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THE JUMP NUMBER OF BIPARTITE POSETS FROM MATROIDS

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1. Introduction

In this paper we try to investigate the connection between matroids and jump numbers. A couple of papers [3,5] are known, but they discuss optimization problems with matroid structure. Here we calculate the jump numbers of some bipartite posets which are induced by matroids.

Let P be a finite poset and let $a, b \in P$ with $a < b$. Then b *covers* a, written $a \prec b$, provided that for any $c \in P$, $a < c \leq b$ implies that $c = b$. A *linear extension* of a poset P is a linear order $L = x_1, x_2, \ldots, x_n$ of the elements of P such that $x_i < x_j$ implies $i < j$. A *(P,L)-chain* is a maximal sequence of elements $z_1, z_2, \ldots, z_k$ such that $z_1 \prec z_2 \prec \cdots \prec z_k$ in both L and P. Let $c(L)$ be the number of (P, L)-chains in L. A consecutive pair (x_i, x_{i+1}) of elements in L is a *jump* of P in L if x_i is incomparable to x_{i+1} in P. Let $s(L, P)$ be the number of jumps of P in L. Then $s(L, P) = c(L) - 1$. Let $s(P)$ be the minimum of $s(L, P)$ over all linear extensions L of P. The number $s(P)$ is called the *jump number* of P.

In the following we will consider the basic properties of matroids as known. However, to fix the terminology we give a brief survey of some definitions. A nice introduction to matroid theory is given in Welsh's book [6]. The cardinality of a set A will be denoted by $|A|$.

A *matroid* is defined on a finite set E by a family of subsets of E, called the *independent* subsets of E, that obey the following axioms :

(i) $\emptyset$ is independent ;

(ii) any subset of an independent set is independent;

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(iii) for any set $A \subseteq E$, all maximal independent subsets of A have the same cardinality.

The common cardinality in (iii) is called the *rank* of A , written $r(A)$. A maximal independent set is called a *base*. A set which is not independent is said to be *dependent*. The minimal dependent sets are called *circuits*. For example, in graphs, bases are spanning trees and circuits are simple closed paths. For all $A \subseteq E$ the maximal set S such that $A \subseteq S \subseteq E$ and $r(A) = r(S)$ is well defined, and this set is called the *span* of A, written $sp(A)$. If $I \subseteq E$ is independent and $e \in sp(I) - I$, then $I \cup \{e\}$ contains a unique circuit, which we shall denote $C(e, I)$.

Let M be a matroid over a set E, and let I be independent. The *(circuit) dependence poset* $DP(I)$ of I is a bipartite poset whose minimal[or maximal] elements are the elements of I[or $E - I$]. There is a comparability in $DP(I)$ between $e_1 \in I$ and $e_2 \in E - I$ if and only if $e_2 \in sp(I)$ and $e_1 \in C(e_2, I)$.

2. Krogdahl's approach

The complete graph on n vertices is denoted by K_n. Although there are different bases for K_n, the jump number of the dependence poset of any base of K_n is unique.

THEOREM 2.1. *For a base B of K_n, we have* $s(DP(B)) = \binom{n-1}{2}$.

Proof. Note that $|B| = n-1$. For a given $b_1 \in B$, there exist $b_2 \in B$, $e_1 \in E - B$ such that $\{b_1, b_2, e_1\}$ makes a circuit in K_n. Since B is a spanning tree in K_n, for a given connected subset $\{b_1, b_2, \ldots, b_k\}$ of B and for $k \geq 2$, there exists $b_{k+1} \in B$ such that b_{k+1} is adjacent to b_i for some $i \in \{1, 2, \ldots, k\}$. Let $e_k \in E - B$ such that $\{b_i, b_{k+1}, e_k\}$ makes a circuit in K_n. Then we can easily construct a linear extension L of $DP(B)$ such that $\{(b_{k+1}, e_k) : k = 1, 2, \ldots, n-2\}$ is a set of 2-element $(DP(B), L)$-chains. Thus $s(DP(B)) \leq \binom{n}{2} - (n-2) - 1 = \binom{n-1}{2}$. On the other hand, since $|B| = n - 1$ for any linear extension L of $DP(B)$ the number of 2-element $(DP(B), L)$-chains is at most $n - 2$. Hence $s(DP(B)) \geq \binom{n}{2} - (n-2) - 1 = \binom{n-1}{2}$.

