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BOUNDARY–VALUED CONDITIONAL YEH–WIENER INTEGRALS AND A KAC–FEYNMAN WIENER INTEGRAL EQUATION

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1. Introduction

For $Q = [0,S] \times [0,T]$ let $C(Q)$ denote Yeh-Wiener space, i.e., the space of all real-valued continuous functions $x(s,t)$ on Q such that $x(0,t) = x(s,0) = 0$ for every (s,t) in Q. Yeh [10] defined a Gaussian measure m_y on $C(Q)$ (later modified in [13]) such that as a stochastic process $\{x(s,t),(s,t)\in Q\}$ has mean $E[x(s,t)] = \int_{C(Q)} x(s,t)m_y(dx) = 0$ and covariance $E[x(s,t)x(u,v)] = \min\{s,u\}\min\{t,v\}$. Let $C_w \equiv C[0,T]$ denote the standard Wiener space on $[0,T]$ with Wiener measure m_w. Yeh [12] introduced the concept of the conditional Wiener integral of F given $X, E(F|X)$, and for the case $X(x) = x(T)$ obtained some very useful results including a Kac-Feynman integral equation.

A very important class of functions in quantum mechanics consists of functions on $C[0,T]$ of the type

$$G(x) = \exp\left\{\int_0^T \theta(s,x(s))ds\right\}$$

where $\theta : [0,T] \times \mathbb{R} \to \mathbb{C}$.

Yeh [12] shows that under suitable regularity conditions on θ, the conditional Wiener integral

(1.1)

$$H(t,\xi) = (2\pi t)^{-\frac{1}{2}} \exp\left\{-\frac{\xi^2}{2t}\right\} E\left(\exp\left\{\int_0^t \theta(s,x(s))ds\right\} |x(t) = \xi\right)$$

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satisfies the Kac-Feynman integral equation

$$(1.2)\quad H(t,\xi) = (2\pi t)^{-\frac{1}{2}} \exp\left\{-\frac{\xi^2}{2t}\right\} + \int_0^t [2\pi(t-s)]^{-\frac{1}{2}} \int_{\mathbb{R}} \theta(s,\eta)H(s,\eta)H(s,\eta)\exp\left\{-\frac{(\eta-\xi)^2}{2(t-s)}\right\} d\eta ds$$

whose solution can be expressed as an infinite series of terms involving Lebesgue integrals. Then using (1.1), one can use the series solution of (1.2) to evaluate the conditional Wiener integral

$$E\left(\exp\left\{\int_0^t \theta(s,x(s))ds\right\} |x(t)=\xi.\right)$$

The corresponding problem in Yeh-Wiener space; namely to evaluate

$$(1.3)\qquad E\left(\exp\left\{\int_0^t\int_0^s \phi(u,v,x(u,v))dudv\right\} |x(s,t)=\xi\right)$$

turned out to be substantially different than the corresponding one-parameter problem. After many attempts to solve this problem by several mathematicians, the first really successful solution was given by Park and Skoug [8] by introducing a sample path-valued conditional Yeh-Wiener integral of the type

$$(1.4)\qquad E\left(\exp\left\{\int_0^t\int_0^s \phi(u,v,x(u,v))dudv\right\} |x(s,\cdot)=\eta(\cdot)\right)$$

which satisfies a Wiener integral equation similar to that of Cameron and Storvick [1]. The Wiener integral equation is then solved to evaluate (1.4), and finally (1.3) is obtained by integrating (1.4) appropriately.

In this paper we consider boundary-valued conditional Yeh-Wiener integrals of the type

$$(1.5)\qquad E(F(x)|x(\cdot,T),x(S,*)),$$

where $F \in L_1(C(Q), m_y)$. Since $x(0,*) = x(\cdot,0) = 0$ for every $x \in C(Q)$, the value of x on ∂Q, the boundary of Q, is completely determined by the value of x on the two edges of Q, namely by $x(\cdot,T)$ and $x(S,*)$.

