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AN INVERSE HOMOGENEOUS INTERPOLATION PROBLEM FOR V–ORTHOGONAL RATIONAL MATRIX FUNCTIONS

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1. Introduction

For a scalar rational function, the spectral data consisting of zeros and poles with their respective multiplicities uniquely determines the function up to a nonzero multiplicative factor. But due to the richness of the spectral structure of a rational matrix function, reconstruction of a rational matrix function from a given spectral data is not that simple. Our purpose here is to use the state space approach as elucidated in [BGR] to interpolation theory to recover a V-orthogonal rational matrix function from a given spectral data. An important basic idea to this approach is to represent a proper (i.e., analytic at infinity) rational matrix function $W(z)$ by

$$W(z) = D + C(zI - A)^{-1}B.$$

Then zero and pole data for $W(z)$ is encoded in constant matrices A, B, C, D. This approach spurred by diverse applications in many engineering context has undergone rapid development in the past couple of decades. Here the spectral data is given by a certain quintuple of matrices $\tau = (C_\pi, A_\pi;\ A_\zeta, B_\zeta;\ \Gamma)$ called a *Sylvester data set* and a rational matrix function $\Theta(z)$ having τ as its local *null-pole triple* is found in terms of given spectral data (see Section 2 for definitions).

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This matrix $\Theta(z)$ is the so called resolvent matrix for the nonhomogeneous interpolation problem. For example, we are asked to find a rational matrix fuction $F(z)$ for which

$$x_i F(z) = y_i, \quad i = 1, \cdots, n_\zeta \tag{1.1}$$

$$F(w_j) u_j = v_j, \quad j = 1, \cdots, n_\pi \tag{1.2}$$

where $\{z_i, \cdots, z_{n_\zeta}, w_1, \cdots, w_{n_\pi}\}$ are given distinct points in $\mathbb{C}$, x_i, u_j are given $1 \times M$, $N \times 1$ nonzero vectors respectively, y_i, v_j are given $1 \times N$, $M \times 1$ vectors. If we organize the data as

$$C_\pi = \begin{bmatrix} u_1 & \cdots & u_{n_\pi} \\ v_1 & \cdots & v_{n_\pi} \end{bmatrix}, \quad A_\pi = \begin{bmatrix} z_1 & & 0 \\ & \ddots & \\ 0 & & z_{n_\pi} \end{bmatrix}$$

$$A_\zeta = \begin{bmatrix} z_1 & & 0 \\ & \ddots & \\ 0 & & z_{n_\zeta} \end{bmatrix}, \quad B_\zeta = \begin{bmatrix} x_1 & -y_1 \\ \vdots & \vdots \\ x_{n_\zeta} & -y_{n_\zeta} \end{bmatrix}$$

$$\Gamma = \left[\frac{x_i u_j - y_i v_j}{z_i - w_j} \right]_{1 \le i \le n_\zeta, 1 \le j \le n_\pi}$$

then $\Theta(z)$ having $\tau = (C_\pi, A_\pi;\ A_\zeta, B_\zeta;\ \Gamma)$ as its local *null-pole triple* provides the coefficients for a linear fractional map which parametrizes the set of all solutions of the nonhomogeneous interpolation problem (1.1)(1.2) in terms of free parameter matrix functions. The details are found in the literature, e.g., [ABKW], [BGR]. If we consider an extra constraints

$$F(z)^T = F(z), \quad \forall z \in \mathbb{C},$$

(correseponding to the transfer function of a reciprocal network in circuit theory)to (1.1)(1.2), then the resolvent matrix $\Theta(z)$ should satisfy the V-orthogonality condition,

$$\Theta(z)^T V \Theta(z) = V, \quad \forall z \in \mathbb{C} \tag{1.3}$$

with $V = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$.

This same state space approach was applied in the series of papers [ABGR1]-[ABGR4] to a variety of factorization and interpolation problems involving other types of symmetries. Also the inverse problem without V-orthogonality constraint is discussed in [GK][GKR1,2] [BKGK].

This paper consists as follows: In section 2, some auxiliary notions and terminologies are introduced. In section 3, the zero-pole structure of rational matrix functions $\Theta(z)$ which satisfy an identity (1.3) for some square matrix V for which $V = \alpha V^T$ where $\alpha = \pm 1$ is developed. The last section includes the existing results on the inverse problem without the extra constraint (1.3). By refining this results further, a local spectral structure of rational matrix functions satisfying (1.3) is understood and finally, an rational matrix function having prescribed local spectral data and satisfying (1.3) is found in terms of given data.

2. Preliminaries

By an $M \times N$ rational matrix function, we understand an $M \times N$ matrix with rational functions as its entries and shall regard it as a meromorphic matrix function over the extended complex plane $\mathbb{C}_\infty$. For an $M \times N$ proper (i.e., analytic at infinity) rational matrix function $W(z)$, we define a *realization* of $W(z)$ to be a representation of the form

$$W(z) = D + C(zI - A)^{-1}B, \quad z \notin \sigma(A) \tag{2.1}$$

where A, B, C, D are matrices of sizes $n \times n, n \times N, M \times n, M \times N$ respectively, and $\sigma(A)$ refers to the spectrum of the matrix A. A realization (2.1) is said to be *minimal* if (C, A) is a *null-kernel pair* and (A, B) is a *full-range pair*, that is

