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ON A MARTINGALE PROBLEM AND A RELAXED CONTROL PROBLEM W.R.T. SDE

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0. Introduction

Let $\mathcal{S}(R^d)$ be the Schwartz space of infinitely differentiable functions on R^d which vanish at ∞ and $\mathcal{S}'(R^d)$ be its dual space. The theory of stochastic differential equations(SDEs) governing processes that takes values in the dual of countably Hilbertian nuclear space such as $\mathcal{S}'(R^d)$ studied by many authors(e.g [M],[KX]). Let M be a martingale measure defined by Walsh[W], then M can be considered as a $\mathcal{S}'(R^d)$-valued process in a certain condition i.e. M has a version of $\mathcal{S}'(R^d)-$valued martingale process.(See [W] for detailed discussion)

In the previous paper[Ch1], we considered the weak convergence of sequences of stochastic integrals with respect to martingale measures: Let $X_n \in D_{C_0(R^d)}[0,\infty)$, M_n be a sequence of martingale measures and let $\mathcal{S}'(R^d)$ be the dual of Schwartz space. If $(X_n, M_n) \Rightarrow (X, M)$ in the Skorohod topology on $D_{C_0(R^d)\times \mathcal{S}'(R^d)}[0,\infty)$, we proved that it implies $(X_n, M_n, \int X_n(x,s)M_n(dx,ds)) \Rightarrow (X, M, \int X(x,s)M(dx,ds))$ on $D_{C_0(R^d)\times \mathcal{S}'(R^d)\times \mathcal{S}'(R^d)}[0,\infty)$. This generalize Theorem 2.2 in Kurtz and Protter[KP] which shows the weak convergence of stochastic integrals driven by semimartingales.

As an extension of the above result, we also proved weak limit theorems for SDEs driven by martingale measures([Ch2]).

In section 1, we begin by giving the definitions of martingale measures as well as our previous theorems for weak convergence of stochastic integrals driven by martingale measures.

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In section 2, we shall show the equivalence of existence of $\mathcal{S}'(R^d)-$ valued solution process between a martingale problem and its corresponding SDE with respect to martingale measures.

In section 3, we shall consider a relaxed control problem in connection with the theory of martingale measure.

1. Preliminaries

We start with the definition of martingale measures established by Walsh[8].

DEFINITION 1.1. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a filtered space, and $\mathcal{B}(R^d)$ be the Borel $\sigma-$field.

Let $M(\cdot,\cdot)$ be a random real-valued function on $R^d \times R_+$. M is called an $(\mathcal{F}_t, P)-$martingale measure if it satisfies the following properties.

(1) For each $A \in \mathcal{B}(R^d)$, $M(A,\cdot)$ is a $(\mathcal{F}_t, P)$-square integrable martingale and $M(A,0) = 0$.
(2) For any $A, B \in \mathcal{B}(R^d)$ s.t. $A \cap B = \emptyset$, and $M(A \cup B, t) = M(A,t) + M(B,t), P$ a.s. for every $t > 0$.
(3) For every $t > 0$, $M(\cdot,t)$ is a $\sigma-$finite L^2-valued measure in the following sesnse: there exists a non-decreasing sequence $\{E_n\}$ of Borel subsets of R^d with $\cup E_n = R^d$ such that
for every $t > 0$, $\sup_{A\in\mathcal{E}_n} E[M(A,t)^2] < \infty$, $\quad \mathcal{E}_n = \mathcal{B}(E_n)$,
for every $t > 0$, $E[M(A_j,t)^2] \to 0$ for all sequence A_j of $\mathcal{E}_n$ decreasing to $\emptyset$. □

For $A, B \in \mathcal{B}(R^d)$, there exists a unique predictable process, $\langle M(A), M(B)\rangle_t$ such tht $M(A,t)M(B,t) - \langle M(A), M(B)\rangle_t$ is a martingale.

A martingale measure, M is orthogonal if $M(A,t)M(B,t)$ is a martingale for $A, B \in \mathcal{B}(R^d), A \cap B = \emptyset$.

