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GENERAL LOCAL COHOMOLOGY MODULES AND COMPLEXES OF MODULES OF GENERALIZED FRACTIONS

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0. Introduction

Throughout this paper, R will be a commutative ring (with non-zero identity) and M will denote an R-module.

The modules of generalized fractions were introduced by Sharp and Zakeri [**16**] and in [**17**, 3.5] they gave a relationship between modules of generalized fractions and local cohomology modules, that is,

$$U_d[1]^{-d-1}M \cong H^d_{\mathfrak{m}}(M),$$

where $(R, \mathfrak{m})$ is a Noetherian local ring of dimension d, $U_d[1]$ is the expansion (see [**16**, 3.2]) of $\{(a_1, \dots, a_d, 1) \in R^{d+1} : a_1, \dots, a_d$ forms a system of parameters for $R\}$ and $H^d_{\mathfrak{m}}(M)$ is the local cohomology module of M.

In [**5**, 2.4], under the same ring as above, when $U_{d'}[1]$ is the expansion of $\{(a_1, \dots, a_{d'}, 1) \in R^{d'+1} : a_1, \dots, a_{d'}$ forms a system of parameters for $M\}$ where M is a finitely generated R-module of $d' = dim\ M$, we had a similar result

$$U_{d'}[1]^{-d'-1}M \cong H^{d'}_{\mathfrak{m}}(M).$$

In [**3**], Bijan-Zadeh studied a generalization of results of Sharp and Zakeri. He proved that, for a fixed sequence of elements $x_1, \dots, x_n$ of a ring R,

$$U(x)[1]^{-n-1}M \cong H^n_{\mathfrak{a}}(M),$$

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where $\mathfrak{a} = (x_1, \ldots, x_n)R$ and $U(x)[1] = \{(x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}, 1) \in R^{n+1} : \alpha_i \in N\}$. Moreover, in [**3**, Theorem], for a given triangular subset U_n of R^n, he obtained

$$U_n[1]^{-n-1}M \cong H^n_{\Phi(U_n)}(M) \cong \varinjlim_{\mathfrak{a}\in\Phi(U_n)} H^n_{\mathfrak{a}}(M),$$

where $\Phi(U_n) = \{(a_1, \ldots, a_n)R : (a_1, \ldots, a_n) \in U_n\}$ and $H^n_{\Phi(U_n)}$ is the n-th right derived functor of the general local cohomology functor $L_{\Phi(U_n)}$ (see Definition 1.4 and Example 1.5(2)).

In [**19**, 5.2.3 and **15**, 3.3], Sharp and Yassi established a relationship between the modules of generalized fractions and the generalized ideal transforms (see Definition 1.4), *i.e.*,

$$U_n^{-n}M \cong \varinjlim_{\mathfrak{a}\in\Phi(U_n)} \mathrm{Hom}_R(\mathfrak{a}, \mathrm{Im}\ e^{n-1})$$

where $e^{n-1} : U_{n-1}^{-n+1}M \to U_n^{-n}M$ is the R-homomorphism for which $e^{n-1}\left(\dfrac{m}{(a_1, \ldots, a_{n-1})}\right) = \dfrac{m}{(a_1, \ldots, a_{n-1}, 1)}$ for $m \in M$ and $(a_1, \ldots, a_{n-1}) \in U_{n-1}$; and

$$U_n^{-n}M \cong \varinjlim_{\mathfrak{a}\in\Phi(U_n)} \mathrm{Hom}_R(\mathfrak{a}, U_{n-1}[1]^{-n}M)$$

where R is Noetherian and $U_{n-1}[1] = \{(a_1, \ldots, a_{n-1}, 1) \in R^n :$ there is $a_n \in R$ such that $(a_1, \ldots, a_n) \in U_n\}$.

Under an arbitrary ring, consider the complex $C(\mathcal{U}, M)$ (see Definition 1.3). In our main results (Theorem 2.2 and 2.4), we investigate the relationship between the modules of generalized fractions ($U_n^{-n}M$, $U_{n-1}[1]^{-n}M$, $\mathrm{Im}\ e^{n-1}$, $\mathrm{Ker}\ e^n$ and $\mathrm{Ker}\ e^{n-1}/\mathrm{Im}\ e^{n-2}$) and the general local cohomology modules of such modules.

That is, we have

$$\mathrm{Ker}\ e^{n-1}/\mathrm{Im}\ e^{n-2} \cong \bigcup_{(a_1,\ldots,a_n)\in U_n} \mathrm{Ann}_{U_{n-1}[1]^{-n}M}(a_1, \ldots, a_n)R$$

and

$$U_n[1]^{-n-1}M \cong H^1_{\Phi(U_n)}(\mathrm{Im}\ e^{n-1}).$$

In particular, under a Noetherian ring, we have for $n \geq 1$

$$U_n[1]^{-n-1}M \cong H^1_{\Phi(U_n)}(U_{n-1}[1]^{-n}M) \cong H^1_{\Phi(U_n)}(\mathrm{Im}\ e^{n-1});$$

$$U_n^{-n}M \cong \varinjlim_{\mathfrak{a}\in\Phi(U_n)} \mathrm{Hom}_R(\mathfrak{a}, U_n^{-n}M) \cong \varinjlim_{\mathfrak{a}\in\Phi(U_n)} \mathrm{Hom}_R(\mathfrak{a}, \mathrm{Ker}\ e^n)$$

$$\cong \varinjlim_{\mathfrak{a}\in\Phi(U_n)} \mathrm{Hom}_R(\mathfrak{a}, \mathrm{Im}\ e^{n-1}) \cong \varinjlim_{\mathfrak{a}\in\Phi(U_n)} \mathrm{Hom}_R(\mathfrak{a}, U_{n-1}[1]^{-n}M);$$

and

$$H^i_{\Phi(U_n)}(U_n^{-n}M) \cong H^i_{\Phi(U_n)}(U_{n-1}[1]^{-n}M) \cong H^i_{\Phi(U_n)}(\mathrm{Im}\ e^{n-1})$$
$$\cong H^i_{\Phi(U_n)}(\mathrm{Ker}\ e^n) \text{ for all } i \geq 2.$$

The notation and terminology about the modules of generalized fractions follow [**16**].

