows that

$$\begin{aligned} \mathcal{E}_\mu(u, v) &= \sum_{i \in \mathbb{Z}^\nu} \sum_{n=1}^\infty (\nabla_n^i u, \nabla_n^i v)_{L^2} \\ &= \sum_{i \in \mathbb{Z}^\nu} \sum_{n=1}^\infty (u, (-\frac{\partial}{\partial e_n^{\{i\}}} - \beta_{e_n^{\{i\}}}) \nabla_n^i v)_{L^2} \\ &= (u, -\Delta v)_{L^2} + (u, -\ll \beta, \nabla v \gg)_{L^2} \\ &= (u, H_\mu v)_{L^2}. \end{aligned}$$

See also the proof of Theorem 1 of [KT] for the proof with arbitrary orthonormal basis. ■

We now turn to the proof of Theorem 2.10.

Proof of Theorem 2.10. Since $\mathcal{F}_{\text{loc}} C_b^\infty \subset C_b^2$ and $(\mathcal{E}_\mu, \mathcal{F}_{\text{loc}} C_b^\infty)$ is associated to the symmetric operator H_μ with domain $\mathcal{F}_{\text{loc}} C_b^\infty$, $(\mathcal{E}_\mu, \mathcal{F}_{\text{loc}} C_b^\infty)$ is closable by Proposition 3.5. The Markov property of $(\mathcal{E}_\mu, \mathcal{F}_{\text{loc}} C_b^\infty)$ follows from a well-known method [AR1, Rö]. See also the proofs of Proposition 4.5, Chapter II, of [Rö], Corollary 3.6 of [AR1], and Lemma 5.5 of [Sch]. ■

We next consider a diffusion process associated with the Dirichlet form for a given Gibbs measure μ . We prove Theorem 2.12. In [AR2], Albeverio and Röckner gave sufficient conditions for the construction of the associated diffusion process. We will check that all the conditions are satisfied in our case. Let $(\tilde{\mathcal{E}}_\mu, D(\tilde{\mathcal{E}}_\mu))$ be the closure of $(\mathcal{E}_\mu, \mathcal{F}_{\text{loc}} C_b^\infty)$, and let $\tilde{\mathcal{E}}_{\mu,1}(u, v) = \tilde{\mathcal{E}}_\mu(u, v) + \int uv d\mu$, $u, v \in D(\tilde{\mathcal{E}}_\mu)$. For any open $U \subset \mathcal{H}_-$, define the capacity [Fu] of U by

$$\text{Cap}(U) \equiv \inf \{ \tilde{\mathcal{E}}_{\mu,1}(u, u) : u \in D(\tilde{\mathcal{E}}_\mu), u \geq 1 \text{ on } U \text{ } \mu\text{-a.e.} \},$$

and for any $A \subset \mathcal{H}_-$

$$\text{Cap}(A) \equiv \inf\{\text{Cap}(U) : A \subset U, U \text{ open}\}.$$

Let us now consider the following conditions introduced in [AR2]:

(i) There exist $K_n \subset \mathcal{H}_-$, $n \in \mathbb{N}$, K_n compact, such that $\lim_{n \rightarrow \infty} \text{Cap}(\mathcal{H}_- \setminus K_n) = 0$.

(ii) There exists a countable set D of bounded continuous functions on \mathcal{H}_- separating the points of \mathcal{H}_- which is dense in $D(\tilde{\mathcal{E}}_\mu)$ with respect to $\tilde{\mathcal{E}}_{\mu,1}$.

(iii) $\tilde{\mathcal{E}}_\mu(u, v) = 0$ if $u, v \in D(\tilde{\mathcal{E}}_\mu)$, continuous, such that $\text{supp } u \cap \text{supp } v = \emptyset$.

(iv) There exist $f_n : \mathcal{H}_- \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, generating the topology of \mathcal{H}_- .

The following is a main result of [AR2]:

THEOREM 3.6 ([AR2], THEOREM 2.7). *Assume that the conditions (i) – (iv) listed above are satisfied. Then, there exists a diffusion process with state space \mathcal{H}_- associated with $(\tilde{\mathcal{E}}_\mu, D(\tilde{\mathcal{E}}_\mu))$.*

Proof of Theorem 2.12. By Theorem 3.6, we only need to check the conditions (i) – (iv) listed above. The condition (ii) is satisfied by Proposition 2.6 of [AR2]. (iii) is obvious and (iv) is satisfied since \mathcal{H}_- is a complete metric space. Thus, it only remains to check the condition (i). Notice that by Lemma 3.4 and Remark 2.9 μ is supported on the Banach space \mathcal{H}_- and \mathcal{H}_- is separable. Hence by Proposition 3.2, Chapter IV, of [Rö], Cap is tight, i.e., there exist compact sets $K_n \subset \mathcal{H}_-$, $n \in \mathbb{N}$, such that $\lim_{n \rightarrow \infty} \text{Cap}(\mathcal{H}_- \setminus K_n) = 0$. This completes the proof. ■

REMARK 3.7. (a) As in the proof of Theorem 2.8 of [LPY], we can also directly construct a sequence of compact sets which satisfies the condition (i). The method is to modify and extend the construction argument in [Kus]. In the way of doing it, we make great use of the regularity of Gibbs measures. However, it is somewhat lengthy and so we decided to follow the contents in [Rö]. Notice that the regularity of Gibbs measures was used in the proof of being supported of the measures in Lemma 3.4.

(b) In their book [MR], Ma and Röckner proved the fundamental existence theorem for Markov processes associated with Dirichlet forms. The condition is the quasi-regularity of the Dirichlet form [MR, Rö], which is similar to the conditions introduced before Theorem 3.6. The closable gradient Dirichlet forms such as in this paper are always quasi-regular [Rö, Sch].

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