$$\bigcap_{j=0}^{n-1} Ker\ CA^j = \{0\}$$

$$\sum_{j=0}^{n-1} Im\ A^j B = \mathbb{C}^n.$$

If D is invertible in (2.1), then

$$W^{-1}(z) = D^{-1} - D^{-1}C(zI - A^\times)^{-1}BD^{-1} \tag{2.2}$$

with $A^\times = A - BD^{-1}C$ is a minimal realization of $W^{-1}(z)$ and (2.1) is minimal if and only if (2.2) is. Realizations for a rational matrix function always exist. In the rest of this section, we assume $M = N$ and $W(z)$ is regular (i.e., $\det W(z) \neq 0$ for some $z \in \mathbb{C}_\infty$). When (2.1) is a minimal realization for $W(z)$, the pair (C, A) is said to be a *(right) pole pair* of $W(z)$ and $(A^\times, B)$ is said to be a *(left) null pair* of $W(z)$, $\Gamma = P^\times | ImP$ is a *null-pole coupling matrix* of $W(z)$, where $P(P^\times)$ represents the Riesz projection of $A(A^\times)$. By a *global null-pole triple* of $W(z)$, we mean a set of matrices $\tau = (C, A; A^\times, B; \Gamma)$.

A collection of matrices

$$\tau = (C_\pi, A_\pi; A_\zeta, B_\zeta; \Gamma)$$

is said to be an *admissible Sylvester data set* if

(C_π, A_π) is a null-kernel pair of matrices of respective sizes $m \times n_\pi, n_\pi \times n_\pi$
(A_ζ, B_ζ) is a full-range pair of matrices of respective sizes $n_\zeta \times n_\zeta, n_\zeta \times m$
and the $n_\zeta \times n_\pi$ matrix Γ satisfies the Sylvester equation $\Gamma A_\pi - A_\zeta \Gamma = B_\zeta C_\pi$.

We note that a null-pole triple for a rational matrix function is an admissible Sylvester data set. From now on , τ denotes an admissible Sylvester data set

$$\tau = (C_\pi, A_\pi; A_\zeta, B_\zeta; \Gamma) \tag{2.3}$$

where the matrices $C_\pi, A_\pi, A_\zeta, B_\zeta, \Gamma$ have respective sizes $m \times n_\pi, n_\pi \times n_\pi, n_\zeta \times n_\zeta, n_\zeta \times m, n_\zeta \times n_\pi$.

THEOREM 2.1. *For a given admissible Sylvester data set τ as in (2.3), there exists a rational matrix function $W(z)$ which has τ as its global-null-pole-triple if and only if Γ is invertible. In this case, such a $W(z)$ is given by*

$$W(z) = I + C_\pi (zI - A_\pi)^{-1} \Gamma^{-1} B_\zeta$$

and

$$W^{-1}(z) = I - C_\pi \Gamma^{-1} (zI - A_\zeta)^{-1} B_\zeta.$$

Proof. See [GK].

Two Sylvester data sets

$$\tau = (C_\pi, A_\pi; A_\zeta, B_\zeta; \Gamma)$$
$$\tau' = (C'_\pi, A'_\pi; A'_\zeta, B'_\zeta; \Gamma')$$

are said to be similar if there exist invertible matrices Φ, Ψ such that

$$C'_\pi = C_\pi \Phi, \quad A'_\pi = \Phi^{-1} A_\pi \Phi$$
$$A'_\zeta = \Psi^{-1} A_\zeta \Psi, \quad B'_\zeta = \Psi^{-1} B_\zeta$$
$$\Gamma' = \Psi^{-1} \Gamma \Phi.$$

Let us denote the similarity of τ and τ' by $\tau \sim \tau'$. When we want to emphasize the matrices Φ and Ψ we say that τ and τ' are (Φ, Ψ) - similar. The proof of the next theorem is found in [BGR].

THEOREM 2.2. *Two admissible Sylvester data sets τ and τ' are the global null-pole triples of a rational matrix function $\Theta(z)$ if and only if $\tau \sim \tau'$.*

3. The spectral structure of V-orthogonal rational matrix functions

Let V be an $m \times m$ invertible constant symmetric or anti-symmetric matrix, that is,

$$V^T = \alpha V$$

where either $\alpha = 1$ or $\alpha = -1$. Here, given an admissible Sylvester data set τ as in (2.3), our goal is to find an $m \times m$ rational matrix function $\Theta(z)$ for which

(3.1) Θ has τ as its global null-pole triple

(3.2) $$\Theta^T(z) V \Theta(z) = V, \quad \forall z \in \mathbb{C}.$$

The first step toward the solution is to understand the null and pole structure of $\Theta(z)$ satisfying (3.2). To a given τ as in (2.3), we associate another set of matrices

(3.3) $$\tau^T = (-V^{-1} B_\zeta^T, A_\zeta^T; A_\pi^T, C_\pi^T V; \Gamma^T).$$

It is easy to check that τ^T is an admissible Sylvester data set if and only if τ is. The next theorem gives a characterization of a global null-pole triple of $\Theta(z)$ satisfying (3.2).

THEOREM 3.1. *Given is a Sylvester data set*

$$\tau = (C_\pi, A_\pi; A_\zeta, B_\zeta; \Gamma)$$

of sizes as in (2.3). *If*

$$\tau \text{ is a global-null-pole triple for } \Theta(z) \tag{3.4}$$

with $\Theta(\infty) = D$ *for an invertible* D *satisfying* $V^{-1}D^{-T}V = D$. *Then,*

$$\Theta^T(z)V\Theta(z) = V, \quad \forall z \in \mathbb{C}$$

if and only if $\tau \sim \tau^T$.

