J. Korean Math. Soc. **33** (1996), No. 4, pp. 735–745

ON PROJECTIVE REPRESENTATIONS OF A FINITE GROUP AND ITS SUBGROUPS II

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1. Introduction

This is the sequel to our paper "On projective representations of a group and its subgroups I" [4]. In Section 4 [4] we proved some global properties on regularity condition. The purpose of this paper is to study local properties, that is, we shall ask how the regularity condition on subgroups is related to that on group. Throughout the paper we use the same notations as in [4].

2. Regularity Conditions on Groups: Local condition

THEOREM 1. *Let $G = H \times K$ with normal subgroups H and K. Let $\beta \in Z^2(H, F^*)$. If every F-class of H is (F,β)-regular then every F-class of G is $(F, Cor\ \beta)$-regular, where $Cor = Cor_{H,G}$.*

Proof. Choose integers n and $m(\sigma)$ for G satisfying the condition (A) in [4]. Then these integers work for H, too. For $g \in G$, choose $(\sigma, x) \in \mathcal{G} \times G$ such that $x^{-1}g^{m(\sigma^{-1})}x = g$, and write $m(\sigma^{-1}) = m$. Choose K as a transversal of H in G with respect to which $\mathrm{Cor}_{H,G}$ is defined. Write $g = h\bar{g}$ and $x = z\bar{x}$ so that $z^{-1}h^m z = h \in H$ and $\bar{x}^{-1}\bar{g}^m\bar{x} = \bar{g} \in K$, for $h, z \in H$, $\bar{g}, \bar{x} \in K$. Then

$$v_H(h)^{\sigma^{-1}} v_H(h)^{-m} \prod_{i=1}^{m-1} \beta(h^i, h)\beta(h^m, z) = \beta(z, h)$$

Received April 13, 1995.
1991 AMS Subject Classification: 18B40, 20Cxx.
Key words: projective representation, regularity.
This paper is supported by KOSEF 1994. # 921-0100-001-2.

where $v_H(h)$ is as in (A) with respect to β. Since

$$v_H(h)^{\mu n} = \prod_{i=1}^{n-1} \beta\left(g^i\overline{g^i}^{-1}, g\overline{g}^{-1}\right)^{\mu} = \prod_{i=1}^{n-1} (\mathrm{Cor}\beta)(g^i, g)$$

(here we have used Theorem 5 [4]), we can choose an n-th root $v(g)$ of $\prod_{i=1}^{n-1}(\mathrm{Cor}\beta)(g^i, g)$ such that $v(g) = v_H(h)^{\mu}$. Then

$$v(g)^{\sigma^{-1}} v(g)^{-m} \prod_{i=1}^{m-1} (\mathrm{Cor}\beta)(g^i, g)(\mathrm{Cor}\beta)(g^m, x)$$

$$= \left[v_H(h)^{\sigma^{-1}} v_H(h)^{-m}\right]^{\mu} \cdot \left[\prod_{i=1}^{m-1} \beta(h^i, h)\beta(h^m, z)\right]^{\mu} = \beta(z, h)^{\mu},$$

which equals $(\mathrm{Cor}\beta)(x, g)$. This completes the proof.

COROLLARY 2. *Suppose that G is a finite nilpotent group which is a direct product $G = Y_{q_1} \times \cdots \times Y_{q_t}$ of normal Sylow subgroups Y_{q_i}. If every F-class of Y_{q_i} is (F, α)-regular for $\alpha \in Z^2(Y_{q_i}, F^*)$ then every F-class of G is $(F, \mathrm{Cor}_{Y_{q_i}, G}\alpha)$-regular.*

Since $G = Y_{q_1} \times (Y_{q_2} \times \cdots \times Y_{q_t})$, it is clear by induction on t.

For a partial answer of the main question, we assume that G is a nilpotent group or G is a central product of two subgroups.