1. Preliminaries

We use ^{T} to denote matrix transpose and $D_n(R)$ to denote the set of all $n \times n$ lower triangular matrices over R. For $H \in D_n(R)$, $|H|$ denotes the determinant of H. Let N denote the set of positive integers.

DEFINITION 1.1. [**16**, 2.1]. Let n be a positive integer. A non-empty subset U_n of R^n is said to be *triangular* if

(i) whenever $(a_1, \ldots, a_n) \in U_n$, then $(a_1^{\alpha_1}, \ldots, a_n^{\alpha_n}) \in U_n$ for all choices of positive integers $\alpha_1, \ldots, \alpha_n$; and

(ii) whenever $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n) \in U_n$, then there exist $(c_1, \ldots, c_n) \in U_n$ and H, $\mathrm{K} \in D_n(R)$ such that $H[a_1\ \ldots\ a_n]^T = [c_1\ \ldots\ c_n]^T = \mathrm{K}[b_1\ \ldots\ b_n]^T$.

LEMMA 1.2. *Let R be a ring and M an R-module. Let U_n be a triangular subset of R^n. Suppose $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are elements of U_n such that $H[a_1\ \ldots\ a_n]^T = [b_1\ \ldots\ b_n]^T$ for some $H \in D_n(R)$. Then we have*

(1) [**16**, 2.8] $\dfrac{m}{(a_1, \ldots, a_n)} = \dfrac{|H|m}{(b_1, \ldots, b_n)}$ *in* $U_n^{-n}M$.

(2) [**16**, 3.3(ii) *and* **15**, 2.2] *If* $m \in (a_1, \ldots, a_{n-1})M$ *then* $\dfrac{m}{(a_1, \ldots, a_n)} = 0$ *in* $U_n^{-n}M$. *In particular, if each element of* U_n *is a poor* M*-sequence, then the converse holds.*

DEFINITION 1.3. [**13**, p. 52]. Let R be a ring and M an R-module. A family $\mathcal{U} = (U_i)_{i\geq 1}$ is called a *chain of triangular subsets* on R if the following conditions are satisfied:

(i) U_i is a triangular subset of R^i for all $i \in N$;

(ii) $(1) \in U_1$;

(iii) whenever $(a_1, \ldots, a_i) \in U_i$ with $i \in N$, then $(a_1, \ldots, a_i, 1) \in U_{i+1}$; and

(iv) whenever $(a_1, \ldots, a_i) \in U_i$ with $1 < i \in N$, then $(a_1, \ldots, a_{i-1}) \in U_{i-1}$.

Each U_i leads to a module of generalized fractions $U_i^{-i}M$ and we can obtain a complex by Lemma 1.2(2);

$$0 \xrightarrow{e^{-1}} M \xrightarrow{e^0} U_1^{-1}M \xrightarrow{e^1} U_2^{-2}M \longrightarrow \cdots$$
$$\longrightarrow U_i^{-i}M \xrightarrow{e^i} U_{i+1}^{-i-1}M \longrightarrow \cdots$$

for which $e^0(m) = \dfrac{m}{(1)}$ for all $m \in M$ and

$$e^i\left(\frac{x}{(a_1, \ldots, a_i)}\right) = \frac{x}{(a_1, \ldots, a_i, 1)}$$

for all $i \in N$, $x \in M$ and $(a_1, \ldots, a_i) \in U_i$.

Let $C(\mathcal{U}, M)$ denote the above complex and $H_U^i(M)$ the i-th cohomology module of this complex. That is $H_U^i(M) = \operatorname{Ker} e^i / \operatorname{Im} e^{i-1}$.

For a given triangular subset U_n of R^n, let

$U_n[1] = \{(a_1, \ldots, a_n, 1) \in R^{n+1} : (a_1, \ldots, a_n) \in U_n\}$ and

$U_{n-1}[1] = \{(a_1, \ldots, a_{n-1}, 1) \in R^n$: there is $a_n \in R$ such that $(a_1, \ldots, a_n) \in U_n\}$.

Then clearly $U_n[1]$ and $U_{n-1}[1]$ are triangular subsets of R^{n+1} and R^n respectively. We interpret $U_0[1]^{-1}M = M$ and $U_0^0 M = M$.

DEFINITION 1.4. [**1**, 2.1 and **15**, 1.2]. A non-empty set Φ of ideals of R is called a *system of ideals of* R if whenever $\mathfrak{a}, \mathfrak{b} \in \Phi$ there is $\mathfrak{c} \in \Phi$ such that $\mathfrak{c} \subset \mathfrak{ab}$.