Proof. Since τ is a global null-pole triple for $\Theta(z)$, by Theorem 2.1 Γ is invertible and

$$\Theta(z) = D + C_\pi(zI - A_\pi)^{-1}\Gamma^{-1}B_\zeta D \tag{3.5a}$$

$$\Theta^{-1}(z) = D^{-1} - D^{-1}C_\pi\Gamma^{-1}(zI - A_\zeta)^{-1}B_\zeta. \tag{3.6a}$$

Take the transpose of (3.5a) and (3.6a) premultiply by V^{-1}, postmultiply by V and then substitute D in place of $V^{-1}D^{-T}V$; the results

$$V^{-1}\Theta^{-T}V = D - V^{-1}B_\zeta^T(zI - A_\zeta^T)^{-1}\Gamma^{-T}C_\pi^T VD \tag{3.5b}$$

and

$$V^{-1}\Theta^T V = D^{-1} + D^{-1}V^{-1}B_\zeta^T\Gamma^{-T}(zI - A_\pi^T)^{-1}C_\pi^T V. \tag{3.6b}$$

If

$$\Gamma^T A_\zeta^T - A_\pi^T\Gamma^T = C_\pi^T V(-V^{-1}B_\zeta^T), \tag{3.7}$$

holds, from (3.5b), (3.6b), and (3.7) we know that

$$\tau^T = (-V^{-1}B_\zeta^T, A_\zeta^T; A_\pi^T, C_\pi^T V, \Gamma^T)$$

is a global null pole triple for $V^{-1}\Theta^{-T}V$. But, (3.7) is equivalent to $\Gamma A_\pi - A_\zeta\Gamma = B_\zeta C_\pi$ which is a part of our hypothesis. Hence

$$\tau^T \text{ is a global-null-pole triple for } V^{-1}\Theta^{-T}V. \tag{3.8}$$

Applying Theorem 2.1 to (3.4) (3.8) we conclude that $\tau \sim \tau^T$ if and only if $V^{-1}\Theta^{-T}(z)V = \Theta(z)$. □

The following theorem gives a canonical form of τ which is similar to τ^T.

THEOREM 3.2. *If a Sylvester data set τ as in (2.3) is similar to τ^T, then τ is similar to τ_c which is in the form*

$$\tau_c = (C_\pi, A_\pi; A_\pi^T, -\alpha C_\pi^T V\ \Gamma_c)$$

with

$$\Gamma_c^T = -\alpha\Gamma_c.$$

Proof. Suppose a given Sylvester data set

$$\tau = (C_\pi, A_\pi; A_\zeta, B_\zeta; \Gamma)$$

is similar to

$$\tau^T = (-V^{-1}B_\zeta^T, A_\zeta^T; A_\pi^T, C_\pi^T V; \Gamma^T).$$

From the similarity, there exist invertible matrices Φ and Ψ such that

$$C_\pi = -V^{-1}B_\zeta^T\Phi, \quad A_\pi = \Phi^{-1}A_\zeta^T\Phi \tag{3.9}$$

$$A_\zeta = \Psi^{-1}A_\pi^T\Psi, \quad B_\zeta = \Psi^{-1}C_\pi^T V \tag{3.10}$$

$$\Gamma = \Psi^{-1}\Gamma^T\Phi. \tag{3.11}$$

Taking transpose in (3.9) (3.10) (3.11), we see that the same equalities are valid with Ψ replaced by $-\alpha\Phi^T$ and Φ by $-\alpha\Psi^T$. By the uniqueness of the similarlity of two Sylvester data sets,

$$\Psi = -\alpha\Phi^T. \tag{3.12}$$

Let

$$\tau_c = (C_\pi, A_\pi; A_\pi^T, -\alpha C_\pi^T V; \Gamma_c)$$

with

$$\Gamma_c = -\alpha\Gamma^T\Phi. \tag{3.13}$$

Upon substituting (3.12) into (3.10) and (3.11), we get

$$A_\zeta = \Phi^{-T}A_\pi^T\Phi^T, \quad B_\zeta = -\alpha\Phi^{-T}C_\pi^T V \tag{3.14}$$

$$\Gamma = -\alpha\Phi^{-T}\Gamma^T\Phi. \tag{3.15}$$

Replacing $\Gamma^T\Phi$ by $-\alpha\Gamma_c$ in (3.15), we can see from (3.13) (3.14) that τ is (I, Φ^T) - similar to τ_c. To complete the proof, the only thing left is to show that $\Gamma_c^T = -\alpha\Gamma_c$. But, the equality is straightforward from (3.13) and (3.14). □

The following more detailed version of Theorem 3.2 is derived by substituting (3.12) in (3.9), (3.10) and (3.11).

COROLLARY 3.3. *Suppose an admissibile Sylvester data set τ as in (2.3) is similar to τ^T. Then there exists an invertible matrix Φ for which τ and τ^T are $(\Phi, -\alpha\Phi^T)$ - similar.*

We get the following theorem from Theorem2.1 and Theorem3.1.

THEOREM3.4. *Given is a σ-admissble Sylvester data set τ. There exists a rational matrix function $W(z)$ satisfying (3.1)(3.2) if and only if $\tau \sim \tau^T$ and Γ is invertible. In this case, such a function is given by*

$$W(z) = I + C_\pi(zI - A_\pi)^{-1}\Gamma^{-1}B_\zeta.$$

4. An inverse homogeneous problem for V-orthogonal functions

Throughout Section 4, τ denotes a σ-admissible Sylvester data set given by (2.3). By an *inverse homogeneous interpolation problem*, we mean a problem of finding a rational matrix function $W(z)$ which has a given σ-Sylvester data set τ as its σ-null-pole triple.

4.1. A minimal complement

The next result comes from [GK] (see also [GKR1], [GKR2]) and is also discussed in [BGR]. A more refined version appears in [BKGK].