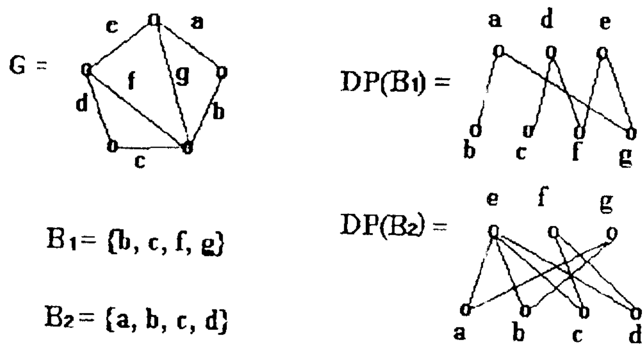

FIG.1

Let $K_{m,n}$ denote the complete bipartite graph. Also, the jump number of the dependence poset of a base for $K_{m,n}$ is independent of the choice of the base.

THEOREM 2.2. *Let* $min\{m,n\} \geq 2$. *For a base* B *of* $K_{m,n}$ *we have*

$$s(DP(B)) = (m-1)(n-1)+1.$$

Proof. Note that $|B| = m+n-1$. Since the complete bipartite graph contains only even circuits, B contains $\{b_1, b_2, b_3\}$ such that $\{b_1, b_2, b_3, e_1\}$ makes a circuit in $K_{m,n}$. Since B is a spanning tree in $K_{m,n}$, for a given connected subset $\{b_1, b_2, \ldots, b_k\}$ of B and for $k \geq 3$, there exists $b_{k+1} \in B$ such that b_{k+1} is adjacent to b_i for some $i \in \{1, 2, \ldots, k\}$. Let $e_{k-1} \in E - B$ such that $\{b_i, b_{k+1}, b_j, e_{k-1}\}$ makes a circuit in $K_{m,n}$ for some $j \in \{1, 2, \ldots, k\}$. Then, by the same technique as in the proof in Theorem 2.1, we can easily construct a linear extension L of $DP(B)$ such that $\{(b_{k+1}, e_{k-1}) : k = 2, 3, \ldots, m+n-2\}$ is a set of 2-element $(DP(B), L)$-chains.

Thus $s(DP(B)) \leq mn - (m+n-3) - 1 = (m-1)(n-1)+1$. Also, by the same reason as in the proof in Theorem 2.1, we get $s(DP(B)) \geq (m-1)(n-1)+1$.

REMARK. We conjectured that $s(DP(B))$ is unique for any base B of a simple graph. But this is not true. Fig.1 gives two different bases B_1, B_2 of a graph G and $s(DP(B_1)) \neq s(DP(B_2))$.

3. New approach

Differently from the Krogdahl's approach, we define a new poset for an independent set in a matroid. Let M be a matroid over a set E, and let I be independent. The ***adjacent dependence poset*** $ADP(I)$ of I is a bipartite poset whose minimal[or maximal] elements are the elements of I[or $E - I$]. There is a comparability in $ADP(I)$ between $e_1 \in I$ and $e_2 \in E - I$ if and only if $e_2 \in sp(I)$, $e_1 \in C(e_2, I)$, and e_1, e_2 are adjacent. In Fig.2, for an independent set I in a given graph G, we give the two posets $DP(I)$, $ADP(I)$.

Although the definition of a dependence poset and that of an adjacent dependence poset are different, for any base B of K_n we get the same jump number.

