J. Korean Math. Soc. **33** (1996), No. 3, pp. 685–692

EQUIVALENCES OF SUBSHIFTS

JUNGSEOB LEE

§1. Introduction

Subshifts of finite type can be classified by various equivalence relations. The most important equivalence relation is undoubtedly strong shift equivalence, i.e., conjugacy. In [W], R. F. Williams introduced shift equivalence which is weaker than conjugacy but still sensitive. Since then it has been conjectured that shift equivalent irreducible subshifts of finite type are conjugate. Krieger [K1] introduced a very important shift equivalence invariant – the dimension group. It is known that the dimension triple is a full invariant of shift equivalence.

In this paper, we extend definitions of shift equivalence and the dimension group for subshifts in general, and establish basic results. In Section 3, it will be shown that shift equivalent subshifts have isomorphic dimension groups. In Section 4, we prove that conjugate subshifts are shift equivalent using Nasu's bipartite decomposition of conjugacy.

§2. Definitions

Consider a finite set $\mathcal{A}$ equipped with the discrete topology. The set $\mathcal{A}$ is called the *alphabet* and elements of $\mathcal{A}$ are called *symbols*. The set of bi-infinite sequences

$$\mathcal{A}^{\mathbb{Z}} = \{x = (x_i)_{i\in\mathbb{Z}} : x_i \in \mathcal{A}\}$$

is called the *full $\mathcal{A}$-shift*. We think of $\mathcal{A}^{\mathbb{Z}}$ as the countable product of $\mathcal{A}$. $\mathcal{A}^{\mathbb{Z}}$ is given the product topology and this becomes a compact

---

Received April 30, 1996.
1991 AMS Subject Classification: 54H20, 58F03.
Key words: subshifts, shift equivalence, dimension group.
This research was partially supported by KOSEF Grant 95-0701-02-01-3.

metrizable space. There is the usual *shift map* $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ defined by

$$\sigma(x)_i = x_{i+1}$$

for each $x = (x_i)_{i\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$.

A subset X of $\mathcal{A}^{\mathbb{Z}}$ is called a *subshift* if it is closed and shift invariant, that is, $\sigma(X) = X$.

A *block* over a subshift X is a finite sequence of symbols which appears on some infinite sequence in X. The *length* of a block is the number of symbols it contains. For $x \in X$ and integers i, j with $i \le j$, the block of coordinates in x from position i to position j is denoted by

$$x_{[i,j]} = x_i x_{i+1} \dots x_j.$$

The collection of finite blocks with length n is denoted by $B_n(X)$.

Let X be a subshift. For each $x = (x_i) \in X$, we define the left infinite sequence $x_- = x_{(-\infty,0]}$ and the right infinite sequence $x_+ = x_{[1,\infty)}$. Let

$$X_- = \{x_- : x \in X\}$$

and

$$X_+ = \{x_+ : x \in X\}.$$

For each $x_- \in X_-$, the *follower set* of x_- is defined by

$$f(x_-) = \{y_+ \in X_+ : x_- y_+ \in X\}.$$

Let

$$F_X = \{f(x_-) : x \in X\}$$

denote the class of follower sets of X, and let $\mathcal{F}_X$ the free abelian group generated by the elements in F_X.

For any $x_- \in X_-$, we will define the set $i_X(x_-)$ of *following symbols*: $a \in i_X(x_-)$ if and only if a is a leading symbol of some y_+ in $f(x_-)$. Also for $\omega = f(x_-) \in F_X$ we set

$$i_X(\omega) = i_X(x_-).$$

We note that the definition is independent from the choice of x_-.

Now we define the linear map $L_X : \mathcal{F}_X \to \mathcal{F}_X$ by

$$L_X(\omega) = \sum_{a \in i_X(\omega)} f(x_{-}a)$$

for $\omega = f(x_{-}) \in F_X$. Again, we can easily see that L_X is well-defined.

We will define the dimension group of a subshift as an analogy to the usual one of a subshift of finite type. See, for example, [BMT]. For a subshift X, we define an equivalence relation on the set $\mathcal{F}_X \times \mathbb{Z}$ by declaring (α, m) and (β, n) with $m \leq n$ to be equivalent when $L_X^{n-m}(\alpha) = \beta$. Let $\mathcal{D}_X$ denote the set of equivalence classes and $[\alpha, m]$ denote the equivalence class that contains (α, m). We provide an abelian group structure on $\mathcal{D}_X$ by defining

$$[\alpha, m] + [\beta, n] = [L_X^{n-m}(\alpha) + \beta, n]$$

for $[\alpha, m]$ and $[\beta, n] \in \mathcal{D}_X$ with $m \leq n$. It is routine to check that this operation is well-defined. Finally, we define a map $d_X : \mathcal{D}_X \to \mathcal{D}_X$ by for every $[\alpha, m] \in \mathcal{D}_X$

$$d_X([\alpha, m]) = [L_X(\alpha), m].$$

Again, it is easily seen that d_X is a well-defined automorphism. We note that $d_X^{-1}([\alpha, m]) = [\alpha, m-1]$. The *dimension pair* of a subshift X is $(\mathcal{D}_X, d_X)$.