THEOREM 4.1.1. *For a given σ-admissible Sylvester data set τ as in (2.3), there always exists a rational matrix function $W(z)$ which has τ as its σ-null-pole-triple.*

Theorem 4.1.1 is obtained as a result of a *completion problem* of a Sylvester data set τ in which τ is augumented to a Sylvester data set $\hat{\tau}$ with matrices of larger size by adding extra zeros and poles in such a way that $\hat{\tau}$ has an invertible coupling matrix. A construction of such a $\hat{\tau}$ due to [GK] is following. Suppose a σ-admissible Sylvester data set τ as in (2.3) is given with $\sigma \subset \mathbb{C}$. Let

$$\tau_0 := (C_0, A_{\pi 0}; A_{\zeta 0}, B_0; \Gamma_0)$$

be a ϵ-admissible Sylvester data set for ϵ, a subset of $\mathbb{C}$, satisfying $\epsilon \cap \sigma = \emptyset$. We call τ_0 a *complement* to τ if the matrix

$$\widetilde{\Gamma} := \begin{bmatrix} \Gamma & \Gamma_{12} \\ \Gamma_{21} & \Gamma_0 \end{bmatrix}$$

is square and invertible, where Γ_{12} and Γ_{21} are the unique solutions of

$$\Gamma_{12}A_{\pi 0} - A_{\zeta}\Gamma_{12} = B_{\zeta}C_0$$
$$\Gamma_{21}A_{\pi} - A_{\zeta 0}\Gamma_{21} = B_0 C_{\pi}.$$

The complement will be called *minimal* if and only if among all complements of τ, the size of the matrix $\widetilde{\Gamma}$ is as small as possible. If τ_0 is a complement to τ, then the function

$$\Theta(z) = I + [C \quad C_0] \begin{bmatrix} (zI - A_{\pi})^{-1} & 0 \\ 0 & (zI - A_{\pi 0})^{-1} \end{bmatrix} \widetilde{\Gamma}^{-1} \begin{bmatrix} B \\ B_0 \end{bmatrix}$$

has τ as a σ-null-pole triple, and if τ_0 is a minimal complement, then $\Theta(z)$ has the minimal possible McMillan degree among all rational matrix functions having τ as a σ-null-pole triple.

To describe such a minimal complement, first we need to introduce some notions. Let N be a complement of $Ker\ \Gamma$ in $\mathbb{C}^{n_\pi}$ and K be a complement of $Im\ \Gamma$ in $\mathbb{C}^{n_\zeta}$, i.e.,

$$\mathbb{C}^{n_\pi} = Ker\ \Gamma \dot{+} N$$

$$\mathbb{C}^{n_\zeta} = Im\ \Gamma \dot{+} K.$$

Let ρ_π be the projection onto $Ker\ \Gamma$ along N and ρ_ζ be the projection onto K along $Im\ \Gamma$. Further, let η_π be the embedding of $Ker\ \Gamma$ into $\mathbb{C}^{n_\pi}$ and η_ζ be the embedding of K into $\mathbb{C}^{n_\zeta}$. The controllability indices of a full-range pair can be defined in many ways. Here the controllability indices of the pair $(\rho_\zeta A_\zeta|_K, \rho_\zeta B)$ are introduced through the following incoming subspaces. Let

$$H_0 := Im\ \Gamma$$
$$H_j := Im\ \Gamma + Im A_\zeta B_\zeta + \cdots + Im A_\zeta^{j-1} B_\zeta,\ j = 1, 2, \cdots .$$

We define the incoming indices $\omega_1 \geq \cdots \geq \omega_s$ by

$$s := dim(H_1/H_0)$$

and

$$\omega_j := \#\{k : dim(H_k/H_{k-1}) \geq j\}, \quad j = 1, \cdots, s.$$

Then, the numbers $\omega_1 \geq \cdots \geq \omega_s$ are the nonzero controllability indices of $(\rho_\zeta A_\zeta|_K, \rho_\zeta B_\zeta)$. Similarly, the observability indices of the null-kernel pair $(C_\pi|_{Ker\ \Gamma}, \rho_\pi A_\pi|_{Ker \Gamma})$ are defined through outgoing subspaces

$$K_0 := Ker\ \Gamma$$
$$K_j := Ker\ \Gamma \cap Ker\ C_\pi A_\pi \cap \cdots \cap Ker\ C_\pi A_\pi^{j-1},\ j = 1, 2, \ldots .$$

We also define outgoing indices $\alpha_1 \geq \cdots \geq \alpha_t$ by $t = dim(K_0/K_1)$ and

$$\alpha_j = \#\{l : dim\ (K_{l-1}/K_l) \geq j\},\ j = 1, \ldots, t.$$

Then $\alpha_1 \geq \ldots \geq \alpha_t$ are the nonzero observability indices of the pair $(C_\pi|_{Ker\ \Gamma}, \rho_\pi A_\pi|_{Ker \Gamma})$.

Choose a point $\epsilon \notin \sigma$. Let $\{d_{jk}\}_{k=1}^{\alpha_j}{}_{j=1}^{t}$ be an outgoing basis for $Ker\ \Gamma$ and $\{f_{jk}\}_{k=1}^{\omega_j}{}_{j=1}^{s}$ be an incoming basis for K. This means
(4.1.1)
$\{f_{j1}\}_{j=1}^{s}$ forms a basis of a complement of $Im\,\Gamma$ in $Im\,\Gamma + Im B$

(4.1.2) $(A_\zeta - \epsilon I)f_{jk} - f_{j,k+1} \in Im\,\Gamma + Im B_\zeta$ for all j, k $(f_{j,\omega_j+1} = 0)$

(4.1.3) $\quad (A_\pi - \epsilon I)d_{jk} = d_{j,k+1},\ k = 1, \ldots, \alpha_j - 1,\ j = 1, \ldots, t$

(4.1.4) $\quad \{d_{jk}\}_{k=1}^{\alpha_j-1}{}_{j=1}^{t}$ forms a basis for $Ker\ \Gamma \cap Ker\ C_\pi$.