If M is an orthogonal martingale measure, one can prove the existence of random positive σ-finite measure $\pi(dx, ds)$ on $R^d \times R^+, \mathcal{F}_t-$ predictable such that for all $A \in \mathcal{B}(R^d), t \in [0,\infty)$, $\pi(A \times (0,t]) = \langle M(A,t), M(A,t)\rangle$. The measure π is called the covariance measure of M. More generally,

$$\langle M(A,t), M(B,t)\rangle = \pi(A \cap B \times (0,t]).$$

We refer more definitions and a construction of stochastic integral with respect to martingale measures to Walsh[W].

Let $\mathcal{S}(R^d)$ be the Schwartz space of infinitly differentiable and rapidly decreasing fuctions at infinity and $\mathcal{S}'(R^d)$ be its dual space. It is necessary to note that $\mathcal{S}(R^d)$ and $\mathcal{S}'(R^d)$ have the following properties as nuclear spaces.

1) The topology of $\mathcal{S}(R^d)$ is determined by a sequence of Hilbertian seminorms, $\|\cdot\|_n, n = 1, 2, \ldots$, which satisfy

$$\|\cdot\|_0 \leq \|\cdot\|_1 \leq \|\cdot\|_2 \leq \cdots.$$

2) If we let $\mathcal{S}_q(R^d)$ denote the completion of $\mathcal{S}(R^d)$ w.r.t. $\|\cdot\|_q$ and $\mathcal{S}'_q(R^d)$ be the dual space of $\mathcal{S}_q(R^d)$ with the dual norm, then

$$\mathcal{S}(R^d) \subset \mathcal{S}_q(R^d) \subset \mathcal{S}_p(R^d) \subset \mathcal{S}_0(R^d) \subset \mathcal{S}'_p(R^d) \subset \mathcal{S}'_q(R^d) \subset \mathcal{S}'(R^d),$$

where $0 < p < q$.

Let $D_{\mathcal{S}'(R^d)}[0,\infty)$ be the space of all mappings of $[0,\infty)$ to $\mathcal{S}'(R^d)$ that are cadlag(right continuous and have left limits) in the strong topology of $\mathcal{S}'(R^d)$ and $D_{\mathcal{S}'_q(R^d)}[0,\infty)$ be the complete separable metric space with the Skorohod topology of all mapping of $[0,\infty)$ to $\mathcal{S}'_q(R^d)$ that are cadlag in the $\|\cdot\|_{-q}$- topology. Since $\mathcal{S}'(R^d)$ is not a metrizable space, we need to find a condition that for any $T > 0$, there exists $\mathcal{S}'_q(R^d)$ such that for $0 < t < T, M^n_t$ has a regular version in $\mathcal{S}'_q(R^d)$ for all n.

Under the following condition, we proved a weak limit theorem for stochastic integrals driven by orthogonal martingale measures[Ch1].

Let $\{M_n\}$ be a sequence of orthogonal martingale measures.

CONDITION 1.1. For each $T > 0$, there exists a $p_0 \geq 0$ such that if M_n, $n = 1, 2\ldots$ are orthogonal and $\pi_n(A,t) = \langle M_n(A,t)\rangle$,

$$\sup_n E\Big[\int_0^T \int_{R^d} \frac{1}{1+|x|^{p_0}} \pi_n(dx, ds)\Big] < \infty$$

THEOREM 1.1[CH1]. *Let M_n $n = 1, 2, \dots$ be orthogonal martingale measures satisfying Condition 1.1. Let (X_n, M_n) be $\{\mathcal{F}_t^n\}$−adapted process with sample paths in $D_{C_0(R^d)\times\mathcal{S}'(R^d)}[0,\infty)$ such that*

$$(X_n, M_n) \Rightarrow (X, M) \quad \text{in the Skorohod topology on } D_{C_0(R^d)\times\mathcal{S}'(R^d)}[0,\infty)$$

and for a Borel set $B \subset R^d$, let

$$Z_n(t,B) \equiv \int_{B\times[0,t]} X_n(x, s-)M_n(dx, ds)$$
$$Z(t,B) \equiv \int_{B\times[0,t]} X(x, s-)M(dx, ds).$$

Then $Z^n \Rightarrow Z$ on $D_{\mathcal{S}'(R^d)}[0,\infty)$. In fact,

$$(X_n, M_n, Z_n) \Rightarrow (X, M, Z)$$

in the Skorohod topology on $D_{C_0(R^d)\times\mathcal{S}'(R^d)\times\mathcal{S}'(R^d)}[0,\infty)$. If $(X_n, M_n) \to (X, M)$ in probability, then the triple converges in probability.