If there exists an isomorphism $\theta : \mathcal{D}_X \to \mathcal{D}_Y$ such that $\theta \circ d_X = d_Y \circ \theta$, we say that $(\mathcal{D}_X, d_X)$ and $(\mathcal{D}_Y, d_Y)$ are isomorphic.

§3. Shift Equivalence

Two subshifts X and Y are called *shift equivalent* if there exist linear maps $S : \mathcal{F}_X \to \mathcal{F}_Y$ and $T : \mathcal{F}_Y \to \mathcal{F}_X$ satisfying the following four equations

$$S \circ L_X = L_Y \circ S, \quad T \circ L_Y = L_X \circ T,$$

$$S \circ T = L_Y^k, \quad T \circ S = L_X^k \tag{1}$$

for some nonnegative integer k. We should remark that the shift equivalence is an equivalence relation.

THEOREM 1. *If two subshifts are shift equivalent, then their dimension pairs are isomorphic.*

Proof. Let X and Y be shift equivalent subshifts, and linear maps S, T and a nonnegative integer k satisfy the equations in (1). We define a map $\tilde{S} : \mathcal{D}_X \to \mathcal{D}_Y$ by $\tilde{S}([\alpha, m]) = [S(\alpha), m]$ for any $[\alpha, m] \in \mathcal{D}_X$. It is easily seen that $\tilde{S}$ is a well-defined abelian group homomorphism. In fact, for $[\alpha, m], [\beta, n] \in \mathcal{D}_X$ with $m \leq n$,

$$\begin{aligned}
\tilde{S}([\alpha, m] + [\beta, n]) &= \tilde{S}([L_X^{n-m}(\alpha) + \beta, n]) \\
&= [S(L_X^{n-m}(\alpha)) + S(\beta), n] \\
&= [L_Y^{n-m}(S(\alpha)) + S(\beta), n] \\
&= [S(\alpha), m] + [S(\beta), n] \\
&= \tilde{S}([\alpha, m]) + \tilde{S}([\beta, n]).
\end{aligned}$$

Now we observe that for $[\alpha, m] \in \mathcal{D}_X$,

$$\begin{aligned}
d_Y \circ \tilde{S}([\alpha, m]) &= d_Y([S(\alpha), m]) \\
&= [L_Y(S(\alpha)), m] \\
&= [S(L_X(\alpha)), m] \\
&= \tilde{S} \circ d_X([\alpha, m]).
\end{aligned}$$

The linear map $\tilde{T} : \mathcal{D}_Y \to \mathcal{D}_X$ is defined analogously from T. Then we see that for $[\alpha, m] \in \mathcal{D}_X$

$$\begin{aligned}
\tilde{T} \circ \tilde{S}([\alpha, m]) &= [T(S(\alpha)), m] \\
&= [L_X^k(\alpha), m] \\
&= d_X^k([\alpha, m]).
\end{aligned}$$

Similarly we can show that $\tilde{S} \circ \tilde{T} = d_Y^k$. Since d_X and d_Y are automorphisms, we conclude that $\tilde{S}$ is an isomorphism. □

§4. Conjugacy and Shift Equivalence

Suppose that X and Y are subshifts over the alphabets $\mathcal{A}_X$ and $\mathcal{A}_Y$ respectively. Fix nonnegative integers m and n. Let $\Phi : B_{m+n+1}(X) \to \mathcal{A}_Y$ be a map from the set of $(m+n+1)$-blocks of X to the alphabet of Y. A function $\varphi : X \to Y$ is called a *sliding block code* with *memory* m and *anticipation* n induced by the block map Φ if

$$\varphi(x)_i = \Phi(x_{[i-m,i+n]}).$$

Such a sliding block code is called of (m,n)-type.