In section 2 we show that the conditional expectation (1.5) is very closely related to the Yeh-Brownian bridge process

$$\{x \in C(Q) | x(s,t) = 0 \text{ for all } (s,t) \in \partial Q\}. \tag{1.6}$$

We also discuss other kinds of two-parameter Brownian bridges in section 2.

In section 4, we evaluate (1.5) for functionals F of the form $F(x) = \exp\{\int_Q \phi(u,v,x(u,v))dudv\}$ by solving a Kac-Feynman Wiener integral equation. Finally, in section 5, a conditional version of the Cameron-Martin translation theorem is obtained for conditional Yeh-Wiener integrals of the type (1.5).

2. Yeh-Brownian Bridges

As is well known, the one-parameter Brownian bridge can be expressed in the form

$$\{w(\cdot) \in C_w | w(T) = 0\} = \{w(\cdot) - \frac{\cdot}{T} w(T), w \in C_w\}. \tag{2.1}$$

Another convenient representation for the Brownian bridge is given by

$$y(t) = \begin{cases} tw\left(\frac{1}{t} - \frac{1}{T}\right) & , \ 0 < t \le T \\ 0 & , \ t = 0 \end{cases}$$

whose covariance is

$$E[y(t)y(t')] = (t \wedge t')\left(1 - \frac{t \vee t'}{T}\right). \tag{2.2}$$

There doesn't seem to be a consensus for the two-parameter version of Brownian bridges. So, we will introduce several version, and then develop further theory for one of these versions.

The first version is given in the form

$$(2.3)\quad \{x \in C(Q)|x(S,T)=0\} = \left\{x(\cdot,*) - \frac{(\cdot)(*)}{ST}x(S,T), x \in C(Q)\right\}.$$

Note that under the conditioning $x(S,T)=0$,

$$x(s,t) - \frac{st}{ST}x(S,T) = x(s,t),$$

and the process $x(s,t) - \frac{st}{ST}x(S,T)$ is independent of $x(S,T)$ for all $(s,t) \in Q$

Other versions of two-parameter Brownian bridges are

$$(2.4)\qquad \{x \in C(Q)|x(\cdot,T)=0\} = \left\{x(\cdot,*) - \frac{*}{T}x(\cdot,T), x \in C(Q)\right\},$$

and

$$(2.5)\qquad \{x \in C(Q)|x(S,*)=0\} = \left\{x(\cdot,*) - \frac{\cdot}{S}x(S,*), x \in C(Q)\right\}.$$

The final version, which we will call the Yeh-Brownian bridge process, is given by

$$(2.6)\quad \{x \in C(Q)|x(S,*)=0, x(\cdot,T)=0\}$$
$$= \left\{x(\cdot,*) - \frac{\cdot}{S}x(S,*) - \frac{*}{T}x(\cdot,T), + \frac{(\cdot)(*)}{ST}x(S,T), x \in C(Q)\right\}.$$

Note that the process

$$(2.7)\qquad z(s,t) = x(s,t) - \frac{s}{S}x(S,t) - \frac{t}{T}x(s,T) + \frac{st}{ST}x(S,T)$$

satisfies the condition $z(S,*)=0$ and $z(\cdot,T)=0$. Furthermore, z is independent of $x(S,*)$ and $x(\cdot,T)$, and the covariance of z is given by

$$(2.8)\quad E[z(s,t)z(s',t')] = (s \wedge s')\left(1 - \frac{s \vee s'}{S}\right)(t \wedge t')\left(1 - \frac{t \vee t'}{T}\right).$$

For $0 < s \le S$ let $Q_s = [0,s] \times [0,T]$, and for $x \in C(Q_s)$ we define $x_{\gamma,s}$ by

$$(2.9)\qquad x_{\gamma,s}(u,v) = \frac{u}{s}x(s,v) + \frac{v}{T}x(u,T) - \frac{uv}{sT}x(s,T).$$

For convenience, if $s=S$, we surpress the S and simply write

$$(2.10)\quad x_\gamma(u,v) \equiv x_{\gamma,S}(u,v) = \frac{u}{S}x(S,v) + \frac{v}{T}x(u,T) - \frac{uv}{ST}x(S,T)$$

for (u,v) in Q. We note that $z = x - x_\gamma$.