Furthermore, using the above theorem, we proved a weak limit theorem for SDE with respect to martingale measures.

CONDITION 1.2. Let $\pi_n(t, A) = \langle M_n(t, A)\rangle$. For every $\phi \in \mathcal{S}(R^d)$, for any $T, \delta > 0$,

$$\sup_{t\le T}\int_t^{t+\delta}\int_{R^d}\phi(x)\pi_n(dx, ds) \Rightarrow \sup_{t\le T}\int_t^{t+\delta}\int_{R^d}\phi(x)\pi(dx, ds).$$

and the sequence $\{\sup_{t\le T}\int_t^{t+\delta}\int_{R^d}\phi(x)\pi_n(dx, ds)\}$ is uniformly integrable.

THEOREM 1.2[CH2]. *Let M_n be orthogonal martingale measures and let*

$F_n : R^d \times D_{\mathcal{S}'(R^d)}[0,\infty) \to D_{C_0(R^d)}[0,\infty)$. Suppose that (Y_n, M_n) satisfies

$$Y_n(t) = Y_n(0) + \int_0^t\int_{R^d} F_n(x, Y_n, s)M_n(dx, ds),$$

$(Y_n(0), M_n) \Rightarrow (Y(0), M)$ and $\{M_n\}$ satisfies Condition 1.1 and 1.2. Also assume that

$$\sup_n \sup_{0 \le s \le T} \|F_n(\cdot,\cdot,s)\|_\infty < \infty.$$

Then $\{(Y_n(0), Y_n, M_n)\}$ is relatively compact and any limit point $(Y(0), Y, M)$ satisfies

$$Y(t) = Y(0) + \int_0^t \int_{R^d} F(x, Y, s-) M(dx, ds)$$

2. A SDE and the corresponding martingale problem

Let $a, b : R^d \times \mathcal{S}'(R^d) \to C_0(R^d)$, $\pi(t, A)$ be a positive measure on $\mathcal{B}[0,\infty) \times \mathcal{B}(R^d)$ and M be a orthogonal martingale measure process satisfying $\langle M(A,t), M(A,t)\rangle = \pi(t, A)$. For $\mathcal{S}'(R^d)-$valued process Y, we denote $Y(t,\phi)$ as the canonical paring of elements $Y(t) \in \mathcal{S}'(R^d)$ and $\phi \in \mathcal{S}(R^d)$.

Martingale problem

Let P be a probability measure and let Y be a process on $(D_{\mathcal{S}'(R^d)}[0,\infty), P)$ with initial distrbution $\mathcal{L}(Y(0))$ satisfying: For every $\phi \in (R^d)$

$$(2.1) \quad N(t,\phi) \equiv Y(t,\phi) - Y(t,0) - \int_0^t \int_{R^d} a(x, Y(s))\phi(x)\pi(dx, ds)$$

is a martingale. The $\langle\,\rangle$ process of $N(t,\phi)$ is

$$(2.2) \qquad V(t,\phi) = \int_0^t \int_{R^d} (b(x, Y(s))\phi(x))^2 \pi(dx, ds).$$

The Stochastic Integral Equation

(2.3)

$$Y(t) = Y(0) + \int_0^t \int_{R^d} a(x, Y(s))\pi(dx, ds) + \int_0^t \int_{R^d} b(x, Y(s)) M(dx, ds)$$

where the equality is in the sense of [KX].

A progressively measurable process Y (with initial distribution $\mathcal{L}(Y(0))$) is said to be a solution to the martingale problem if (2.1) and (2.2) hold with respect to the measure P and the filtration $\mathcal{F}_t = \sigma\{Y(s); s \le t\}$.

Our formulation of the martingale problem is a little different from that of Watkins[Wt]. He considers a Banach space-valued orthogonal martingale measure process with continuous trajectory. He shows the equivalence of existence of solution between a martingale problem and a stochastic integral equation in Theorem 3.1[Wt].

By a similar argument with his proof and the proof of Th.3.3(p. 293 [EK]), we can prove the following proposition.