THEOREM 3. *Let G be a nilpotent group as in Corollary 2, and let $f \in Z^2(G, F^*)$. If every F-class of Y_q is (F, f_{Y_q})-regular for all q, then every F-class of G is (F, f)-regular.*

Proof. Let $K_j = Y_{q_1} \times \cdots \times Y_{q_{j-1}} \times Y_{q_{j+1}} \times Y_{q_t}$ with $|K_j| = \mu_j$, and take K_j as a transversal of Y_{q_j} in G where $\mathrm{Cor}_{Y_{q_j}, G}$ (write Cor_j for it) is defined. If every F-class of Y_{q_j} is $(F, f_{Y_{q_j}})$-regular, then every F-class of G is $(F, \mathrm{Cor}_j(f_{Y_{q_j}}))$-regular (by Theorem 1) thus every F-class of G is (F, f^{μ_j})-regular, since $\mathrm{Cor}_j(f_{Y_{q_j}})$ is cohomologous to f^{μ_j} (by Lemma 2 and 3 [4]). Choose n and $m(\sigma)$ for G as in (A), and write $m = m(\sigma^{-1})$. Then these integers work for subgroups of G. For $g \in G$, choose $(\sigma, x) \in \mathcal{G} \times G$ with $x^{-1}g^{m(\sigma^{-1})}x = g$. With $v_j(g)$ which is analogues of $v(g)$ for f^{μ_j},

$$v_j(g)^{\sigma^{-1}} v_j(g)^{-m} \prod_{i=1}^{m-1} f^{\mu_j}(g^i, g) f^{\mu_j}(g^m, x) = f^{\mu_j}(x, g).$$

Since $v_j(g)^n = \prod_{i=1}^{n-1} f(g^i, g)^{\mu_j} = v(g)^{n\mu_j}$, there is $v_j'(g)$ such that $v_j'(g) = v(g)^{\mu_j}$, so we may let $v_j(g) = v_j'(g)$. Using integers $c_1, \cdots, c_t$ with $\sum_{j=1}^t \mu_j c_j = 1$, $\prod_{j=1}^t v_j(g)^{c_j} = v(g)^{\sum_{j=1}^t \mu_j c_j} = v(g)$, hence

$$f(x,g) = \left[f^{\sum_{j=1}^t \mu_j c_j}\right](x,g) = \prod_{j=1}^t f^{\mu_j c_j}(x,g)$$

$$= \left[\prod_{j=1}^t v_j(g)^{c_j}\right]^{\sigma^{-1}} \left[\prod_{j=1}^t v_j(g)^{c_j}\right]^{-m}$$

$$\prod_{i=1}^{m-1} \left[\prod_{j=1}^t f^{\mu_j c_j}(g^i, g)\right] \prod_{j=1}^t f^{\mu_j c_j}(g^m, x)$$

$$= v(g)^{\sigma^{-1}} v(g)^{-m} \prod_{i=1}^{m-1} f(g^i, g) f(g^m, x);$$

thus every F-class of G is (F, f)-regular.

Theorem 3 implies that when G is a nilpotent group and $f \in Z^2(G, F^*)$, if the number of irreducible representations of each Sylow subgroup Y_q of G (over F) equals that of irreducible f_{Y_q}-representations of Y_q then the number of irreducible representations of G equals that of irreducible f-representations of G.

COROLLARY 4. *Suppose that $G = HK$ is a central product of two subgroups H and K, and suppose that we choose a transversal S of H in G as a subgroup of K with respect to which $Cor_{H,G} = Cor$ is defined. If every F-class of H is (F, α)-regular for $\alpha \in Z^2(H, F^*)$ then every F-class of G is $(F, Cor\ \alpha)$-regular.*

Proof. Since S is a subgroup of K and $H \cap K \subseteq Z(G)$, we may apply Theorem 5 [4] to this case.

3. Regularity Conditions on Semi-direct Product

For a general finite group G, it is not known whether the converse of Theorem 6 [4] is true, even when G is a semi-direct product and even when F is the complex field C. In this section we shall consider the same question over certain semi-direct cases and over C; such as Frobenius groups and dihedral groups.

Let G be a Frobenius group with Frobenius kernel N and complement H. Then $G = NH$, $N \cap H = \{1\}$ and H acts on the normal subgroup N without fixed points, i.e., $H \cap H^x = 1$ for $x \in G - H$ (here $H^x = x^{-1}Hx$). And we have $C_G(n) \subseteq N$ and $C_G(h) \subseteq H$ for $1 \neq n \in N$, $1 \neq h \in H$. If $|H|$ is even then N is abelian and H posesses a unique involution which is in $Z(H)$.