Given such a system of ideals Φ, for every R-module M, we define

$$L_\Phi(M) = \{m \in M : m\mathfrak{a} = 0 \text{ for some } \mathfrak{a} \in \Phi\} = \bigcup_{\mathfrak{a}\in\Phi} (0 :_M \mathfrak{a})$$

and

$$G_{\Phi}(M) = \varinjlim_{\mathfrak{a}\in\Phi} \operatorname{Hom}_R(\mathfrak{a}, M).$$

Then L_{Φ} and G_{Φ} are additive, left exact functors from the category of all R-modules and R-homomorphisms to itself. The functor L_{Φ} is called the *general local cohomology functor with respect to* Φ and G_{Φ} the *generalized ideal transform determined by* Φ, or, more briefly, the Φ-*transform*.

For any R-module M, the modules $H^i_{\Phi}(M)$ are called *general local cohomology modules* of M, where H^i_{Φ} is the i-th right derived functor of L_{Φ}. That is, by [**1**, 2.3 and **2**, 2.1] we have

$$H^i_{\Phi}(\ ) = \varinjlim_{\mathfrak{a}\in\Phi} \operatorname{Ext}^i_R(R/\mathfrak{a},\ ) = \varinjlim_{\mathfrak{a}\in\Phi} H^i_{\mathfrak{a}}(\ ).$$

We say that an R-module M is a Φ-*torsion module* if $L_{\Phi}(M) = M$ [**14**, 1.4(i)].

EXAMPLE 1.5. (1) $\Phi = \{\mathfrak{a}^i : \mathfrak{a}$ is an ideal of R and $i \in N\}$ is a system of ideals of R.

(2) [**3**, Theorem] $\Phi(U_n) = \{(a_1, \ldots, a_n)R : (a_1, \ldots, a_n) \in U_n\}$ is a system of ideals of R, where U_n is a triangular subset of R^n.

LEMMA 1.6. *Let R be Noetherian and M an R-module. Then we have the following.*

(1) [**19**, 3.1.6 *and* **14**, 1.4] *If M is a Φ-torsion module, then $H^i_{\Phi}(M) = 0$ for all $i > 0$.*
(2) [**1**, 2.7] *If dim $M = d$, then $H^i_{\Phi}(M) = 0$ for all $i > d$.*

PROPOSITION 1.7. *Let R be Noetherian and M an R-module. Then we have the following.*

(1) $\operatorname{Supp}(H^i_{\Phi}(M)) \subset \bigcup_{\mathfrak{a}\in\Phi} V(\mathfrak{a})$.
(2) *If* $\operatorname{Supp}(M) \subset \bigcup_{\mathfrak{a}\in\Phi} V(\mathfrak{a})$, *then $H^i_{\Phi}(M) = 0$ for all $i > 0$.*

Proof. (1) By [**12**, p.85 3.13] and [**9**, 35.5] we have

$$\operatorname{Supp}(H^i_{\Phi}(M)) = \operatorname{Supp}(\varinjlim_{\mathfrak{a}\in\Phi} H^i_{\mathfrak{a}}(M)) \subset \bigcup_{\mathfrak{a}\in\Phi} \operatorname{Supp}(H^i_{\mathfrak{a}}(M)) \subset \bigcup_{\mathfrak{a}\in\Phi} V(\mathfrak{a}).$$

(2) Since M is Φ-torsion module, this follows from Lemma 1.6(1). □

LEMMA 1.8. *Let R be a ring and M an R-module. Then, in the complex $C(\mathcal{U}, M)$, for $n \geq 0$ we have the following.*

(1) [**6**, 2.4] $\operatorname{Supp}(U_{n+1}^{-n-1}M) \subset \operatorname{Supp}(U_n[1]^{-n-1}M) \subset \{\mathfrak{p} \in \operatorname{Supp}(M) : ht_M\mathfrak{p} \geq n\}$.

(2) [**6**, 2.8] $\operatorname{Ass}(U_{n+1}^{-n-1}M) = \operatorname{Ass}(\operatorname{Im} e^n) = \operatorname{Ass}(\operatorname{Ker} e^{n+1})$.

(3) [**6**, 2.7] *For each* $\frac{m}{(a_1, \ldots, a_n)} + \operatorname{Im} e^{n-1} \in H_U^n(M)$, *there are* $(b_1, \ldots, b_{n+1}) \in U_{n+1}$ *and* $H \in D_n(R)$ *such that* $H[a_1 \ \ldots \ a_n]^T = [b_1 \ \ldots \ b_n]^T$ *and*

$$(b_1, \ldots, b_{n+1})R \subset \left(\operatorname{Im} e^{n-1} : \frac{m}{(a_1, \ldots, a_n)}\right).$$

PROPOSITION 1.9. *Let R be Noetherian and M an R-module. Let Φ be a system of ideals of R and $d = dim\ M$. Then, in the complex $C(\mathcal{U}, M)$, for $n \geq 0$ we have the following.*

(1) $H_\Phi^i(\operatorname{Ker} e^n / \operatorname{Im} e^{n-1}) = 0$ *for all* $i > d - n$.

(2) $H_\Phi^i(U_{n+1}^{-n-1}M) = H_\Phi^i(U_n[1]^{-n-1}M) = H_\Phi^i(\operatorname{Ker} e^{n+1})$
$= H_\Phi^i(\operatorname{Im} e^n) = 0$ *for all* $i > d - n$.