PROPOSITION 2.1. *The stochastic integral equation(2.3) has a solution if and only if the martingale problem has a solution(2.2).*

Proof. It is enough to show the converse. Assume that we have a solution to the martingale problem i.e. for each $\phi \in \mathcal{S}(R^d)$

$$N(t,\phi) = Y(t,\phi) - Y(0,\phi) - \int_0^t \int_{R^d} a(x, Y(s))\phi(x)\pi(dx, ds)$$

is a martingale and

$$\langle N(t,\phi)\rangle = V(t,\phi) = \int_0^t \int_{R^d} (b(x, Y(s)\phi(x))^2 \pi(dx, ds).$$

If b in equation (2.2) is never zero, then $N(t,\phi), V(t,\phi)$ provide us with enough randomness to recover M in (2.3). We can define Z by

$$Z(t,A) = \int_0^t \int_A \frac{1}{b(x, Y(s))} N(dx, ds).$$

for $A \in \mathcal{B}(R^d)$. Then $\langle Z(t,A)\rangle = \int_0^t \int_A \frac{1}{b^2(x,Y(s))} b^2(x, Y(s))\pi(dx, ds) = \pi(t, A)$ and $Z(t, A_1)Z(t, A_2)$ is a martingale if $A_1 \cap A_2 = \emptyset$. Since π is sufficiently restricted to determine the distribution, we have $\mathcal{L}(M) = \mathcal{L}(Z)$. And

$$\int_0^t \int_{R^d} b(x, Y(s))\phi(x)Z(dx, ds) = \int_0^t \int_{R^d} \phi(x)N(dx, ds)$$

$$= N(t,\phi) = Y(t,\phi) - Y(0,\phi) - \int_0^t \int_{R^d} a(x, Y(s))\phi(x)\pi(dx, ds)$$

and the stochastic integral equation holds.

If b is sometimes zero, the processes N and V may lack the randomness necessary to recover M. To this end, let M' be a martingale measure on a space $(\Omega', \mathcal{F}', P')$ with respect to the filtration $\mathcal{F}'_t$. M' has covariance measure π.

Define $Y'(t,\omega,\omega') = Y(t,\omega)$ and $M'(A,t,\omega,\omega') = M(A,t,\omega')$. On the augmented probability space $(\Omega\times\Omega', \mathcal{F}\times\mathcal{F}', P\times P')$ with filtration $\mathcal{F}_t \times \mathcal{F}'_t$,

$$N'(t,A) = Y'(t,A) - Y'(0,A) - \int_0^t \int_A a(x, Y'(s))\pi(dx, ds)$$

is a martingale measure. The covariance measure of $N'(t,A)$ is

$$V'(t,A) = \int_0^t \int_A b^2(x, Y'(s))\pi(dx, ds).$$

Define $\eta' : R^d \times S'(R^d) \to C_0(R^d)$ in the following way; If $b(x,y) = 0$, set $\eta'(x,y) \equiv 0$. If $b(x,y) \neq 0$ $\eta'(x,y) \equiv \frac{1}{b(x,y)}$.

Let $\rho(x,y) = I_{\{b(x,y)=0\}}$. As before, we can check that

$$Z'(t,A) = \int_0^t \int_A \eta'(x, Y'(s))N'(dx, ds) + \int_0^t \int_A \rho(x, Y(s))M'(dx, ds),$$

is a martingale measure. It is easy to see that

$$\langle Z'(t,A), Z'(t,B)\rangle = \pi(t, A\cap B) \text{ and } \langle Z'(t,A), Z'(t,A)\rangle = \pi(t,A)$$

and $Z'(t,A)Z'(t,B)$ is a martingale if $A\cap B = \emptyset$. Again, we have recovered M satisfying $\mathcal{L}(M)=\mathcal{L}(Z')$.

$$\begin{aligned}
\int_0^t\int_{R^d} & b(x,Y'(s))\phi(x)Z'(dx,ds)\\
&=\int_0^t\int_{R^d} I_{\{b(x,Y'(s))\neq 0\}}b(x,Y'(s))\phi(x)Z'(dx,ds)\\
&=\int_0^t\int_{R^d} I_{\{b(x,Y'(s)\phi(x)\neq 0\}}b(x,Y'(s))\eta'(x,Y'(s))\phi(x)N'(dx,ds)\\
&\quad+\int_0^t\int_{R^d} I_{\{b(x,Y'(s))\neq 0\}}\rho(x,Y'(s))\phi(x)M'(dx,ds)\\
&=\int_0^t\int_{R^d} I_{\{b(x,Y'(s))\neq 0\}}\phi(x)N'(dx,ds)+0\\
&=N'(t,\phi(x))-\int_0^t\int_{R^d} I_{\{b(x,Y'(s))=0\}}\phi(x)N'(dx,ds)
\end{aligned}$$