THEOREM 5. *Let $G = NH$ be a Frobenius group with Frobenius kernel N and Frobenius complement H. Suppose that $|H| = 2$ and every class of N is f-regular for $f \in Z^2(N, C^*)$. Then every class of G is $Cor_{N,G}f$-regular, where $Cor_{N,G}$ is defined by the transversal H of N in G.*

Proof. Since $|H|$ is even, N is abelian and so we have that $C_G(n) = N$ for all $n \in N - \{1\}$. Now let $H = \{1, h\}$. Since $G = N \cup (\cup_{x \in G} H^x)$, it follows that

$$G - N = \{h^x \mid x \in G\} = Nh = \{nh \mid n \in N\},$$

and $C_G(h^x) = H^x = \{1, h^x\}$ for $x \in G$. Thus the pair (g_1, g_2) of elements in G which commutes each other is one of the following:

$$(n, m),\ (h^x, 1),\ (h^x, h^x)$$

where $n, m \in N$. Hence $(\mathrm{Cor}f)(g_1, g_2) = (\mathrm{Cor}f)(g_2, g_1)$ when $g_1 g_2 = g_2 g_1$, thus every class of G is $\mathrm{Cor}f$-regular.

THEOREM 6. *Let $D_n = \langle a, b \mid a^n = b^2 = 1,\ ab^{-1} = a^{-1} \rangle$ be a dihedral group of order 2^n. If every class of $\langle a \rangle$ is f-regular for $f \in Z^2(\langle a \rangle, C^*)$, then every class of G is $Cor_{\langle a \rangle, G}f$-regular, where $Cor_{\langle a \rangle, G}f$ is defined by a transversal $\langle b \rangle$ of $\langle a \rangle$ in G.*

Proof. For odd n, D_n is a Frobenius group with Frobenius kernel $\langle a \rangle$ and Frobenius complement $\langle b \rangle$ where $|\langle b \rangle| = 2$, and the assertion follows from Theorem 5. Suppose $n = 2m$ is even. Then $D_{2m} = \langle a \rangle \langle b \rangle$ is a semi-direct product but not a Frobenius group since $Z(D_{2m}) = \langle a^m \rangle = \{1, a^m\}$. It is easy to see that

$C_G(a^i) = D_{2m}$ if $a = 0$ or $i = m$,
$C_G(a^i) = \langle a \rangle$ if $a^i \notin Z(D_{2m})$,
$C_G(b) = C_G(a^m b) = \{1, b, a^m, a^m b\}$,
$C_G(ab) = C_G(a^{m+1}b) = \{1, ab, a^m, a^{m+1}b\}, \cdots,$
and $C_G(a^{m-1}b) = C_G(a^{2m-1}b) = \{1, a^{m-1}b, a^m, a^{2m-1}b\}$.

In the case that $g = a^m$, $x = a^i b$, $0 \le i < 2m$, we have

$$(\mathrm{Cor} f)(g, x) = f(a^m, a^i) f(a^m, a^{-i}) = (\mathrm{Cor} f)(x, g).$$

If $g = a^i b$, $x = a^{m+i} b$, $0 \le i < m$, we have

$$(\mathrm{Cor} f)(g, x) = f(a^i, a^{m-i}) f(a^{-i}, a^{m+i}) = (\mathrm{Cor} f)(x, g).$$

Hence $(\mathrm{Cor} f)(g, x) = (\mathrm{Cor} f)(x, g)$ for all $g, x \in D_{2m}$ such that $gx = xg$. This completes the proof.

4. Examples

The present section is devoted to constructing explicit examples of groups for which the converse of Theorem 6 in [4] holds. Most of previously known examples of the situation have been nilpotent groups, the groups in this section are not nilpotent.