Proof. By Lemma 1.8 we have $\operatorname{Supp}(\operatorname{Im} e^n) = \operatorname{Supp}(\operatorname{Ker} e^{n+1}) = \operatorname{Supp}(U_{n+1}^{-n-1}M) \subset \operatorname{Supp}(U_n[1]^{-n-1}M) \subset \{\mathfrak{p} \in \operatorname{Supp}(M) : ht_M\mathfrak{p} \geq n\}$ and $\operatorname{Supp}(\operatorname{Ker} e^n/\operatorname{Im} e^{n-1}) \subset \operatorname{Supp}(U_n[1]^{-n-1}M)$ by [**6**, 2.8(5)]. Therefore the results follow from Lemma 1.6(2). □

REMARK. Let R be Noetherian and M a finitely generated R-module. In the complex $C(\mathcal{U}, M)$, assume that $\mathcal{U} = ((U_s)_i)_{i\geq 1}$, $((U_h)_i)_{i\geq 1}$, $((U_r)_i)_{i\geq 1}$ or $((U_f)_i)_{i\geq 1}$ where R is local in the case $((U_f)_i)_{i\geq 1}$ (see [**6**, Example 1.3]). Then we have

$$H_\Phi^i(\operatorname{Ker} e^n/\operatorname{Im} e^{n-1}) = 0 \quad \text{for all } i \geq d - n.$$

For, from Lemma 1.8(3) we have easily $dim(\operatorname{Ker} e^n/\operatorname{Im} e^{n-1}) < d - n$.

PROPOSITION 1.10. *Let R be a ring and M an R-module. For a fixed positive integer n, assume U_n is a triangular subsets of R^n. Let $\operatorname{Ass}_f(U_n^{-n}M) = \{\mathfrak{q} \in \operatorname{Supp}(M) : \mathfrak{q}$ is a weakly associated prime ideal of $U_n^{-n}M$ in the sense of* [**4**, p.289 Exercise 17]$\}$ *and* $\mathfrak{p} \in \operatorname{Ass}_f(U_n^{-n}M)$, *that is, $\mathfrak{p}$ is a minimal prime over $(0:x)$ for some $0 \neq x \in U_n^{-n}M$. Then we have, for all $(a_1, \dots, a_n) \in U_n$,*

$$(a_1, \dots, a_n)R \not\subset \mathfrak{p}.$$

In particular, for all $0 \neq y \in U_n^{-n}M$, we have $(a_1, \dots, a_n)R \not\subset (0:y)$.

Proof. By assumption there is $\dfrac{m}{(b_1, \dots, b_n)} \in U_n^{-n}M$ such that

$$\left(0 : \frac{m}{(b_1, \dots, b_n)}\right) \subset \mathfrak{p} \quad \text{and} \quad \sqrt{\left(0 : \frac{m}{(b_1, \dots, b_n)}\right)_{\mathfrak{p}}} = \mathfrak{p}R_{\mathfrak{p}}.$$

Suppose that $(a_1, \dots, a_n)R \subset \mathfrak{p}$ for some $(a_1, \dots, a_n) \in U_n$. Then by the definition of the triangular subsets, there is $(c_1, \dots, c_n) \in U_n$ and H, $\mathrm{K} \in D_n(R)$ such that $H[a_1 \ \dots \ a_n]^T = [c_1 \ \dots \ c_n]^T = \mathrm{K}[b_1 \ \dots \ b_n]^T$. Hence we obtain $(c_1, \dots, c_n)R \subset \mathfrak{p}$. By Lemma 1.2(1) we have

$$\sqrt{\left(0 :_{R_{\mathfrak{p}}} \frac{m}{(b_1, \dots, b_n)}\right)} = \sqrt{\left(0 :_{R_{\mathfrak{p}}} \frac{|\mathrm{K}|m}{(c_1, \dots, c_n)}\right)} = \mathfrak{p}R_{\mathfrak{p}},$$

where $\dfrac{m}{(b_1, \dots, b_n)} = \dfrac{|\mathrm{K}|m}{(c_1, \dots, c_n)}$ is regarded as the canonical image in the $R_{\mathfrak{p}}$-module. Since $c_n \in \mathfrak{p}$, there are $r \in R \setminus \mathfrak{p}$ and $t \in N$ such that

$$\frac{{c_n}^t r|\mathrm{K}|m}{(c_1, \dots, c_n)} = 0.$$

Then by [**17**, 2.1] we get $\dfrac{r|\mathrm{K}|m}{(c_1, \dots, c_n)} = 0$. That is, we have the following contradiction.

$$r \in \left(0 : \frac{|\mathrm{K}|m}{(c_1, \dots, c_n)}\right) = \left(0 : \frac{m}{(b_1, \dots, b_n)}\right) \subset \mathfrak{p}. \ \square$$

From now on, let $\Phi_U = (\Phi(U_i))_{i \geq 1}$ be the family of systems of ideals of R induced by a chain $\mathcal{U} = (U_i)_{i \geq 1}$ of triangular subsets on R as in Example 1.5(2).

REMARK. In [**6**, 2.8], using Proposition 1.10, we have the same results with $\mathrm{Ass}_f(U_{n+1}^{-n-1}M)$ instead of $\mathrm{Ass}(U_{n+1}^{-n-1}M)$.