$\int_0^t\int_{R^d} I_{\{b(x,Y'(s))=0\}}\phi(x)N'(dx,ds)=0$ because the covariance measure for this martingale is

$$\int_0^t\int_{R^d} I_{\{b^2(x,Y'(s))=0\}}\phi^2(x)b^2(x,Y'(s))\pi(dx,ds)=0.$$

Therefore

$$\begin{aligned}
\int_0^t\int_{R^d} b(x,Y'(s))\phi(x)Z'(dx,ds) &= \int_0^t\int_{R^d}\phi(x)N'(dx,ds)\\
&=N'(t,\phi)\\
&=Y'(t,\phi)-Y'(0,\phi)-\int_0^t\int_{R^d} a(x,Y'(s))\phi(x)\pi(dx,ds)
\end{aligned}$$

i.e. the stochastic integral equation holds. □

The following is a main theorem in this paper.

THEOREM 2.2. *Let $a,b:R^d\times\mathcal{S}'(R^d)\to C_0(R^d)$ be continuous and bounded, and let Y_n be a solution to the martingale problem (2.1)(2.2) w.r.t. M_n and π_n defined as before and $M_n\Rightarrow M$ in $D_{\mathcal{S}'(R^d)}[0,\infty)$. Suppose that M_n satisfies Condition 1.1 and 1.2, and $Y_n(0)\Rightarrow Y(0)$.*

Then (M_n, Y_n) *is relatively compact in* $D_{S'(R^d)\times S'(R^d)}[0,\infty)$, *and a limit point* (M,Y) *satisfies*
(2.4)

$$Y(t) = Y(0)+\int_0^t\int_{R^d} a(x,Y(s))\pi(dx,ds)+\int_0^t\int_{R^d} b(x,Y(s))M(dx,ds).$$

And hence, $Y(t)$ *is a solution to martingale problem corresponding to* (2.4).

Proof. By Proposition 2.1, Y_n is a solution to the corresponding stochastic integral equation;

$$\begin{aligned} Y_n(t,\phi) = Y_n(0,\phi) &+ \int_0^t\int_{R^d} a(x,Y_n(s))\phi(x)\pi_n(dx,ds) \\ &+ \int_0^t b(x,Y_n(s))\phi(x)M_n(dx,ds) \end{aligned}$$

For the relative compactness of $\{Y_n\}$, it is enough to show that for each $\phi\in\mathcal{S}(R^d)$, $\{Y_n(t,\phi)\}$ is relatively compact by Theorem 4.1 in [M]. For convenience, let for each $\phi\in\mathcal{S}(R^d)$,

$$\begin{aligned} V_n(t,\phi) &= \int_0^t\int_{R^d} b(x,Y_n(s))\phi(x)M_n(dx,ds) \\ U_n(t,\phi) &= \int_0^t\int_{R^d} a(x,Y_n(s))\phi(x)\pi_n(dx,ds) \end{aligned}$$

Then

$$Y_n(t,\phi) = Y_n(0,\phi) + U_n(t,\phi) + V_n(t,\phi). \tag{2.5}$$

Let

$$z_n(t,\phi) = \int_0^t\int_{R^d} \phi(x)\pi_n(dx,ds) \quad z(t,\phi) = \int_0^t\int_{R^d}\phi(x)\pi(dx,ds).$$

Condition 1.2 implies that $z_n(\cdot,\phi)\Rightarrow z(\cdot,\phi)$ uniformly on $[0,T]$, for any $T>0$. Hence we can have

$$(z_n(\cdot,\phi)), M_n(\cdot,\phi)) \Rightarrow (z(\cdot,\phi), M(\cdot,\phi)) \text{ in } D_{R^2}[0,\infty)$$