By a notation $[g, x]_\sigma = g^{-1} x^{-1} g^{m(\sigma^{-1})} x$, $(\sigma \in \mathcal{G})$, we mean an *F-commutator* of g and x in G. A subgroup generated by all F-commutators is called *F-commutator subgroup* of G and denoted by $G'(F)$. If $F = E$ an algebraic closure over F then $[g, x]_\sigma = [g, x]$ a commutator, and $G'(F) = G'$ the commutator subgroup. Further for any subgroup H of G and $\sigma \in \mathcal{G} = \mathrm{Gal}(E/F)$ we write $[H, G]_\sigma$ for $\langle [h, g]_\sigma \mid h \in H,\ g \in G \rangle$; if $H = G$ then $[H, G]_\sigma = G'(F)$. The following lemma is important for application.

LEMMA 7. ([2]) *For a field F, the following are equivalent.*

(1) *Finding a group G with $f \in Z^2(G, F^*)$ such that f is not cohomologous to 1, $o(f) < \infty$, and every F-class of G is D_Γ-regular for $\Gamma = F^f G$.*

(2) *Finding a group H with a p'-cyclic subgroup A such that $\zeta_{|A|} \in F^*$, $1 \neq A \subseteq Z(H) \cap H'(F)$, and such that A contains no F-commutator of H except identity.*

In case $F = C$, the lemma was studied in [5]. We construct an example of class of groups, which was studied while the second author was in Tufts University. The author would like to express sincere thank to Professor W.F. Reynolds. Let $F = C$ and

$$H = \langle h_1, \dots, h_{10} \mid [h_2, h_1] = h_5,\ [h_3, h_1] = h_6,\ [h_4, h_1] = h_7,$$
$$[h_3, h_2] = h_8,\ [h_4, h_2] = h_9,\ [h_4, h_3] = h_{10},\ o(h_i) = q\rangle.$$

This group has been introduced in [1, § 3], and studied in [5]. Let $A = \langle h_5 h_{10}\rangle$. Then $|H| = q^{10}$, $|A| = q$ and $A \subseteq Z(H) = H' = \langle h_5, h_6, h_7, h_8, h_9, h_{10}\rangle$. For the sake of clarity, we divide this example into five steps.

STEP 1. *A does not contain any commutator of H except 1.*
Take any elements h, x in H. We may let, once and for all,

$$(4.1)\qquad h = h_1^{a_1} \cdots h_{10}^{a_{10}}, \quad x = h_1^{b_1} \cdots h_{10}^{b_{10}}, \quad 0 \le a_i, b_i < q.$$

Let $k_{ij} = a_i b_j - a_j b_i$ for $1 \le i < j \le 4$. Since $H' \subseteq Z(H)$,

$$(4.2)\qquad [h, x] = h_5^{-k_{12}} h_6^{-k_{13}} h_7^{-k_{14}} h_8^{-k_{23}} h_9^{-k_{24}} h_{10}^{-k_{34}}.$$

If $1 \ne [h, x] \in A$ then $[h, x] = h_5^i h_{10}^i$ for some $0 < i < q$. Comparing this with (4.2), we have (by modulo q) $k_{12} \equiv k_{34} \equiv -i$, $k_{13} \equiv k_{14} \equiv k_{23} \equiv k_{24} \equiv 0$. Thus $k_{12}k_{34} - k_{13}k_{24} + k_{14}k_{23} \equiv i^2$, whereas $k_{12}k_{34} - k_{13}k_{24} + k_{14}k_{23} = 0$, hence $i^2 \equiv 0$. However q does not divide i^2, this yields a contradiction.

Let $L \cong H/A$ and consider the central cyclic group extension

$$(4.3)\qquad 1 \to A \to H \to \frac{H}{A} \to 1.$$

Then by Lemma 7, there is a noncobounding $f \in Z^2(L, C^*)$ such that every class of L is f-regular.