LEMMA 1.11. *Let R be a ring and M an R-module. Let $\Phi_U = (\Phi(U_i))_{i\geq 1}$ be as above. Then, in the complex $C(\mathcal{U}, M)$, we have the following.*

(1) *For $1 \leq n \leq i$, we have* $H^0_{\Phi(U_i)}(U_n^{-n}M) = H^0_{\Phi(U_i)}(\mathrm{Ker}\ e^n) = H^0_{\Phi(U_i)}(\mathrm{Im}\ e^{n-1}) = 0.$

(2) *For $1 \leq i < n$, we have $U_n^{-n}M$, $\mathrm{Im}\ e^{n-1}$, $\mathrm{Ker}\ e^n, U_{n-1}[1]^{-n}M$ and $H_U^{n-2}(M)$ are $\Phi(U_i)$-torsion modules.*

Proof. (1) Since $\mathrm{Im}\ e^{n-1} \subset \mathrm{Ker}\ e^n \subset U_n^{-n}M$, the results follow immediately from Proposition 1.10.

(2) Since by Lemma 1.2(2) for all $\dfrac{m}{(a_1, \ldots, a_n)} \in U_n^{-n}M$

$$(a_1, \ldots, a_i)R \cdot \frac{m}{(a_1, \ldots, a_i, \ldots, a_n)} = 0$$

where the ideal $(a_1, \ldots, a_i)R \in \Phi(U_i)$ is induced from $(a_1, \ldots, a_n) \in U_n$, we get $U_n^{-n}M$, $\mathrm{Im}\ e^{n-1}$ and $\mathrm{Ker}\ e^n$ are $\Phi(U_i)$-torsion modules.

Next, using the same method, we have $U_{n-1}[1]^{-n}M$ is a $\Phi(U_i)$-torsion module.

For $H_U^{n-2}(M)$, by Lemma 1.8(3) for all $x \in H_U^{n-2}(M)$ we have

$$(a_1, \ldots, a_{n-1})R \subset (0 : x)$$

for some $(a_1, \ldots, a_{n-1})R \in \Phi(U_{n-1})$. Hence $H_U^{n-2}(M)$ is a $\Phi(U_i)$-torsion module for $i < n$. □

COROLLARY 1.12. *Let R be Noetherian and M an R-module. Let $\Phi_U = (\Phi(U_i))_{i\geq 1}$ be as above. Then, in the complex $C(\mathcal{U}, M)$, we have the following.*

(1) *If M is a $\Phi(U_i)$-torsion module, then $G_{\Phi(U_i)}(M) = 0$.*

(2) *For $1 \leq i < n$, we have $G_{\Phi(U_i)}(U_n^{-n}M) = G_{\Phi(U_i)}(\mathrm{Ker}\ e^n) = G_{\Phi(U_i)}(\mathrm{Im}\ e^{n-1}) = G_{\Phi(U_i)}(U_{n-1}[1]^{-n}M) = G_{\Phi(U_i)}(H_U^{n-2}(M)) = 0$.*

Proof. (1) By [**19**, 3.1.10] we have the following exact sequence;

$$0 \longrightarrow M/L_{\Phi(U_i)}(M) \longrightarrow G_{\Phi(U_i)}(M) \longrightarrow H^1_{\Phi(U_i)}(M) \longrightarrow 0.$$

Hence the assertion follows from Lemma 1.6(1).

(2) These immediately follow from (1) and Lemma 1.11(2). □

2. Main results

LEMMA 2.1. *Let R be a ring and M an R-module. Let $\Phi_U = (\Phi(U_i))_{i\geq 1}$ be the family of systems of ideals of R induced by a chain $\mathcal{U} = (U_i)_{i\geq 1}$ of triangular subsets on R. Then, in the complex $C(\mathcal{U}, M)$, for $n \geq 1$ we have the following.*

(1) [**3**, Theorem] $U_n[1]^{-n-1}M \cong H^n_{\Phi(U_n)}(M)$.

(2) [**7**, 3.3] $U_n[1]^{-n-1}M \cong U_n^{-n}M/\mathrm{Im}\ e^{n-1}$.

(3) [**19**, 3.3.8] $U_n^{-n}M \cong G_{\Phi(U_n)}(\mathrm{Im}\ e^{n-1})$.

THEOREM 2.2. *Let R be a ring and M an R-module. Let $\Phi_U = (\Phi(U_i))_{i\geq 1}$ be the family of systems of ideals of R induced by a chain $\mathcal{U} = (U_i)_{i\geq 1}$ of triangular subsets on R. Then, in the complex $C(\mathcal{U}, M)$, for $n \geq 1$ we have the following.*

(1) $H^{n-1}_U(M) \cong \bigcup_{(a_1,\dots,a_n)\in U_n} \mathrm{Ann}_{U_{n-1}[1]^{-n}M}(a_1, \dots, a_n)R$.

(2) $U_n[1]^{-n-1}M \cong H^1_{\Phi(U_n)}(\mathrm{Im}\ e^{n-1})$.