Our first result may be stated as follows:

THEOREM 1. *If $\{x(s,t),(s,t) \in Q\}$ is the standard Yeh-Wiener process, then $x - x_\gamma$ and x_γ are independent Gaussian processes on Q, and so are $x - x_{\gamma,s}$ and $x_{\gamma,s}$ on Q_s. Furthermore, $x - x_\gamma$ is independent of $x(\cdot,T)$ and $x(S,*)$ on Q, and $x - x_{\gamma,s}$ is independent of $x(\cdot,T)$ and $x(s,*)$ on Q_s.*

Proof. Using the formula

$$E[x(s,t)x(u,v)] = (s \wedge u)(t \wedge v)$$

it is easy to establish that

$$E[\{x(s,t) - x_\gamma(s,t)\}x_\gamma(u,v)] = 0.$$

Since uncorrelated Gaussian processes are independent, we may conclude that $x - x_\gamma$ and x_γ are independent processes on Q. The rest of the proof can be established in a similar manner.

■

3. Boundary-Valued Conditional Yeh-Wiener Integrals

For $x \in C(Q)$, define $X(x)$ and $X_s(x)$ by

(3.1) $$X(x) \equiv X_S(x) = (x(\cdot,T), x(S,*)), \quad \text{and}$$

(3.2) $$X_s(x) = (x(\cdot,T), x(s,*)).$$

Thus if $\eta \in C(Q)$, then $X(x) = X(\eta)$ means that x and η agree on ∂Q. Similarly, $X_s(x) = X(\eta)$ means that x and η agree on ∂Q_s.

The following theorem plays a key role throughout this paper.

THEOREM 2. *Let $F \in L_1(C(Q))$. Then for each $\eta \in C(Q)$,*

(3.3) $$E(F(x)|X(x) = X(\eta)) = E[F(x - x_\gamma + \eta_\gamma)].$$

If $F \in L_1(C(Q_s))$, then for each $\eta \in C(Q_s)$,

(3.4) $$E(F(x)|X_s(x) = X_s(\eta)) = E[F(x - x_{\gamma,s} + \eta_{\gamma,s})].$$

Proof. Under the conditioning $X(x) = X(\eta)$, we have $x_\gamma = \eta_\gamma$ on Q, and hence $x = x - x_\gamma + \eta_\gamma$. Therefore

$$E(F(x)|X(x) = X(\eta)) = E(F(x - x_\gamma + \eta_\gamma)|X(x) = X(\eta)).$$

Thus (3.3) follows from the fact that the process $x - x_\gamma$ is independent of each component of $X(x)$ by Theorem 1. Equation (3.4) follows in a similar manner. ■

The following case is singled out as it corresponds to the Yeh-Brownian bridge.

COROLLARY. *Let $F \in L_1(C(Q))$. Then*

$$(3.5) \qquad E(F(x)|X(x) = (0,0)) = E[F(x - x_\gamma)];$$

that is to say,

$$(3.5) \qquad E(F(x)|x(s,t) = 0 \text{ on } \partial Q) = E[F(x(\cdot,*) - \frac{\cdot}{S}x(S,*) - \frac{*}{T}x(\cdot,T) + \frac{(\cdot)(*)}{ST}x(S,T))].$$

The following example illustrates Theorem 2 and its Corollary.