Such a basis can be constructed (see [BGK]). The next theorem gives a minimal complement of τ. For the proof, see [GKR1].

THEOREM 4.1.2. *Let $\tau = (C_\pi, A_\pi; A_\zeta, B_\zeta; \Gamma)$ be a σ-admissible Sylvester data set. Then a minimal complement to τ is given by*

$$\tau_0 = (-C_\pi X - F, T; S, -YB_\zeta + G; Y\Gamma X - Y\eta_\zeta - \rho_\pi X).$$

Here $S : Ker\ \Gamma \to Ker\ \Gamma$ and $T : K \to K$ are given by

(4.1.5) $\quad (S - \epsilon)d_{jk} = d_{j,k+1}\ (e_{j,\alpha_j+1} = 0),$

$$(4.1.6) \qquad (T-\epsilon)f_{jk} = f_{j,k+1} \ (f_{j,\omega_j+1}=0).$$

Furthermore, $G:\mathbb{C}^m \to Ker\ \Gamma$, $F: K \to \mathbb{C}^m$ *are chosen so that*

$$(4.1.7) \qquad \rho_\zeta(A_\zeta|_K + B_\zeta F) = T$$

$$(4.1.8) \qquad (\rho_\pi A_\pi - GC_\pi)_{|Ker\Gamma} = S$$

(*For the details of the choices of* G, F *see* [$GKR1$]). *Finally,*

$$(4.1.9) \qquad X = \sum_{\nu=1}^{\omega_1}(A_\pi-\epsilon)^{-\nu}\Gamma^+(A_\zeta+B_\zeta F)(T-\epsilon)^{\nu-1}$$

$$(4.1.10) \qquad Y = \sum_{\nu=1}^{\alpha_1}(S-\epsilon)^{\nu-1}\rho_\pi(A_\pi - GC_\pi)\Gamma^+(A_\zeta-\epsilon)^{-\nu}$$

where Γ^+ *is a generalized inverse of* Γ *such that* $\Gamma\Gamma^+ = I-\rho_\zeta$, $\Gamma^+\Gamma = I-\rho_\pi$, $Ker\ \Gamma^+ = K$ *and* $Im\ \Gamma^+ = N$.

REMARK. In Theorem 4.1.2, a $\sigma\cup\{\epsilon\}$-admissible Sylvester data set $\tau\oplus\tau_0$ is given by

$$(4.1.11) \qquad \left([C_\pi \quad -C_\pi X - F], \begin{bmatrix} A_\pi & 0 \\ 0 & T\end{bmatrix}; \begin{bmatrix} A_\zeta & 0 \\ 0 & S\end{bmatrix}, \begin{bmatrix} B_\zeta \\ -YB_\zeta + G\end{bmatrix}; \widetilde{\Gamma}\right)$$

where the null-pole coupling matrix $\widetilde{\Gamma}$ is an $(n_\pi + n_\zeta - \text{rank}\ \Gamma)\times(n_\pi + n_\zeta - \text{rank}\ \Gamma)$ invertible matrix given by

$$\widetilde{\Gamma} = \begin{bmatrix} \Gamma & -\Gamma X + \eta_\zeta \\ -Y\Gamma + \rho_\pi & Y\Gamma X - \rho_\pi - Y\eta_\zeta \end{bmatrix}$$

and

$$\widetilde{\Gamma}^{-1} = \begin{bmatrix} \eta_\pi Y + X\rho_\zeta + \Gamma^+ & \eta_\pi \\ \rho_\zeta & 0 \end{bmatrix}.$$

In this case, an $n \times n$ rational matrix function $\widetilde{\Theta}(z)$ with $\widetilde{\Theta}(\infty) = I$ and having τ as its σ-null-pole triple is given by

(4.1.12)
$$\begin{aligned}\widetilde{\Theta}(z) = I + C_\pi(zI - A_\pi)^{-1}\{(\Gamma^+ + X\rho_\zeta)B_\zeta + \eta_\pi G\} \\ - (C_\pi X + F)(zI - T)^{-1}B_\zeta\end{aligned}$$

and

$$\begin{aligned}\widetilde{\Theta}^{-1}(z) = I - \{C_\pi(\Gamma^+ + \eta_\pi Y) - F\rho_\zeta\}(zI - A_\zeta)^{-1}B_\zeta \\ - C_\pi(zI - S)^{-1}(-YB_\zeta + G).\end{aligned}$$

4.2. An homogeneous problem with a symmetric spectral data

Assume now that σ is a proper subset of $\mathbb{C}$. An important step to the main result Theorem 4.2.2 of this section is to construct a minimal complement