STEP 2. *Any permutation τ of $\{h_1, h_2, h_3, h_4\}$ extends to an automorphism of H.*

Define $\bar{\tau} : H \to H$ by $\bar{\tau}(h)$ equals

$$\tau(h_1)^{a_1}\tau(h_2)^{a_2}\tau(h_3)^{a_3}\tau(h_4)^{a_4} \cdot [\tau(h_2), \tau(h_1)]^{a_5}[\tau(h_3), \tau(h_1)]^{a_6}$$
$$\cdot [\tau(h_4), \tau(h_1)]^{a_7}[\tau(h_3), \tau(h_2)]^{a_8}[\tau(h_4), \tau(h_2)]^{a_9}[\tau(h_4), \tau(h_3)]^{a_{10}}.$$

Clearly $\bar{\tau}$ is a well-defined bijection, and for $x \in H$,

(4.4)

$$hx = h_1^{a_1+b_1} h_2^{a_2+b_2} h_3^{a_3+b_3} h_4^{a_4+b_4} h_5^{a_2 b_1+a_5+b_5} \cdot h_6^{a_3 b_1+a_6+b_6} h_7^{a_4 b_1+a_7+b_7} h_8^{a_3 b_2+a_8+b_8} h_9^{a_4 b_2+a_9+b_9} h_{10}^{a_4 b_3+a_{10}+b_{10}}.$$

Hence by applying $\bar{\tau}$ to hx, we have

$$\begin{aligned}\bar{\tau}(hx) = {} & \tau(h_1)^{a_1+b_1} \tau(h_2)^{a_2+b_2} \tau(h_3)^{a_3+b_3} \tau(h_4)^{a_4+b_4} \\ & \cdot [\tau(h_2), \tau(h_1)]^{a_2 b_1+a_5+b_5} [\tau(h_3), \tau(h_1)]^{a_3 b_1+a_6+b_6} \\ & \cdot [\tau(h_4), \tau(h_1)]^{a_4 b_1+a_7+b_7} [\tau(h_3), \tau(h_2)]^{a_3 b_2+a_8+b_8} \\ & \cdot [\tau(h_4), \tau(h_2)]^{a_4 b_2+a_9+b_9} [\tau(h_4), \tau(h_3)]^{a_4 b_3+a_{10}+b_{10}}.\end{aligned}$$

On the other hand, using the calculation similar to the one that yielded (4.4), we have $\bar{\tau}(hx) = \bar{\tau}(h)\bar{\tau}(x)$.

Due to STEP 2, we define an automorphism e of H such that

(4.5) $$e(h_1) = h_3,\ e(h_2) = h_4,\ e(h_3) = h_1,\ e(h_4) = h_2.$$

Then $e(h_5) = h_{10}$, $e(h_6) = h_6^{-1}$, $e(h_7) = h_8^{-1}$, $e(h_8) = h_7^{-1}$, $e(h_9) = h_9^{-1}$ and $e(h_{10}) = h_5$, hence $e^2 = 1$, $e(h_5 h_{10}) = h_5 h_{10}$.

Let $T = H\langle e\rangle$ be the semi-direct product with multiplication $hu \cdot xy = hx^u \cdot uy$ for $h, x \in H$, $u, y \in \langle e\rangle$, where x^u means $u^{-1}xu$. Then $(hu)^{-1} = (h^{-1})^u u$, and we may consider H and $\langle e\rangle$ as subgroups of T. T is not nilpotent, $A \subseteq Z(T)$, and e fixes $h_5 h_{10}$.

Consider another central cyclic group extension

(4.6) $$1 \to A \to T \to \frac{T}{A} \to 1.$$

Since $\langle e\rangle$ is an operator group on H and since A is an $\langle e\rangle$-invariant normal subgroup of H, T/A is isomorphic to the semi-direct product of H/A and $\langle e\rangle$ with respect to the induced action on H/A. Hence the exact sequence (4.6) is isomorphic to

$$1 \to A \to H\langle e\rangle \to \frac{H}{A}\langle e\rangle \to 1.$$

STEP 3. *A does not contain any commutators of T of the form $[h,t]$ or $[t,h]$ except the identity, for $h \in H,\ t \in T$.*