The celebrated theorem of Curtis-Hedlund-Lyndon asserts that sliding block codes are the only continous functions between subshifts which intertwine shift maps. A bijective sliding block code is called a *conjugacy.* We shoule note that the inverse of a conjugacy is also a conjugacy. When a conjuagy and its inverse are both of $(0,0)$-type, it is called a *symbolic conjugacy.*

THEOREM 2. *Let X and Y be subshifts. Suppose that $\varphi : X \to Y$ is a symbolic conjugacy. Then there is an isomorphism $S : \mathcal{F}_X \to \mathcal{F}_Y$ such that $S \circ L_X = L_Y \circ S$. In particular, $L_X = T \circ S$ and $L_Y = S \circ T$ for some linear map $T : \mathcal{F}_Y \to \mathcal{F}_X$.*

Proof. We define a map $S : F_X \to F_Y$ by

$$S(\omega) = f(\varphi(x)_-)$$

for $\omega = f(x_-) \in F_X$. First we will show that the definition of S is independent of the choice of x. Suppose that $f(x_-) = f(y_-) \in F_X$. If t_+ is in $f(\varphi(x)_-)$ then there exists an element $s \in X$ such that $\varphi(s) = \varphi(x)_- t_+$. Since φ is a symbolic conjugacy, $s_- = x_-$ and so $s_+ \in f(x_-) = f(y_-)$. Thus $\varphi(y_- s_+) = \varphi(y)_- t_+$ and $t_+ \in f(\varphi(y)_-)$. This shows that $f(\varphi(x)_-) \subset f(\varphi(y)_-)$. The other side inclusion can be shown similarly.

Now we extend S as a linear map from $\mathcal{F}_X$ to $\mathcal{F}_Y$. The linear map $S' : \mathcal{F}_Y \to \mathcal{F}_X$ is analogously defined via φ^{-1}. We see that for every $\omega = f(x_-) \in F_X$

$$\begin{aligned} S' \circ S(\omega) &= S'(f(\varphi(x)_-)) \\ &= f(\varphi^{-1} \circ \varphi(x)_-) \\ &= f(x_-) \\ &= \omega, \end{aligned}$$

and so $S' \circ S$ is the identity on $\mathcal{F}_X$. A similar argument yields that $S \circ S'$ is the identity on $\mathcal{F}_Y$ and hence $S' = S^{-1}$.

Now we prove that $S \circ L_X = L_Y \circ S$. Suppose that the symbolic conjugacy φ is induced by the 1-block map Φ. Then it is easily seen that for each $x_- \in X_-$

$$\{\Phi(a) : a \in i_X(x_-)\} = \{b : b \in i_Y(\varphi(x)_-)\}.$$

Thus we obtain that for $\omega = f(x_-) \in F_X$,

$$\begin{aligned} S(L_X(\omega)) &= S(\sum_{a \in i_X(x_-)} f(x_-a)) \\ &= \sum_{a \in i_X(x_-)} S(f(x_-a)) \\ &= \sum_{a \in i_X(x_-)} f(\varphi(x)_-\Phi(a)) \\ &= \sum_{b \in i_Y(\varphi(x)_-)} f(\varphi(x)_-b) \\ &= L_Y(f(\varphi(x)_-)) \\ &= L_Y(S(\omega)). \end{aligned}$$

In order to prove the last statement of the theorem, it suffices to put $T = L_X \circ S^{-1}$. $\square$

A subshift X over the alphabet $\mathcal{A}$ is said to be *bipartite* if there are disjoint subsets C and D of $\mathcal{A}$ such that for any $(x_i)_{i \in \mathbb{Z}} \in X$ either $x_i \in C$ and $x_{i+1} \in D$ or $x_i \in D$ and $x_{i+1} \in C$. Then the second power subshift $X^{(2)}$ is divided into two disjoint subshifts X_{CD} and X_{DC} where X_{CD} is defined to be the set of sequences $(c_i d_i)_{i \in \mathbb{Z}} \in X^{(2)}$ such that $c_i \in C$ and $d_i \in D$ for each $i \in \mathbb{Z}$, and X_{DC} is the set of sequences $(d_i c_i)_{i \in \mathbb{Z}} \in X^{(2)}$ such that $d_i \in D$ and $c_i \in C$ for each $i \in \mathbb{Z}$.

The conjugacy $\zeta : X_{CD} \to X_{DC}$ defined by

$$\zeta((c_i d_i)_{i \in \mathbb{Z}}) = (d_i c_{i+1})_{i \in \mathbb{Z}}$$

is said to be a *forward bipartite conjugacy*. The *backward bipartite conjugacy* $\tau : X_{CD} \to X_{DC}$ is defined by

$$\tau((c_i d_i)_{i \in \mathbb{Z}}) = (d_{i-1} c_i)_{i \in \mathbb{Z}}.$$

In [N], it was proved that symbolic and bipartite conjugacies are basic constituents of any conjugacy. We state the result in the following.