As in the proof of Theorem 1.1[Ch1], we can show that (U_n, V_n) are relatively compact in $D_R[0,\infty) \times D_R[0,\infty)$. Observe that at any discontinuities of U_n, V_n,

$$|U_n(t,\phi) - U_n(t-,\phi)| \le \|a\|_\infty |z_n(t,\phi) - z_n(t-,\phi)|$$

$$\begin{aligned}|V_n(t,\phi) - V_n(t-,\phi)| &\le \|b\|_\infty \Big|\int_{t-}^{t}\int_{R^d} \phi(x) M_n(dx,ds)\Big| \\ &\le \|b\|_\infty |M_n(t,\phi) - M_n(t-,\phi)|\end{aligned}$$

Hence, in (2.5), by the following well known Lemma 2.3,

$$(U_n(\cdot,\phi), V_n(\cdot,\phi)) \Rightarrow (U(\cdot,\phi), V(\cdot,\phi)) \text{ in } D_{R^2}[0,T]$$

Therefore, $(M_n(\cdot,\phi), Y_n(0,\phi), U_n(\cdot,\phi), Y_n(\cdot,\phi))$ is relatively compact in $D_{R^4}[0,T]$. Let Y be any limit point of $\{Y_n\}$. Note that

$$(M_n(\cdot,\phi), z_n(\cdot,\phi)) \Rightarrow (M(\cdot,\phi), z(\cdot,\phi))$$

and $(M_n, Y_n) \Rightarrow (M, Y)$ implies that

$$(2.6) \qquad (Y_n, z_n(\cdot,\phi)) \Rightarrow (Y, z(\cdot,\phi)) \text{ in } D_{\mathcal{S}'(R^d)\times R}[0,T].$$

For each ϕ, define

$$\nu_n^\phi(dx,ds) = \phi(x)\pi_n(dx,ds) \qquad \nu^\phi(dx,ds) = \phi(x)\pi(dx,ds)$$

Let $l(R^d)$ be the space of measures on $[0,\infty)\times R^d$ such that $\nu([0,t]\times R^d) < \infty$ for each $t \ge 0$. Since Condition 1.2 implies that $\sup_n E[|\int_0^t \int_{R^d} \phi(x)\pi_n(dx,ds)|] < \infty$, we can consider $\nu_n^\phi, \nu^\phi \in l(R^d)$ a.s.. Applying Lemma 1.5[K1], we have by (2.6)

$$(Y_n, z_n(\cdot,\phi), U_n(\cdot,\phi)) \Rightarrow (Y, z(\cdot,\phi), U(\cdot,\phi))$$

in $D_{\mathcal{S}'(R^d)\times R\times R}[0,T]$. And by Theorem 1.1

$$\int_0^\cdot \int_{R^d} b(x, Y_n(s)) M_n(dx,ds) \Rightarrow \int_0^\cdot \int_{R^d} b(x, Y(s)) M(dx,ds).$$

Hence, $(M, Y(0), U, Y)$ satisfies the stochastic integral equation (2.4) and

$$N(t,\phi) \equiv Y(t,\phi) - Y(0,\varphi) - \int_0^t \int_{R^d} a(x, Y(s))\phi(x)\pi(dx, ds),$$

is an orthogonal martingale measure with the covariance measure

$$V(t,\phi) = \int_0^t \int_{R^d} b^2(x, Y(s))\phi(x)\pi(dx, ds)$$

It implies that Y is a solution of the martingale problem. □

LEMMA 2.3. *Let $X_n, Y_n \in D_R[0,\infty)$, and $X'_n, Y'_n \in D_R[0,\infty)$. Assume that $(X_n, Y_n) \Rightarrow (X, Y)$ in $D_R[0,\infty) \times D_R[0,\infty)$ and $(X'_n, Y'_n) \Rightarrow (X', Y')$ in $D_{R^2}[0,\infty)$. Suppose the discontinuities of X_n are bounded by the discontinuities of X'_n and the discontinuities of Y_n are bounded by those of Y'_n. Then*

$$(X_n, Y_n) \Rightarrow (X, Y) \qquad \text{in } D_{R^2}[0,\infty)$$

3. Application to a relaxed control problem

We now consider a model of controlled diffusion which is the solution of the equation.