(4.2.1)
$$\tau_0 = (C_0, A_{\pi 0}; A_{\zeta 0}, B_0; \Gamma_0)$$

to a given σ-admissible Sylvester data set

$$\tau = (C_\pi, A_\pi; A_\zeta, B_\zeta; \Gamma)$$

similar to τ^T so that

(4.2.2)
$$\begin{aligned}\hat{\tau} &:= \tau \oplus \tau_0 \\ &:= \left(\begin{bmatrix} C_\pi & C_0 \end{bmatrix}, \begin{bmatrix} A_\pi & 0 \\ 0 & A_{\pi 0} \end{bmatrix}; \begin{bmatrix} A_\zeta & 0 \\ 0 & A_{\zeta 0} \end{bmatrix}, \begin{bmatrix} B_\zeta \\ B_0 \end{bmatrix}; \begin{bmatrix} \Gamma & \Gamma_{12} \\ \Gamma_{21} & \Gamma_0 \end{bmatrix} \right)\end{aligned}$$

is similar to $(\tau \oplus \tau_0)^T$, where Γ_{12}, Γ_{21} are the unique solutions of

$$\begin{aligned}\Gamma_{12}A_{\pi 0} - A_\zeta\Gamma_{12} = B_\zeta C_0 \\ \Gamma_{21}A_\pi - A_{\zeta 0}\Gamma_{21} = B_0 C_\pi.\end{aligned}$$

To establish the existence of such a $\hat{\tau}$, it is enough to show that we can construct a minimal complement τ_0 of τ such that $\tau_0 \sim \tau_0^T$.

THEOREM 4.2.1. *If τ as in (2.3) is a σ-admissible Sylvester data set which is similar to τ^T, then there exists a minimal complement τ_0 to τ which is similar to τ_0^T.*

Proof. Since τ is similar to τ^T, $n_\pi = n_\zeta$ and there exists an invertible matrix Φ for which

$$C_\pi = -V^{-1}B_\zeta^T\Phi, \quad A_\pi = \Phi^{-1}A_\zeta^T\Phi \tag{4.2.3}$$

$$\Gamma = -\alpha\Phi^{-T}\Gamma^T\Phi \tag{4.2.4}$$

by Corollary 3.3. Let $\{d_{jk}\}_{k=1}^{\alpha_j}{}_{j=1}^{t}$ be an incoming basis for $Ker\,\Gamma$ chosen as in Theorem 4.1.2 and U be an $n_\pi \times n_\pi$ invertible matrix having $\{d_{jk}\}_{k=1}^{\alpha_j}{}_{j=1}^{t}$ as the entries in column $(\alpha_1 + \cdots \alpha_{j-1} + k)$. Postmultiplying (4.2.4) by U and premultiplying by Φ^T and taking transpose, we have

$$(\Gamma U)^T\Phi = -\alpha U^T\Phi^T\Gamma.$$

By the choice of U, the first $(\alpha_1 + \cdots \alpha_t)$ rows of $(\Gamma U)^T$ are equal to zero. Thus

$$U^T\Phi^T(Im\,\Gamma) \subset \text{ linear span of } \{e_i | i > (\alpha_1 + \cdots + \alpha_t)\},$$

where $\{e_i\}_{i=1}^{n_\pi}$ is the standard orthonormal basis for $\mathbb{C}^{n_\pi}$. If we choose f_{jk} so that f_{jk} satisfies the relation

$$U^T\Phi^T f_{jk} = e_{\alpha_1+\cdots+\alpha_{j-1}+\alpha_j+k-1} \tag{4.2.5}$$

then, $K := \mathcal{L}(\{f_{jk}\}_{k=1}^{\alpha_j}{}_{j=1}^{t})$ is a complement of $Im\ \Gamma$, where $\mathcal{L}(A)$ is the linear span of the set A. Moreover $\{f_{jk}\}_{k=1}^{\alpha_j}{}_{j=1}^{t}$ is an incoming basis for K. The details are checked in Lemma A at the end of this section.

Choose $\eta_\zeta, \rho_\zeta, \eta_\pi, \rho_\pi$ and define S, G, T as in Theorem 4.1.2. Then

$$T = \Phi^{-T}S^T\Phi^T|_K. \tag{4.2.6}$$

Indeed, for $1 \le j \le t$, $1 \le k \le \alpha_j$,

$$\Phi^{-T}(S^T - \epsilon I)\Phi^T f_{jk} = \Phi^{-T}(S^T - \epsilon I)\widetilde{u}_{j,\alpha_j-k+1}$$

by (4.2.5). From (4.1.5), the right hand side of the above equality is the same as $\Phi^{-T}\widetilde{u}_{j,\alpha_j-k}$ ($\widetilde{u}_{j0}=0$) which turns out to be $f_{j,k+1}$ by (4.2.5) ($f_{j,\alpha_j+1}=0$). So, we have shown (4.2.6).

If we put

$$F=-\alpha V^{-1}G^T\Phi^T, \tag{4.2.7}$$

then the condition (4.1.7) is satisfied.

Define X and Y as in (4.1.9) and (4.1.10) respectively. Then by Theorem 4.1.2,

$$\tau_0=(-C_\pi X-F,T;S,-YB_\zeta+G;\Gamma_0) \tag{4.2.8}$$

with

$$\Gamma_0=Y\Gamma X-Y\eta_\zeta-\rho_\pi X \tag{4.2.9}$$

is a minimal complement to τ.

Now, we show that $\tau_0\sim\tau_0^T$. To this end, first we derive

$$\begin{aligned} A_\zeta &=\Phi^{-T}A_\pi^T\Phi^T, & A_\pi &=\Phi^{-1}A_\zeta^T\Phi,\\ C_\pi &=-V^{-1}B_\zeta^T\Phi, & \Gamma^+ &=-\alpha\Phi^{-1}(\Gamma^+)^T\Phi^T \end{aligned}$$

from (4.2.3) and (4.2.4), and substitute the above equalities and (4.2.6) (4.2.7) in places of A_ζ, A_π, C_π, Γ^+, S, G of (4.1.10). Then

$$\begin{aligned} Y &=\Phi^{-1}\sum_{\nu=1}^{\omega_1}(T^T-\epsilon I)^{\nu-1}(A_\zeta^T+F^TB_\zeta^T)(-\alpha)\Gamma^{+T}(A_\pi^T-\epsilon I)^{-\nu}\Phi^T\\ &=-\alpha\Phi^{-1}\{\sum_{\nu=1}^{\omega_1}(A_\pi-\epsilon I)^{-\nu}\Gamma^+(A_\zeta+B_\zeta F)(T-\epsilon I)^{\nu-1}\}^T\Phi^T. \end{aligned}$$