Proof. (1) Consider the following exact sequence

$$0 \longrightarrow \mathrm{Ker}\ e^{n-1}/\mathrm{Im}\ e^{n-2} \longrightarrow U_{n-1}^{-n+1}M/\mathrm{Im}\ e^{n-2} \longrightarrow U_{n-1}^{-n+1}M/\mathrm{Ker}\ e^{n-1} \longrightarrow 0.$$

Since $\mathrm{Ker}\ e^{n-1}/\mathrm{Im}\ e^{n-2} \cong H^{n-1}_U(M)$, $U_{n-1}^{-n+1}M/\mathrm{Im}\ e^{n-2} \cong U_{n-1}[1]^{-n}M$ by Lemma 2.1(2) and $U_{n-1}^{-n+1}M/\mathrm{Ker}\ e^{n-1} \cong \mathrm{Im}\ e^{n-1}$, we have the following long exact sequence

(*)

$$\begin{aligned} 0 &\longrightarrow H^0_{\Phi(U_n)}(H^{n-1}_U(M)) \longrightarrow H^0_{\Phi(U_n)}(U_{n-1}[1]^{-n}M) \\ &\longrightarrow H^0_{\Phi(U_n)}(\mathrm{Im}\ e^{n-1}) \longrightarrow H^1_{\Phi(U_n)}(H^{n-1}_U(M)) \\ &\longrightarrow H^1_{\Phi(U_n)}(U_{n-1}[1]^{-n}M) \longrightarrow H^1_{\Phi(U_n)}(\mathrm{Im}\ e^{n-1}) \longrightarrow \cdots . \end{aligned}$$

Since $H_U^{n-1}(M)$ is a $\Phi(U_n)$-torsion module and $H^0_{\Phi(U_n)}(\text{Im } e^{n-1}) = 0$ by Lemma 1.11, we easily have the conclusion.

(2) From [**19**, 3.1.10], we have the following exact sequence

$$0 \longrightarrow \text{Im } e^{n-1}/H^0_{\Phi(U_n)}(\text{Im } e^{n-1}) \longrightarrow G_{\Phi(U_n)}(\text{Im } e^{n-1}) \\ \longrightarrow H^1_{\Phi(U_n)}(\text{Im } e^{n-1}) \longrightarrow 0.$$

Since $H^0_{\Phi(U_n)}(\text{Im } e^{n-1}) = 0$ by Lemma 1.11(1) and $G_{\Phi(U_n)}(\text{Im } e^{n-1}) \cong U_n^{-n}M$ by Lemma 2.1(3), we have

$$H^1_{\Phi(U_n)}(\text{Im } e^{n-1}) \cong U_n^{-n}M/\text{Im } e^{n-1} \cong U_n[1]^{-n-1}M.$$

by Lemma 2.1(2). □

The next Exactness theorem was proved by Sharp and Zakeri [**18**, 3.3] under the condition of a Noetherian ring and O'carroll [**11**, 3.1] gave a simple proof for an arbitrary ring. We had shown a refinement of the result of O'carroll [**6**, 2.13]. We describe another proof of this Exactness theorem using Theorem 2.2(1).

COROLLARY 2.3. *Let R be a ring and M an R-module. Let $\mathcal{U} = (U_i)_{i\geq 1}$ be a chain of triangular subsets on R. Then $C(\mathcal{U}, M)$ is exact if and only if for all $i \geq 1$ each element of U_i is a poor M-sequence.*

Proof. (⇒) We prove by induction on i. In case $i = 1$, by the hypothesis and Theorem 2.2(1), we have $H^0_U(M) \cong \bigcup_{(a_1)\in U_1} \text{Ann}_{U_0[1]^{-1}M}(a_1) = \bigcup_{(a_1)\in U_1} \text{Ann}_M(a_1) = 0$. Hence each element of U_1 is a poor M-sequence.

Suppose that each element of U_{i-1} is a poor M-sequence and hence each element of $U_{i-1}[1]$ is a poor M-sequence. Note that by Lemma 1.2(2) for all $(b_1, \ldots, b_{i-1}) \in U_{i-1}$

$$\frac{m}{(b_1, \ldots, b_{i-1}, 1)} \neq 0 \text{ in } U_{i-1}[1]^{-i}M \Longleftrightarrow m \notin (b_1, \ldots, b_{i-1})M.$$

Let $(a_1, \ldots, a_i) \in U_i$. Then we may assume that $\{a_1, \ldots, a_{i-1}\}$ is an M-sequence. Therefore it is sufficient to show that if $m \notin (a_1, \ldots, a_{i-1})M$, then $a_i m \notin (a_1, \ldots, a_{i-1})M$.

Assume that $m \notin (a_1, \ldots, a_{i-1})M$. Hence by the above note we have $\dfrac{m}{(a_1, \ldots, a_{i-1}, 1)} \neq 0$ in $U_{i-1}[1]^{-i}M$.

On the other hand, by the hypothesis and Theorem 2.2(1) we have

$$H_U^{i-1}(M) \cong \bigcup_{(b_1, \ldots, b_i) \in U_i} \operatorname{Ann}_{U_{i-1}[1]^{-i}M}(b_1, \ldots, b_i)R = 0.$$

Hence we have $(a_1, \ldots, a_i)R \cdot \dfrac{m}{(b_1, \ldots, b_{i-1}, 1)} \neq 0$ for all non-zero element $\dfrac{m}{(b_1, \ldots, b_{i-1}, 1)}$ of $U_{i-1}[1]^{-i}M$. In particular, we have

$$\frac{a_i m}{(a_1, \ldots, a_{i-1}, 1)} \neq 0 \quad \text{for} \quad \frac{m}{(a_1, \ldots, a_{i-1}, 1)} \in U_{i-1}[1]^{-i}M,$$

since $(a_1, \ldots, a_{i-1})R \cdot \dfrac{m}{(a_1, \ldots, a_{i-1}, 1)} = 0$ by Lemma 1.2(2). Hence we obtain $a_i m \notin (a_1, \ldots, a_{i-1})M$ by the above note.