$$\text{(3.1)} \qquad \begin{aligned} dX_t^u &= b(t, X_t^u, u_t)dt + \sigma(t, X_t^u, u_t)dB_t \\ X_0^u &= z, \end{aligned}$$

where B_t is a Brownian motion, b, σ are continuous, bounded functions on $R_+ \times R^d \times E$ and uniformly Lipshitz continuous in the R^d variable, and the control process u is $\mathcal{F}_t$-predictable with values in a compact metric space E. Let

$$\mathcal{U} = \{u; \mathcal{F}_t - \text{predictable with values in } E\}$$
$$\mathcal{R} = \{\mu; \text{probability measures on } E \times [0,T], \mu(dx, dt) = q_t(dx)dt\}$$

where q_t is a transition probability measure. $\mathcal{R}$ is called the space of relaxed controls and is compact for the weak topology. The cost function C_T on the time interval $[0,T]$ is defined by

$$C_T(z,u) = E[\int_0^T h(s, X_s^u, u_s)ds + g(X_T^u)] \tag{3.2}$$

where $g : R^d \to R^d$, $h : R_+ \times R^d \times E \to R^d$ are continuous and bounded. Now, the object of the control problem is to minimize this cost function over the process u in $\mathcal{U}$. The value function V is defined by

$$V(z) = \inf_{u \in \mathcal{U}} C_T(z,u),$$

and the optimal control is the control on which this infimum is obtained if there exists one.

It is well known that an optimal control does not necessarily exist in $\mathcal{U}$, which is not endowed with a compact topology. The usual idea in control theory (e.g.[ENP]) is to embed this set $\mathcal{U}$ in the set $\mathcal{R}$. The following lemma has been proved in [ENP],

CHATTERING LEMMA. *Let $\{q_t\}$ be a predictable process with values in $\mathcal{P}(E)$. Then there exists a sequence of E-valued predictable process $\{u_t^k\}$ such that $\delta_{u_t^k}(dx)dt$ converges weakly to $q_t(dx)dt$,* P a.s.

We can view this control problem in connection with the theory of martingale measure. Motivated by the work of Meleard[Me], we apply our results of weak convergence of SDEs with respect to martingale measures to a relaxed control problem w.r.t. martingale measures. Let $\{u_n\}$ be a sequence in $\mathcal{U}$. We rewrite (3.1) using the notation of martingale measures. Letting $M_t^n(A) = \int_0^t I_A(u_s^n)dB_s$, (3.1) becomes

(3.3)

$$X_t^n = X_0 + \int_0^t \int_E b(s, X_s^n, x)\delta_{u_s^n}(dx)ds + \int_0^t \int_E \sigma(s, X_s^n, x)M^n(dx, ds)$$

Note that M^n is an orthogonal martingale measure, whose covariance measures is $\delta_{u_t^n}(dx)dt$.

Then it is well known that there exists a unique solution process X^q, which is the solution to the SDE

(3.4)

$$X_t^q = X_0 + \int_0^t \int_E \sigma(s, X_s^q, x)M(dx, ds) + \int_0^t \int_E b(s, X_s^q, x)q_s(dx)ds,$$

where M is a continuous martingale measure with covariance measure $q_t(dx)dt$. (The existence of the martingale measure satisfying (3.4) is proved as a representation theorem in [Me].) Under this situation, we can show the following approximation result.

THEOREM 3.1. *Given a relaxed control $q_t(dx)dt$,*

(1) *there exists a sequence of controlled diffusions X_t^n s.t.*

$$X_\cdot^n \Rightarrow X_\cdot^q \qquad \text{in } D_{R^d}[0,\infty),$$

where X^q is the associated relaxed diffusion to $q_t(dx)dt$, satisfying (3.4).

(2) *Let C_T^n, C_T be the cost functions associated with X^n, X^q respectively. Then there exists a subsequence $C_T^{n_k}$ which converges to C_T*

Proof. By the chattering lemma, there exists a sequence of $E-$valued predictable process $\{u_t^n\}$ such that the sequence of random measures $\delta_{u_t^n}(dx)dt$ converges weakly on $E \times [0,1]$ to $q_t(dx)dt$ a.s. Let

$$\mu_t^n(dx)dt = \delta_{u_t^n}(dx)dt \quad M^n(dx,ds) = \delta_{u_t^n}(dx)dB_t$$