EXAMPLE. For $x \in C(Q)$, let $F(x) = x^2(S/2, T/2)$. Then by use of equations (3.3) and (3.5), the fact that $E[x(s,t)] = 0$, and $E[x(s,t)x(u, v)] = \min\{s,u\}\min\{t,v\}$, it follows that

$$\begin{aligned} &E\left[x^2\left(S/2,T/2\right)\right] = ST/4, \\ &E\left(x^2\left(S/2,T/2\right)\right)\left|X(x) = X(\eta)\right) \\ &= E\left\{\left[x\left(S/2,T/2\right) - x_\gamma\left(S/2,T/2\right) + \eta_\gamma\left(S/2,T/2\right)\right]^2\right\} \\ &= ST/16 + \eta_\gamma^2\left(S/2,T/2\right), \end{aligned}$$

and

$$E\left(x^2\left(S/2,T/2\right)\left|x(s,t) = 0 \text{ on } \partial Q\right.\right) = ST/16.$$

In section 5, we need to consider stochastic integrals of $h \in L_2(Q)$ with respect to x_γ, defined by (2.10). Such stochastic integrals may be expressed in terms of the functions h_1, h_2andh_3 defined below.

DEFINITION. For each function $h \epsilon L_2(Q)$, define h_1, h_2 and h_3 on Q by

$$h_1(s,t) = \frac{1}{S}\int_0^S h(u,t)du,$$

$$(3.6) \qquad h_2(s,t) = \frac{1}{T}\int_0^T h(s,v)dv, \quad \text{and}$$

$$h_3(s,t) = \frac{1}{ST}\int_Q h(u,v)dudv.$$

The following theorem gives some useful and interesting formulas involving the h_j and x_γ. The proof is rather straightforward and hence omitted. Some similar observations were made by Park and Skoug [7, p.456].

THEOREM 3. *Let $h \epsilon L_2(Q)$. Then for $j = 1, 2, 3$,*

$$(3.7) \qquad \int_Q hh_j = \int_Q h_j^2.$$

$$(3.8) \qquad \|h - h_j\|_2^2 = \|h\|_2^2 - \|h_j\|_2^2 \geq 0, \quad \textit{and}$$

$$(3.9) \qquad \|h_1 + h_2 - h_3\|_2^2 = \|h_1\|_2^2 + \|h_2\|_2^2 - \|h_3\|_2^2.$$

Furthermore, for every $x \epsilon C(Q)$,

$$(3.10) \qquad \int_Q h_j dx_\gamma = \int_Q h_j dx \quad \text{for} \quad j = 1, 2, 3, \quad \text{and}$$

$$(3.11) \qquad \int_Q hdx_\gamma = \int_Q (h_1 + h_2 - h_3)dx.$$

4. Evaluation of $E\left(\exp\left\{\int_Q \phi(s,t,x(s,t))dsdt\right\} |X(x) = X(\eta)\right)$

Let $\phi(s,t,u)$ be a bounded continuous function on $Q \times \mathbb{R}$, and let

$$(4.1) \qquad \theta(s,\ x(s,\cdot)) = \int_0^T \phi(s,t,x(s,t))dt.$$

Then

$$F(s,x) \equiv \exp\left\{\int_0^s \int_0^T \varphi(u,t,x(u,t))dtdu\right\}$$
$$= \exp\left\{\int_0^s \theta(u,x(u,\cdot))du\right\}.$$

Since $\partial F(s,x)/\partial s = \theta(s,x(s,\cdot))F(s,x)$, by integrating over $[0,s], 0 < s \geq S$, we obtain

$$F(s,x) - 1 = \int_0^s \theta(\sigma,x(\sigma,\cdot))F(\sigma,x)d\sigma.$$

Next take the conditional expectation of both sides above and then use the Fubini theorem to obtain

(4.2) $E(F(s,x)|X_s(x) = X_s(\eta))$
$$= 1 + \int_0^s E(\theta(\sigma,x(\sigma,\cdot))F(\sigma,x)|X_s(x) = X_s(\eta))d\sigma.$$

But by Theorem 2, for $0 < \sigma \leq s \leq S$.