Let $t = xy$ ($x \in H,\ y \in \langle e \rangle$), then y must be either 1 or e, and

$$[h,t] = h^{-1}((x^{-1})^y y)h(xy) = \ h^{-1}h^y \cdot [h,x]^y \in H.$$

If $y = 1$ then $[h,t] = [h,x]$ a commutator of H; this is reduced to STEP 1. Let $y = e$. Then for any h and x as in (4.1),

$$h^{-1}h^e = h_1^{-(a_1-a_3)} h_2^{-(a_2-a_4)} h_3^{a_1-a_3} h_4^{a_2-a_4}$$
$$\cdot h_5^{a_2(a_1-a_3)-a_5+a_{10}} h_6^{a_3(a_1-a_3)+a_1a_3-2a_6} h_7^{a_4(a_1-a_3)+a_2a_3-a_7-a_8}$$
$$\cdot h_8^{a_3(a_2-a_4)+a_1a_4-a_7-a_8} h_9^{a_4(a_2-a_4)+a_2a_4-2a_9} h_{10}^{-a_4(a_1-a_3)+a_5-a_{10}}.$$

Moreover by (4.2),

$$[h,x]^e = h_{10}^{-k_{12}} h_6^{k_{13}} h_8^{k_{14}} h_7^{k_{23}} h_9^{k_{24}} h_5^{-k_{34}}.$$

Multiplying the above two equations, we have

$$[h,t] = h^{-1}h^e[h,x]^e = h_1^{a_3-a_1} h_2^{a_4-a_2} h_3^{a_1-a_3} h_4^{a_2-a_4}$$
$$\cdot h_5^{a_2(a_1-a_3)-a_5+a_{10}-k_{34}} \cdot h_6^{a_3(a_1-a_3)+a_1a_3-2a_6+k_{13}}$$
$$\cdot h_7^{a_4(a_1-a_3)+a_2a_3-a_7-a_8+k_{23}} \cdot h_8^{a_3(a_2-a_4)+a_1a_4-a_7-a_8+k_{14}}$$
$$\cdot h_9^{a_4(a_2-a_4)+a_2a_4-2a_9+k_{24}} \cdot h_{10}^{-a_4(a_1-a_3)+a_5-a_{10}-k_{12}}.$$

Suppose that A contains a nonidentity generator of $[H,T]$, i.e., $1 \neq [h,t] = h_5^i h_{10}^i \in A$. Comparing exponents, we have

(i) $a_1 - a_3 \equiv a_2 - a_4 \equiv 0$

(ii) $a_5 - a_{10} - k_{12} \equiv i$, $\quad -a_5 + a_{10} - k_{34} \equiv i$

(iii) $a_1a_3 - 2a_6 + k_{13} \equiv 0$, $\quad a_2a_4 - 2a_9 + k_{24} \equiv 0$

(iv) $a_1a_4 - a_7 - a_8 + k_{14} \equiv 0$, $\quad a_2a_3 - a_7 - a_8 + k_{23} \equiv 0$

Here, (i) has been used to simplify the other congruences. The definition of k_{ij}, and (i), (ii) and (iv) give rise to

$$-2i \equiv k_{12} + k_{34} \equiv a_1b_2 - a_2b_1 + a_1b_4 - a_2b_3$$
$$0 \equiv k_{14} - k_{23} \equiv a_1b_4 - a_2b_1 - a_2b_3 + a_1b_2,$$

hence $2i \equiv 0 \pmod q$. This is a contradiction since q is odd. Thus A does not contain any commutators of T of the form $[h,t]$ except 1. Also since $[t,h]=[h,t]^{-1}$, the fact that $1\neq[h,t]\notin A$ implies $1\neq[t,h]\notin A$. Moreover A contains no generators of $[H,T]$ or $[T,H]$ except 1.

STEP 4. *A does not contain any commutators of T except 1.*

Let $[r,t]$ for some $r,t\in T$ be any commutator in T and let $r=hu$ and $t=xy$ for $h,x\in H$, $u,y\in\langle e\rangle$. Then

$$[r,t]=(h^{-1})^u(x^{-1})^{yu}h^{yu}x^y\in H., \tag{4.7}$$

There are 4 cases for choosing y and u. If either y or u is 1, then $[r,t]$ is $[r,x]$ or $[h,t]$; this is STEP 3. Assume $y=u=e$. From (4.7) and $H'\subseteq Z(H)$,

$$[r,t]=\left(h^{-1}h^e\right)^e\cdot\left(x^{-1}x^e\right)\cdot[x,h]^{-1}=\left(h^{-1}h^e\right)^e\cdot\left(x^{-1}x^e\right)\cdot[h,x].$$