($\Leftarrow$) By Theorem 2.2(1), it is enough to show that

$$\bigcup_{(a_1, \ldots, a_i) \in U_i} \operatorname{Ann}_{U_{i-1}[1]^{-i}M}(a_1, \ldots, a_i)R = 0 \quad \text{for all } i \geq 1.$$

Assume that, for some $0 \neq \dfrac{m}{(b_1, \ldots, b_{i-1}, 1)} \in U_{i-1}[1]^{-i}M$ and $(a_1, \ldots, a_i) \in U_i$,

$$(a_1, \ldots, a_i)R \cdot \frac{m}{(b_1, \ldots, b_{i-1}, 1)} = 0 \quad \text{in } U_{i-1}[1]^{-i}M.$$

Then from Lemma 1.2(2) we have, in $U_i^{-i}M$,

$$\frac{m}{(b_1, \ldots, b_{i-1}, 1)} \neq 0 \quad \text{and} \quad (a_1, \ldots, a_i)R \ \frac{m}{(b_1, \ldots, b_{i-1}, 1)} = 0.$$

On the other hand, by the definition of triangular subset there are $(c_1, \ldots, c_i) \in U_i$ and H, K $\in D_i(R)$ such that $\mathrm{H}[a_1 \ \ldots \ a_i]^T = [c_1 \ \ldots \ c_i]^T = \mathrm{K}[b_1 \ \ldots \ b_{i-1} \ 1]^T$, since $(b_1, \ldots, b_{i-1}, 1) \in U_i$. Hence we have $(c_1, \ldots, c_i)R \subset (a_1, \ldots, a_i)R$ and then

$$(c_1, \ldots, c_i)R \cdot \frac{m}{(b_1, \ldots, b_{i-1}, 1)} = 0 \quad \text{in } U_i^{-i}M.$$

In particular,

$$\frac{c_i m}{(b_1, \ldots, b_{i-1}, 1)} = \frac{c_i|\mathrm{K}|m}{(c_1, \ldots, c_{i-1}, c_i)} = 0 \quad \text{in } U_i^{-i}M$$

by Lemma 1.2(1). Therefore we get $c_i|\mathrm{K}|m \in (c_1, \ldots, c_{i-1})M$ by Lemma 1.2(2). Hence we obtain $|\mathrm{K}|m \in (c_1, \ldots, c_{i-1})M$, since $\{c_1, \ldots, c_i\}$ is a poor M-sequence. This leads to the following contradiction;

$$\frac{m}{(b_1, \ldots, b_{i-1}, 1)} = \frac{|\mathrm{K}|m}{(c_1, \ldots, c_{i-1}, c_i)} = 0 \text{ in } U_i^{-i}M$$

by Lemma 1.2(2). □

THEOREM 2.4. *Let R be Noetherian and M an R-module. Let $\Phi_U = (\Phi(U_i))_{i\geq 1}$ be the family of systems of ideals of R induced by a chain $\mathcal{U} = (U_i)_{i\geq 1}$ of triangular subsets on R. Then, in the complex $C(\mathcal{U}, M)$, for $n \geq 1$ we have the following.*

(1) $U_n[1]^{-n-1}M \cong H^1_{\Phi(U_n)}(U_{n-1}[1]^{-n}M) \cong H^1_{\Phi(U_n)}(\mathrm{Im}\ e^{n-1})$.

(2) $H^1_{\Phi(U_n)}(U_n^{-n}M) = 0$.

(3) $\mathrm{Im}\ e^n \cong H^1_{\Phi(U_n)}(\mathrm{Ker}\ e^n)$.

(4) $U_n^{-n}M \cong G_{\Phi(U_n)}(U_n^{-n}M) \cong G_{\Phi(U_n)}(\mathrm{Ker}\ e^n)$
$\cong G_{\Phi(U_n)}(\mathrm{Im}\ e^{n-1}) \cong G_{\Phi(U_n)}(U_{n-1}[1]^{-n}M)$.

(5) $H^i_{\Phi(U_n)}(U_{n-1}[1]^{-n}M) \cong H^i_{\Phi(U_n)}(\mathrm{Im}\ e^{n-1}) \cong H^i_{\Phi(U_n)}(U_n^{-n}M)$
$\cong H^i_{\Phi(U_n)}(\mathrm{Ker}\ e^n)$, *for all* $i \geq 2$.

(6) $0 \longrightarrow H^n_U(M) \longrightarrow H^1_{\Phi(U_{n+1})}(\mathrm{Im}\ e^{n-1}) \longrightarrow H^1_{\Phi(U_{n+1})}(\mathrm{Ker}\ e^n)$
$\longrightarrow 0$ *is a short exact sequence.*

(7) $H^i_{\Phi(U_{n+1})}(\mathrm{Im}\ e^{n-1}) \cong H^i_{\Phi(U_{n+1})}(\mathrm{Ker}\ e^n)$, *for all* $i \geq 2$.

Proof. ((1) and the first isomorphism of (5)) For (1), by Theorem 2.2(2), it is enough to show that

$$H^1_{\Phi(U_n)}(U_{n-1}[1]^{-n}M) \cong H^1_{\Phi(U_n)}(\mathrm{Im}\ e^{n-1}).$$

But this and the first isomorphism of (5) follow from the above long exact sequence (∗) and Lemma 1.6(1), since $H^{n-1}_U(M)$ is a $\Phi(U_n)$-torsion module.