(4.3)
$$E(\theta(\sigma,x(\sigma,\cdot))F(\sigma,x)|X_s(x) = X_s(\eta))$$
$$= E\left[\theta(\sigma,(x - x_{\gamma,s} + \eta_{\gamma,s})(\sigma,\cdot))F(\sigma,x - x_{\gamma,s} + \eta_{\gamma,s})\right]$$
$$= E\left[\theta(\sigma,(x - x_{\gamma,s} + \eta_{\gamma,s})(\sigma,\cdot)) \exp\left\{\int_0^\sigma \theta(u,(x - x_{\gamma,s} + \eta_{\gamma,s})(u,\cdot))du\right\}\right].$$

Note that for $0 \leq u \leq \sigma \leq s \leq S$ and fixed σ and s,

(4.4) $$(\eta_{\gamma,s})_{\gamma,\sigma}(u,\cdot) = \eta_{\gamma,s}(u,\cdot)$$

(4.5)
$$(x - x_{\gamma,s})(u,\cdot) = (x - x_{\gamma,\sigma})(u,\cdot) + \frac{u}{\sigma}(x - x_{\gamma,s})(\sigma,\cdot).$$

Easy, but lengthy computations show that $(x - x_{\gamma,\sigma})(u,\cdot)$ and $(x - x_{\gamma,s})(\sigma,\cdot)$ are independent processes, and $(x - x_{\gamma,s})(\sigma,\cdot)$ is equivalent to $\sqrt{\sigma(1-\frac{\sigma}{s})}y(\cdot)$ for fixed σ and s, where $y(\cdot)$ is the Brownian bridge on $[0,T]$, namely,

$$y(t) = w(t) - \frac{t}{T}w(T). \tag{4.6}$$

Since $y(T) = 0$, it follows that

$$\frac{u}{\sigma}\sqrt{\sigma(1-\frac{\sigma}{s})}y(\cdot) = \left[\sqrt{*(1-\frac{*}{s})}y(\cdot)\right]_{\gamma,\sigma}(u,\cdot). \tag{4.7}$$

Hence, using (4.4) through (4.7) and the above comments in (4.3), we obtain

(4.8)

$$\begin{aligned}
&E[\theta(\sigma,x(\sigma,\cdot))F(\sigma,x)|X_s(x) = X_s(\eta)]\\
=&E\Big[\theta(\sigma,(x - x_{\gamma,s} + \eta_{\gamma,s})(\sigma,\cdot))\\
&\cdot \exp\{\int_0^\sigma \theta[u,(x - x_{\gamma,\sigma})(u,\cdot) + \frac{u}{\sigma}(x - x_{\gamma,s})(\sigma,\cdot) + \eta_{\gamma,s}(u,\cdot)\Big]du\}]\\
=&E_y\Big[\theta(\sigma,\sqrt{\sigma(1-\frac{\sigma}{s})}\ y(\cdot) + \eta_{\gamma,s}(\sigma,\cdot))\\
&\cdot E_x[\exp\{\int_0^\sigma \theta(u,(x - x_{\gamma,\sigma})(u,\cdot) + \frac{u}{\sigma}\sqrt{\sigma(1-\frac{\sigma}{s})}\ y(\cdot) + \eta_{\gamma,s}(u,\cdot))du\}]\Big]\\
=&E_y\Big[\theta(\sigma,\sqrt{\sigma(1-\frac{\sigma}{s})}\ y(\cdot) + \eta_{\gamma,s}(\sigma,\cdot))\\
&\cdot E_x\Big[\exp\{\int_0^\sigma \theta(u,(x - x_{\gamma,\sigma})(u,\cdot) + [\sqrt{*(1-\frac{*}{s})}\ y(\cdot) + \eta_{\gamma,s}(*,\cdot)]_{\gamma,\sigma}(u,\cdot))du\}]\Big]\\
=&E_y\Big[\theta(\sigma,\sqrt{\sigma(1-\frac{\sigma}{s})}\ y(\cdot) + \eta_{\gamma,s}(\sigma,\cdot))\\
&\cdot E_x(\exp\{\int_0^\sigma \theta(u,x(u,\cdot))du\}|X_\sigma(x) = X_\sigma(\sqrt{*(1-\frac{*}{s})}\ y(\cdot) + \eta_{\gamma,s}(*,\cdot)))\Big].
\end{aligned}$$