From (4.1.9), we see that the formula inside of the braces is X. Hence, we have

$$Y=-\alpha\Phi^{-1}X^T\Phi^T. \tag{4.2.10}$$

Now, we prove the similarity between τ_0 in (4.2.8) and

$$\tau_0^T=(-V^{-1}(-YB_\zeta+G)^T,S^T;T^T,(-C_\pi X-F)^TV;\Gamma_0^T).$$

On substituting (4.2.3), $F = \alpha V^{-1} G^T \Phi^T{}_{|K}$ obtained from (4.2.6) and $X = -\alpha\Phi^{-1} Y^T \Phi^T{}_{|K}$ from (4.2.10) into $-C_\pi X - F$, we have

$$(4.2.11) \qquad -C_\pi X - F = -\alpha V^{-1}(-YB_\zeta + G)^T \Phi^T{}_{|K}.$$

Upon taking transpose of (4.2.7) and postmultiplying by V, premultiplying by $-\Phi^{-1}$, we get

$$(4.2.12) \qquad -YB_\zeta + G = -\Phi^{-1}(-C_\pi X - F)^T V.$$

Note that (4.2.6) is equivalent to

$$(4.2.13) \qquad T = \Phi^{-T} S^T \Phi^T{}_{|K}.$$

From (4.2.11)$\sim$ (4.2.13) and (4.2.6), we see that if

$$(4.2.14) \qquad \Gamma_0 = -\alpha\Phi^{-1}\Gamma_0^T \Phi^T{}_{|K}$$

then τ_0 and τ_0^T are $(\alpha\Phi^T{}_{|K}, -\Phi_{|Ker\Gamma})$-similar. We replace (4.2.10) in (4.2.9) and take out common factors $-\alpha\Phi^{-1}$, $\Phi^T_{|K}$ to get

$$(4.2.15) \quad \Gamma_0 = -\alpha\Phi^{-1}(X^T \Phi^T \Gamma X \Phi^{-T} - X^T + \alpha\rho_\zeta \Phi X \Phi^{-T})\Phi^T{}_{|K}.$$

Upon taking the transpose of (4.2.9) and substituting $Y^T = -\alpha\Phi X \Phi^{-T}$ obtained from (4.2.10) and $\Gamma^T \Phi = -\alpha\Phi^T\Gamma$ obtained from (4.2.4), we get

$$\begin{aligned}
\Gamma_0^T &= X^T \Gamma^T Y^T \rho_\pi - \rho_\zeta Y^T \rho_\pi - X^T \rho_\pi \\
&= -\alpha X^T \Gamma^T \Phi X \Phi^{-T} \rho_\pi + \alpha\rho_\zeta \Phi X \Phi^{-T} \rho_\pi - X^T \rho_\pi \\
&= (X^T \Phi^T \Gamma X \Phi^{-T} + \alpha\rho_\zeta \Phi X \Phi^{-T} - X^T)\rho_\pi.
\end{aligned}$$

Since the last line of the above equalities coincides with the formula in the parenthesis of (4.2.15), we conclude

$$\Gamma_0 = -\alpha\Phi^{-1}\Gamma_0^T \Phi^T{}_{|K}.$$

as we desired. This completes the proof. □

The following theorem characterizes a local null-pole triple for a rational matrix function satisfying
$\Theta^T(z) V \Theta(z) = V$.

THEOREM 4.2.2. *Suppose τ as in (2.3) is a σ-admissible Sylvester data set, where $\sigma \subset \mathbf{C}$. Then there exists an $m \times m$ rational matrix function $\Theta(z)$ for which*

(i) $\Theta^T(z)V\Theta(z) = V$ *for all* $z \in \mathbf{C}$

(ii) Θ *has* τ *as its* σ*-null-pole triple*

(iii) $\Theta^{\pm 1}$ *is analytic at infinity if and only if* τ *is similar to* τ^T. *In this case such a function* $\Theta(z)$ *is given by* (4.1.12) *with* Θ *in place of* $\widetilde{\Theta}$ *where* Γ^+, X, ρ_ζ, η_π, G, F, T *are chosen as in theorem 4.2.1.*

Proof. Suppose τ is a given σ-admissible Sylvester data set which is similar to τ^T. By Theorem 4.2.1, there exists a minimal complement τ_0 given by (4.2.8) to τ which is also similar to τ_0^T. According to Corollary 3.3, there exists an invertible matrix $\Phi(\Phi_0)$ such that $\tau(\tau_0)$ is $(\Phi, -\alpha\Phi^T)$-similar $((\Phi_0, -\alpha\Phi_0^T)$-similar) to $\tau^T(\tau_0^T)$. Thus, $\tau \oplus \tau_0$ is $\left(\begin{bmatrix} \Phi & 0 \\ 0 & \Phi_0 \end{bmatrix}, \begin{bmatrix} -\alpha\Phi^T & 0 \\ 0 & -\alpha{\Phi_0}^T \end{bmatrix}\right)$-similar to $\tau^T \oplus \tau_0^T$. But it is easy to check that $\tau^T \oplus \tau_0^T = (\tau \oplus \tau_0)^T$. Hence if we let $\Theta(z)$ be an rational $m \times m$ matrix function which has $\tau \oplus \tau_0$ as its global null-pole triple with $\Theta(\infty) = I$, then $\Theta(z)$ satisfies the conditions (i)-(iii) by its construction and such a function given by (4.1.12) with Θ in place of $\widetilde{\Theta}$. Applying Theorem 3.1, we conclude that $\Theta^T V \Theta = V$. Conversely, suppose there exists an $m \times m$ rational matrix function $\Theta(z)$ which satisfies (i)- (iii). If $\hat{\tau}$ is a global null-pole triple for $\Theta(z)$, by Theorem 3.1, $\hat{\tau}$ is similar to $\hat{\tau}^T$. By applying Theorem 3.2 and Theorem 2.2 consecutively, we can assume that $\hat{\tau}$ is in form of