THEOREM 3. *Any conjugacy φ between subshifts is factorized into a composition of the form*

$$\varphi = \kappa_n \zeta_n \kappa_{n-1} \zeta_{n-1} \dots \kappa_1 \zeta_1 \kappa_0,$$

where $\kappa_0, \dots, \kappa_n$ are symbolic conjugacies, and $\zeta_1, \dots, \zeta_n$ are either forward or backward bipartite conjugacies.

THEOREM 4. *Let $\zeta : X_{CD} \to X_{DC}$ be a forward or backward bipartite conjugacy. Then there exist linear maps $S : \mathcal{F}_{X_{CD}} \to \mathcal{F}_{X_{DC}}$ and $T : \mathcal{F}_{X_{DC}} \to \mathcal{F}_{X_{CD}}$ such that $T \circ S = L_{X_{CD}}$ and $S \circ T = L_{X_{DC}}$.*

Proof. Let $\omega = f(\dots(c_{-1}d_{-1})(c_0 d_0)) \in F_{X_{CD}}$. Define

$$S(\omega) = \sum_c f(\dots(d_{-1}c_0)(d_0 c))$$

where the sum runs over all the elements $c \in i_X(\dots d_{-1}c_0 d_0)$. Clearly S is a well-defined map from $F_{X_{CD}}$ to $F_{X_{DC}}$. Now S can be extended as a linear map from $\mathcal{F}_{X_{CD}}$ to $\mathcal{F}_{X_{DC}}$. The linear map $T : \mathcal{F}_{X_{DC}} \to \mathcal{F}_{X_{CD}}$ is defined by

$$T(\tau) = \sum_{d \in i_X(\dots c_0 d_0 c_1)} f(\dots(c_0 d_0)(c_1 d))$$

for $\tau = f(\dots(d_{-1}c_0)(d_0 c_1)) \in F_{X_{DC}}$. We find that

$$\begin{aligned}
T(S(\omega)) &= T\Big(\sum_{c \in i_X(\dots d_{-1} c_0 d_0)} f(\dots(d_{-1}c_0)(d_0 c))\Big) \\
&= \sum_{c \in i_X(\dots d_{-1} c_0 d_0)} T(f(\dots(d_{-1}c_0)(d_0 c))) \\
&= \sum_{c \in i_X(\dots d_{-1} c_0 d_0)} \sum_{d \in i_X(\dots c_0 d_0 c)} f(\dots(c_0 d_0)(cd)) \\
&= \sum_{cd \in i_{X_{CD}}(\dots(c_{-1}d_{-1})(c_0 d_0))} f(\dots(c_0 d_0)(cd)) \\
&= L_{X_{CD}}(\omega)
\end{aligned}$$

for every $\omega = f(\dots(c_{-1}d_{-1})(c_0 d_0)) \in F_{X_{CD}}$, and a similar argument shows that $S \circ T = L_{X_{DC}}$. □

THEOREM 5. *If two subshifts X and Y are conjugate then there are linear maps S_i's and T_i's such that*

$$L_X = T_1 \circ S_1, S_1 \circ T_1 = T_2 \circ S_2, \dots, S_{n-1} \circ T_{n-1} = T_n \circ S_n, S_n \circ T_n = L_Y.$$

Proof. Using Theorem 3, we can decompose the conjugacy into symbolic and bipartite conjugacies. Then the result follows immediately from Theorem 2 and Theorem 4. □

Two subshifts of finite type are called strongly shift equivalent if the relation in Theorem 5 is satisfied, and it is known that strongly shift equivalent subshifts of finite type are conjugate. We should note that this is not the case in general.

Combining Theorem 1 and Theorem 5, we easily get the following result.

COROLLARY 6. *If two subshifts are conjugate, then they are shift equivalent and therefore their dimension pairs are isomorphic.*

References

[BMT] M. Boyle, B. Marcus and P. Trow, *Resolving maps and the dimension group for shifts of finite type*, Memoirs of A. M. S. **70** (1987).

[K1] W. Krieger, *On a dimension for a class of homeomorphism groups*, Math. Ann. **252** (1980), 87–95.

[K2] W. Krieger, *On dimension functions and topological Markov chains*, Invent. Math. **56** (1980), 239–250.

[K3] W. Krieger, *On sofic systems I*, Israel J. Math. **48** (1984), 305–330.

[N] M. Nasu, *Topological conjugacy for sofic systems*, Ergod. Th. & Dynam. Sys. **6** (1986), 265–280.

[W] R. F. Williams, *Classification of subshifts of finite type*, Annals of Math. **98** (1973), 120–153.

Department of Mathematics
Ajou University
Suwon 442-749, Korea

E-mail: jslee@madang.ajou.ac.kr