If we set

$$G(s,\eta) = E(F(s,x)|X_s(x) = X_s(\eta)), \tag{4.9}$$

then it follows from (4.2) and (4.8) that

$$\begin{aligned}
G(s,\eta) = 1 + \int_0^s &E_y[\theta(\sigma\sqrt{\sigma(1-\frac{\sigma}{s})}\ y(\cdot) + \eta_{\gamma,s}(\sigma,\cdot))\\
&\cdot G(\sigma,(\sqrt{*(1-\frac{*}{s})}\ y(\cdot) + \eta_{\gamma,s}(*,\cdot))]d\sigma.
\end{aligned} \tag{4.10}$$

Since $y(t) = w(t) - \frac{t}{T}w(T)$, we may express (4.10) in terms of the Brownian motion $w(\cdot)$ and obtain

(4.11)

$$G(s,\eta) = 1 + \int_0^s E_w[\theta(\sigma\sqrt{\sigma(1-\frac{\sigma}{s})}\,[w(\cdot) - \frac{\cdot}{T}w(T)] + \eta_{\gamma,s}(\sigma,\cdot)) \cdot G(\sigma,(\sqrt{*(1-\frac{*}{s})}\,[w(\cdot) - \frac{\cdot}{T}w(T)] + \eta_{\gamma,s}(*,\cdot))]d\sigma.$$

This Wiener integral equation is very similar to the Cameron-Storvick integral equation [1, equation (4.3)] and the Park-Skoug integral equation [8, equation (4.5)]. Thus the Wiener integral equation (4.11) has a series solution

$$G(s,\eta) = \sum_{k=0}^{\infty} H_k(s,\eta), \tag{4.12}$$

where the sequence $\{H_k\}$ is given inductively by

$$H_0(s,\eta) = 1,$$

and

$$H_{k+1}(s,\eta) = \int_0^s E_w[\theta(\sigma,\sqrt{\sigma(1-\frac{\sigma}{s})}[w(\cdot) - \frac{\cdot}{T}w(T)] + \eta_{\gamma,s}(\sigma,\cdot)) \cdot H_k(\sigma,\sqrt{*(1-\frac{*}{s})}[w(\cdot) - \frac{\cdot}{T}w(T)] + \eta_{\gamma,s}(*,\cdot))]d\sigma.$$

Using the same method used by Park and Skoug [8, section 4], one can show that the series in (4.12) converges uniformly on [0,S], and that it is the only bounded continuous solution of (4.11).

5. Translation of Boundary-Valued conditional Yeh-Wiener integrals

The Cameron-Martin translation theorem for Yeh-Wiener integrals, see Yeh [11], states that if $x_0(s,t) = \int_0^t \int_0^s h(u,v)dudv$ on Q for $h \in L_2(Q)$, and if T_1 is the transformation of $C(Q)$ into itself defined by

$$z = T_1(x) = x + x_0$$

for $x \in C(Q)$, then for any Yeh-Wiener integrable function F on $C(Q)$,

$$E[F(z)] = E[F(x+x_0)J(x_0,x)], \tag{5.1}$$

where

$$J(x_0.x) = \exp\{-\frac{1}{2}\int_Q h^2(u,v)dudv\}\exp\{-\int_Q h(u,v)dx(u,v)\}.$$

The following is the boundary-valued conditional version of (5.1).

THEOREM 4. *Let $x_0(s,t) = \int_0^t\int_0^s h(u,v)dudv$ on Q for some $h \in L_2(Q)$, and let $F \in L_1(C(Q),m_y)$. Then, for each $\eta \in C(Q)$,*

$$\begin{aligned} E(F(z)|X(z)=X(\eta)) &= E(F(x+x_0)J(x_0,x)|X(x+x_0) \\ &=X(\eta))\exp\Big\{-\frac{1}{2}\|h_1+h_2-h_3\|_2^2 + \int_Q hd\eta_\gamma\Big\} \end{aligned}$$

where h_1, h_2, and h_3 are given by (3.6).