$$\hat{\tau} = (\hat{C}_\pi, \hat{A}_\pi; \hat{A}_\pi^T, \hat{C}_\pi^T V; \hat{\Gamma})$$

with

$$\hat{\Gamma}^T = -\alpha\hat{\Gamma}.$$

If we represent the Riesz projection of $\hat{A}_\pi$ corresponding to the eigenvalues in σ by ρ, then

$$\widetilde{\tau} = (\widetilde{C_\pi}, \widetilde{A_\pi}; \widetilde{A_\zeta}, \widetilde{B_\zeta}; \widetilde{\Gamma})$$

is a σ-null-pole triple for $\Theta(z)$ where

$$(\widetilde{C}_\pi, \widetilde{A}_\pi) = (\hat{C}_\pi|_{Im\,\rho}, \hat{A}_\pi|_{Im\,\rho}),$$

$$(\widetilde{A}_\zeta, \widetilde{B}_\zeta) = (\hat{A}_\pi^T|_{Im\,\rho^T}, -\alpha\rho^T\hat{C}_\pi^T V) \text{ and } \widetilde{\Gamma} = \rho^T\hat{\Gamma}|_{Im\rho}.$$

From the above equalities, we observe that $(\widetilde{A}_\zeta, \widetilde{B}_\zeta) = (\widetilde{A}_\pi^T, -\alpha \widetilde{C}_\pi^T V)$ and $\widetilde{\Gamma}^T = -\alpha\widetilde{\Gamma}$. By applying Theorem 3.2 to the previous observation, we see $\widetilde{\tau}$ is similar to $\widetilde{\tau}^T$. On the other hand, two σ-null-pole triples $\tau, \widetilde{\tau}$ for $\Theta(z)$ are similar by Theorem 2.2. If τ and $\widetilde{\tau}$ are (Φ, Ψ)-similar, upon taking transpose of the similarity relations, we can see that τ^T and $\widetilde{\tau}^T$ are (Ψ^T, Φ^T)-similar. Since the similarity relation on Sylvester data sets is an equivalence relation, we can conclude that τ is similar to τ^T. □

LEMMA A. *If we take* $\{f_{jk}\}_{j=1}^{t}{}_{k=1}^{\alpha_j}$ *as in* (4.2.5), *then* $\{f_{jk}\}_{j=1}^{t}{}_{k=1}^{\alpha_j}$ *is an incoming basis for* K.

Proof. To prove $\{f_{jk}\}_{j=1}^{t}{}_{k=1}^{\alpha_j}$ is an incoming basis for K, we need to show that the conditions (4.1.1) and (4.1.2) are fulfilled. To compute $(A_\zeta - \epsilon I) f_{jk}$, we replace A_ζ, f_{jk} by (4.2.3) and (4.2.5) respectively. Then we have

$$(A_\zeta - \epsilon I) f_{jk} = \Phi^{-T}(A_\pi^T - \epsilon I)\widetilde{u}_{j,\alpha_j - k+1} \tag{4.2.16}$$

for $j = 1, \ldots, t$, $k = 1, \ldots, \alpha_j$, where $\widetilde{u}_{jk}$ represents the $(\alpha_1 + \cdots + \alpha_{j-1} + k)^{th}$ column of U^{-T}. By (4.1.3) and the choice of $\widetilde{u}_{jk}$, we have that

$$\Phi^{-T}(A_\pi^T - \epsilon I)\widetilde{u}_{jk} \in f_{j,k+1} + \mathcal{L}(\{f_{j1}\}_{j=1}^t)$$

for $1 \le j \le t$, $1 \le k \le \alpha_j$, where $f_{j,\alpha_j+1} = 0$ and $\mathcal{L}(A)$ denotes the linear span of of the set A. This proves (4.1.2).

To check the condition (4.1.1), we note that (4.1.4) implies that the $(\alpha_1 + \ldots \alpha_{j-1} + k)^{th}$ column of $C_\pi U$ is 0. Since $C_\pi = -VB_\zeta^T \Phi$ by (4.2.3), this observation implies that

the $(\alpha_1 + \cdots \alpha_{j-1} + k)^{th}$ of $U^T \Phi^T B_\zeta V$ is 0, $\quad 1 \le k \le \alpha_j - 1$, $1 \le j \le t$.

Thus,

$$\begin{aligned}
&\text{the column space of } \rho_\zeta B_\zeta \\
&= \text{ the column space of } \rho_\zeta B_\zeta V \\
&= \mathcal{L}(\{(\alpha_1 + \cdots + \alpha_j)^{th} \text{ of } \Phi^{-T}U^{-T}\}_{j=1}^t) \\
&= \mathcal{L}(\{\Phi^{-T}\widetilde{u}_{j\alpha_j}\}_{j=1}^t) \\
&= \mathcal{L}(\{f_{jk}\}_{j=1}^t).
\end{aligned}$$

The condition (4.1.1) now follows. □

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