Then

$$\mu^n \to q \text{ in } D_{\mathcal{M}_p}[0,\infty) \text{ a.s.}$$

implies that

$$M^n \Rightarrow M \text{ in } D_{\mathcal{S}'(R^d)}[0,\infty)$$

We can find $X_n, n = 1, 2\dots$ which are solutions of the following;

$$\begin{aligned} X_t^n &= X_0 + \int b(s,X_s^n,x)\delta_{u_s^n}(dx)ds + \int \sigma(s,X_s^n,x)\delta_{u_s^n}(dx)dB_s \\ &= X_0 + \int b(s,X_s^n,x)\mu_s^n(dx)ds + \int \sigma(s,X_s^n,x)M^n(dx,ds) \end{aligned}$$

We claim that $\{X^n\}$ is relative compact in $D_{R^d}[0,\infty)$ and $X^n \Rightarrow X^q$. For $0 \le t \le T$, $\quad 0 \le u \le \delta, \delta > 0$,

$$\begin{aligned} &E[|X_{t+u}^n - X_t^n||\mathcal{F}_t] \\ &\quad \le E[\int_t^{t+u}\int_E \sigma^2(s,X_s,x)\mu_s^n(dx)ds \\ &\qquad + \int_t^{t+u}\int_E |b(s,X_s,x)|\mu_s^n(dx)ds|\mathcal{F}_t] \\ &\quad \le C_1 \cdot \delta \cdot E[\mu^n(E)] \le C_1 \cdot \delta, \end{aligned}$$

where $C_1 \geq \sup_{s \leq T} \|\sigma^2(s,\cdot,\cdot)\|_\infty, \sup_{s\leq T} \|b(s,\cdot,\cdot)\|_\infty$. Choosing $\gamma_n(\delta) \equiv C_1 \cdot \delta$, $\{X_n\}$ satisfies the criteria of Theorem 3.8.6[EK] for the relative compactness. In fact,

$$X^n_\cdot \to X^q \text{ in } D_{R^d}[0,\infty) \text{ in probability},$$

by the uniqueness of solution of (3.4). Furthermore it is easy to see that

$$(X_n, M_n) \Rightarrow (X^q, M) \quad \text{in } D_{R^d \times S'(R^d)}[0,\infty).$$

Applying Theorem 1.1 and Lemma 1.5[K1], we have

$$X_t = X_0 + \int_0^t \int_E \sigma(s, X_s, x) M(dx, ds) + \int_0^t \int_E b(s, X_s, x) q_s(dx) ds$$

The convergence of a subsequence of C_T^n is a direct consequence of (1). □

References

[Ch1] N. Cho, *Weak convergence of stochastic integrals driven by martingale measures*, Stoch. process. appl. **59** (1995), 55-79.

[Ch2] N. Cho, *Weak limit theorems for stochastic differential equations driven by martingale measures*, to appear in Stochastics (1996).

[EK] S. Ethier and T. Kurtz, Markov process: Characterizatioon and convergence, Wiely, 1986.

[ENP] N. Karoui, D.Huu Nguyen, M. Jeanblanc-Picque, *Compactifications methods in the control of degenerate diffusions: existence of an optimal control*, Stochastics **20** (1987), 169-219.

[IW] N. Ikeda and S. Watanabe, Stochastic differential equations and diffusion processes, North-Holland, 1981.

[K1] T. Kurtz, *Averaging for martingale problems and stochastic application*, Lect. notes in Control and Inf.sci. **177** (1992), 186-209.

[KP] T.Kurtz and P.Protter, *Weak limit theorems for stochastic integrals and stochastic differential equations*, Ann. Prob **19** (1991), 1035-1070.

[M] I. Mitoma, *Tightness of probabilities on* $(C0,1];S')$ *and* $(D0,1];S')$. Ann. Prob **11** (1983), 989-999.

[KX] G. Kallianpur and J. Xiong, *A nuclear space-valued stochastic differential equation driven bt Poisson random measure.*, Lec. notes. in control and info. sci. **176** (1992), 135-143.

[Mc] H. P.McKean, *Propogation of chaos for a class of non-linear parabolic equations*, Lecture series in Differntial Equations 7 Catholic Univ. (1967), 41-57.

[Me] S. Meleard, *Representation and Approximation of martingale measures*, Lecture series in Bio. mathematics (1992), 188-199

[W] J. Walsh, *An introduction to stochastic P.D.E.*, Lect. notes in Math. **1180** (1986), 265-439.

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