Proof. First, using Theorem 2, we see that

$$E(F(z)|X(z)=X(\eta)) = E[F(z-z_\gamma+\eta_\gamma)].$$

Since $(x+x_0)_\gamma = x_\gamma + (x_0)_\gamma$, we may apply (5.1) to get

$$E[F(z-z_\gamma+\eta_\gamma)] = E[F(x+x_0-x_\gamma-(x_0)_\gamma+\eta_\gamma)J(x_0,x)]. \tag{5.2}$$

Next, we write $J(x_0,x)$ in the form

$$\begin{aligned} J(x_0,x) &= \exp\{-\frac{1}{2}\|h\|_2^2\}\exp\{-\int_Q hd(x-x_\gamma+\eta_\gamma-(x_0)_\gamma)\} \\ &\cdot \exp\{-\int_Q hdx_\gamma\}\exp\{\int_Q hd\eta_\gamma\}\exp\{-\int_Q hd(x_0)_\gamma\}. \end{aligned} \tag{5.3}$$

Since $x - x_\gamma$ and x_γ are independent processes on Q by Theorem 1, it follows from (5.2), (5.3) and (3.11) that

(5.4)

$$\begin{aligned} E[F(z-z_\gamma+\eta_\gamma)] &= \exp\{-\frac{1}{2}\|h\|_2^2 + \int_Q hd\eta_\gamma - \int_Q hd(x_0)_\gamma\} \\ &\cdot E[F(x+x_0-x_\gamma-(x_0)_\gamma+\eta_\gamma) \\ &\quad \exp\{-\int_Q hd(x-x_\gamma+\eta_\gamma-(x_0)_\gamma)\}] \\ &\cdot E[\exp\{-\int_Q hdx_\gamma\}]. \end{aligned}$$

As $\int_Q h dx_\gamma = \int_Q (h_1 + h_2 - h_3) dx$ by (3.11), $\int_Q h dx_\gamma$ is a normal random variable with mean zero and variance $\|h_1 + h_2 - h_3\|_2^2$. Thus

$$E[\exp\{-\int_Q h dx_\gamma\}] = \exp\{\frac{1}{2}\|h_1 + h_2 - h_3\|_2^2\}. \tag{5.5}$$

Next, using (3.11), (3.7), and the definition of x_0, we see that

$$\begin{aligned}\int_Q h d(x_0)_\gamma &= \int_Q (h_1 + h_2 - h_3) dx_0 \\ &= \int_Q (h_1 + h_2 - h_3) h \\ &= \|h_1\|_2^2 + \|h_2\|_2^2 - \|h_3\|_2^2.\end{aligned}$$

Using Theorem 2 again, we obtain

$$\begin{aligned}&E(F(x + x_0) J(x_0, x) | X(x + x_0) = X(\eta)) \\ &= E[F(x + x_0 - x_\gamma - (x_0)_\gamma + \eta_\gamma) \\ &\qquad \exp\{-\frac{1}{2}\|h\|_2^2 - \int_Q h d(x - x_\gamma + \eta_\gamma - (x_0)_\gamma)\}].\end{aligned} \tag{5.6}$$

Finally, we substitute (5.5) and (5.6) into (5.4), and then use (3.9) to obtain

$$\begin{aligned}E[F(z - z_\gamma + \eta_\gamma)] &= \exp\{-\frac{1}{2}\|h_1 + h_2 - h_3\|_2^2 + \int_Q h d\eta_\gamma\} \\ &\quad \cdot E(F(x + x_0) J(x_0, x) | X(x + x_0) = X(\eta)),\end{aligned}$$

which completes the proof of Theorem 4. ■

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