We computed $[h,x]$, $h^{-1}h^e$ and $x^{-1}x^e$. Further we calculate

$$\begin{aligned}&\left(h^{-1}h^e\right)^e\\&=h_1^{a_1-a_3}h_2^{a_2-a_4}h_3^{-(a_1-a_3)}h_4^{-(a_2-a_4)}h_5^{-a_4(a_1-a_3)+a_5-a_{10}}h_6^{-a_1^2+2a_6}\\&\quad\cdot h_7^{-a_1a_2+a_7+a_8}h_8^{-a_1a_2+a_7+a_8}h_9^{-a_2^2+2a_9}h_{10}^{a_2(a_1-a_3)-a_5+a_{10}},\end{aligned}$$

and

$$\begin{aligned}&\left(h^{-1}h^e\right)^e\cdot\left(x^{-1}x^e\right)\\&=h_1^{a_1-a_3-b_1+b_3}h_2^{a_2-a_4-b_2+b_4}h_3^{-a_1+a_3+b_1-b_3}h_4^{-a_2+a_4+b_2-b_4}\\&\quad\cdot h_5^{-(a_2-a_4)(b_1-b_3)-a_4(a_1-a_3)+a_5-a_{10}+b_2(b_1-b_3)-b_5+b_{10}}\\&\quad\cdot h_6^{(a_1-a_3)(b_1-b_3)-a_1^2+2a_6+b_3(b_1-b_3)+b_1b_3-2b_6}\\&\quad\cdot h_7^{(a_2-a_4)(b_1-b_3)-a_1a_2+a_7+a_8+b_4(b_1-b_3)+b_2b_3-b_7-b_8}\\&\quad\cdot h_8^{(a_1-a_3)(b_2-b_4)-a_1a_2+a_7+a_8+b_3(b_2-b_4)+b_1b_4-b_7-b_8}\\&\quad\cdot h_9^{(a_2-a_4)(b_2-b_4)-a_2^2+2a_9+b_4(b_2-b_4)+b_2b_4-2b_9}\\&\quad\cdot h_{10}^{-(a_2-a_4)(b_1-b_3)+a_2(a_1-a_3)-a_5+a_{10}-b_4(b_1-b_3)+b_5-b_{10}}.\end{aligned}$$

Multiply this by $[h,x]$ as in (4.2), then we have

$$\begin{aligned}&[r,t]=\left(h^{-1}h^e\right)^e\cdot\left(x^{-1}x^e\right)\cdot[h,x]\\&=h_1^{a_1-a_3-b_1+b_3}h_2^{a_2-a_4-b_2+b_4}h_3^{-a_1+a_3+b_1-b_3}h_4^{-a_2+a_4+b_2-b_4}\\&\quad\cdot h_5^{-(a_2-a_4)(b_1-b_3)-a_4(a_1-a_3)+a_5-a_{10}+b_2(b_1-b_3)-b_5+b_{10}-k_{12}}\\&\quad\cdot h_6^{(a_1-a_3)(b_1-b_3)-a_1^2+2a_6+b_3(b_1-b_3)+b_1b_3-2b_6-k_{13}}\\&\quad\cdot h_7^{(a_2-a_4)(b_1-b_3)-a_1a_2+a_7+a_8+b_4(b_1-b_3)+b_2b_3-b_7-b_8-k_{14}}\end{aligned}$$

$$\cdot h_8^{(a_1-a_3)(b_2-b_4)-a_1a_2+a_7+a_8+b_3(b_2-b_4)+b_1b_4-b_7-b_8-k_{23}}$$
$$\cdot h_9^{(a_2-a_4)(b_2-b_4)-a_2^2+2a_9+b_4(b_2-b_4)+b_2b_4-2b_9-k_{24}}$$
$$\cdot h_{10}^{-(a_2-a_4)(b_1-b_3)+a_2(a_1-a_3)-a_5+a_{10}-b_4(b_1-b_3)+b_5-b_{10}-k_{34}}.$$