((2) and the second isomorphism of (5)) Consider the following short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Im}\ e^{n-1} & \longrightarrow & U_n^{-n}M & \longrightarrow & U_n^{-n}M/\mathrm{Im}\ e^{n-1} & \longrightarrow & 0 \\ & & & & & & \| & & \\ & & & & & & U_n[1]^{-n-1}M & & \end{array}$$

where the isomorphism follows from Lemma 2.1(2) again. Then, since $U_n[1]^{-n-1}M$ is a $\Phi(U_n)$-torsion module and $H^0_{\Phi(U_n)}(U_n^{-n}M) = 0$ by Lemma 1.11, using the long exact sequence induced from the above short exact sequence, we have the second isomorphism of (5) and the following short exact sequence

$$0 \longrightarrow U_n[1]^{-n-1}M \longrightarrow H^1_{\Phi(U_n)}(\mathrm{Im}\ e^{n-1}) \longrightarrow H^1_{\Phi(U_n)}(U_n^{-n}M) \longrightarrow 0.$$

Then, from (1) and the above short exact sequence, we have $H^1_{\Phi(U_n)}(U_n^{-n}M) = 0$.

((3) and the third isomorphism of (5)) From (2) and the following short exact sequence

$$0 \longrightarrow \mathrm{Ker}\ e^n \longrightarrow U_n^{-n}M \longrightarrow \mathrm{Im}\ e^n \longrightarrow 0,$$

we have the results by means of the long exact sequence induced by this short exact sequence, since $\mathrm{Im}\ e^n$ is a $\Phi(U_n)$-torsion module (Lemma 1.11(2)) and $H^0_{\Phi(U_n)}(U_n^{-n}M) = 0$ (Lemma 1.11(1)).

(4) By [**15**, p. 176] , we have the following exact sequence

$$(**) \qquad 0 \longrightarrow L_{\Phi(U_n)}(U_n^{-n}M) \longrightarrow U_n^{-n}M \longrightarrow G_{\Phi(U_n)}(U_n^{-n}M) \longrightarrow H^1_{\Phi(U_n)}(U_n^{-n}M) \longrightarrow 0.$$

From the above sequence, we have

$$U_n^{-n}M \cong G_{\Phi(U_n)}(U_n^{-n}M),$$

since $L_{\Phi(U_n)}(U_n^{-n}M) = H^1_{\Phi(U_n)}(U_n^{-n}M) = 0$ by (2) and Lemma 1.11(1).

Next, since $L_{\Phi(U_n)}(\text{Ker } e^n) = 0$ and $H^1_{\Phi(U_n)}(\text{Ker } e^n) \cong \text{Im } e^n$ by (3), from the above sequence (∗∗) with $U_n^{-n}M$ replaced by Ker e^n we have the following exact sequence

$$0 \longrightarrow \text{Ker } e^n \longrightarrow G_{\Phi(U_n)}(\text{Ker } e^n) \longrightarrow \text{Im } e^n \longrightarrow 0.$$

Hence we get

$$G_{\Phi(U_n)}(\text{Ker } e^n)/\text{Ker } e^n \cong \text{Im } e^n \cong U_n^{-n}M/\text{Ker } e^n.$$

That is $U_n^{-n}M \cong G_{\Phi(U_n)}(\text{Ker } e^n)$.

The third isomorphism is Lemma 2.1(3).

For the last isomorphism, using [**19**, 3.1.10] again, we have the following exact sequence

$$0 \longrightarrow U_{n-1}[1]^{-n}M/L_{\Phi(U_n)}(U_{n-1}[1]^{-n}M) \longrightarrow G_{\Phi(U_n)}(U_{n-1}[1]^{-n}M) \longrightarrow H^1_{\Phi(U_n)}(U_{n-1}[1]^{-n}M) \longrightarrow 0.$$

On the other hand, we have

$$U_{n-1}[1]^{-n}M/L_{\Phi(U_n)}(U_{n-1}[1]^{-n}M) \cong \text{Im } e^{n-1}$$

by Lemma 2.1(2) and Theorem 2.2(1). We also get

$$H^1_{\Phi(U_n)}(U_{n-1}[1]^{-n}M) \cong U_n[1]^{-n-1}M \cong U_n^{-n}M/\text{Im } e^{n-1}$$

by (1) and Lemma 2.1(2). Hence by the five lemma we have

$$U_n^{-n}M \cong G_{\Phi(U_n)}(U_{n-1}[1]^{-n}M).$$

((6) and (7)) From the following short exact sequence

$$0 \longrightarrow \text{Im } e^{n-1} \longrightarrow \text{Ker } e^n \longrightarrow H^n_U(M) \longrightarrow 0$$

we have the results, since $H^n_U(M)$ is a $\Phi(U_{n+1})$-torsion module and $H^0_{\Phi(U_{n+1})}(\text{Ker } e^n) = 0$ by Lemma 1.11. □

Remark. The last proof of Theorem 2.4(4) is another simple proof of [**19**, 5.2.3] and [**15**, 3.3].

Corollary 2.5. *Let R and M be as above. Then for $n \geq 1$ we have the following.*

$$H^n_U(M) = 0 \Leftrightarrow H^1_{\Phi(U_{n+1})}(\text{Im } e^{n-1}) \hookrightarrow H^1_{\Phi(U_{n+1})}(\text{Ker } e^n)$$
$$\Leftrightarrow H^1_{\Phi(U_{n+1})}(\text{Im } e^{n-1}) \cong H^1_{\Phi(U_{n+1})}(\text{Ker } e^n)$$

Proof. These immediately follow from Theorem 2.4(6). □

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