Suppose that $1 \neq [r,t] = h_5^i h_{10}^i$ for some i. Then

1) $a_1 - a_3 - b_1 + b_3 \equiv a_2 - a_4 - b_2 + b_4 \equiv 0$

2) $(a_4-a_2)(b_1-b_3)-a_4(a_1-a_3)+a_5-a_{10}+b_2(b_1-b_3)-b_5+b_{10}-k_{12} \equiv i$
$(a_4-a_2)(b_1-b_3)+a_2(a_1-a_3)-a_5+a_{10}-b_4(b_1-b_3)+b_5-b_{10}-k_{34} \equiv i$

3) $(a_1-a_3)(b_1-b_3) - a_1^2 + 2a_6 + b_3(b_1-b_3) + b_1b_3 - 2b_6 - k_{13} \equiv 0$
$(a_2-a_4)(b_2-b_4) - a_2^2 + 2a_9 + b_4(b_2-b_4) + b_2b_4 - 2b_9 - k_{24} \equiv 0$

4) $(a_2-a_4)(b_1-b_3)-a_1a_2+a_7+a_8+b_2b_3+b_4(b_1-b_3)-b_7-b_8-k_{14} \equiv 0$
$(a_1-a_3)(b_2-b_4)-a_1a_2+a_7+a_8+b_1b_4+b_3(b_2-b_4)-b_7-b_8-k_{23} \equiv 0$

From 1), we have $a_1 - a_3 \equiv b_1 - b_3$ and $a_2 - a_4 \equiv b_2 - b_4 \pmod q$. By applying this to 2) and 4), we have

$$2i \equiv -2(a_2-a_4)(a_1-a_3) + (a_2-a_4)(a_1-a_3) + (b_2-b_4)(a_1-a_3)$$
$$\equiv -k_{12} - k_{34},$$
$$0 \equiv b_4(b_1-b_3) + b_2b_3 - b_3(b_2-b_4) - b_1b_4 - k_{14} + k_{23} \equiv -k_{14} + k_{23}.$$

However since $0 \equiv (a_1-a_3)(b_2-b_4) - (a_2-a_4)(b_1-b_3) = k_{12} + k_{23} - k_{14} + k_{34}$ (by 1)), we have $-k_{14}+k_{23} \equiv -k_{12}-k_{34}$, and so $2i \equiv 0 \pmod q$, which is contradict to $q \geq 3$. This means that A contains no generators of T except 1.

STEP 5. *Finally, we have a situation of Theorem 3 for this group.*

In (4.6), let $T/A \cong G$. Then by Lemma 7, there is a noncobounding $f' \in Z^2(G, C^*)$ where every class of G is f'-regular. Considering f' together with f as in below (4.3); define isomorphisms $H/A \to L$ by $h_l A \mapsto l$ $(l \in L, h_l \in H)$ and $T/A \to G$ by $t_g A \mapsto g$ $(g \in G, t_g \in T)$. Then the 2-cocycles f and f' are indeed constructed by $f = \chi\alpha$ and $f' = \chi\alpha'$, where χ is a generator of $\mathrm{Hom}(A, C^*)$, and α and α' are 2-cocycles in $Z^2(L, A)$ and $Z^2(G, A)$ respectively, defined by $\alpha(l,l') = h_l h_{l'} h_{ll'}^{-1}$, $\alpha'(g,g') = t_g t_{g'} t_{gg'}^{-1}$ (refer to [2]). For any $l \in L$, we may choose $h_l = t_l$, so that $\alpha = \alpha'_L$. Hence $f = \chi\alpha = \chi(\alpha'_L) = (\chi\alpha')_L = f'_L$. Therefore, we have the situation of Theorem 3 that for $f' \in Z^2(G, C^*)$, every class of G is f'-regular and every class of L is $f(= f'_L)$-regular, while L is a Sylow q-subgroup of G.

REMARK. For any element $t = xy \in T$ ($x \in H, y \in \langle \epsilon \rangle$), $t^{2q} = (xx^y)^q = 1$. Hence $\exp(T) = 2q$. Therefore, if F is any field of characteristic 0 that contains ζ_{2q} then by choosing integer n divisible by $2q$ and $m(\sigma)$, $t^{m(\sigma)} = t$ for all $t \in T$, $\sigma \in \mathcal{G}$. Hence A contains no F-commutators of T except 1; this is a family of examples with the F.

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