THEOREM 3.1. *For any base B of K_n, we have*

$$s(ADP(B)) = \binom{n-1}{2}.$$

Proof. Let $B = \{b_1, b_2, \ldots, b_{n-1}\}$ and let $1 \le i < j \le n$. For any $b_i, b_j \in B$, there exists a unique $e_{ij} \in E - B$ such that $\{b_i, b_j, e_{ij}\}$ or $\{b_i, b_j, e_{ij}, b_k, \ldots, b_l\}$ makes a circuit in K_n.

Then, by the same method as in the proof in Theorem 2.1, we can easily construct a linear extension L of $DP(B)$ such that $\{(b_i, e_{1j}) : j = 2, 3, \ldots, n-1\}$ is a set of 2-element $(ADP(B), L)$-chains. Thus $s(ADP(B)) \le \binom{n}{2} - (n-2) - 1 = \binom{n-1}{2}$. Also, by the same reason as in the proof in Theorem 2.1, we get $s(ADP(B)) \ge \binom{n-1}{2}$.

REMARK. Unlike the result in Theorem 2.2, $s(ADP(B))$ is not unique for a base B of $K_{m,n}$, where $min\{m, n\} \ge 2$. Fig.3 gives two different bases B_1, B_2 of $K_{3,5}$ and $s(ADP(B_1)) = 10 \ne 9 = s(ADP(B_2))$.

So far we have studied bipartite posets from matroids. What about the other direction? That is, for a given bipartite poset P is there a base B of a matroid such that $P = DP(B)$ or $P = ADP(B)$? This seems to be very difficult. But we can easily show that there are some structural properties. Let X be the minimal elements of P. If

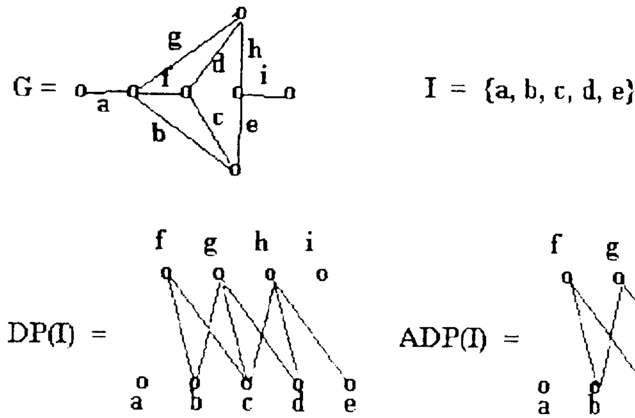

FIG.2

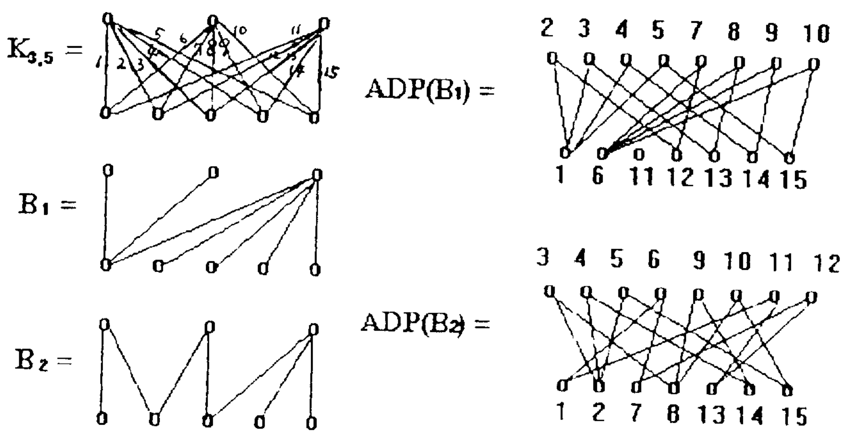

FIG.3

$P = DP(B)$ for some base B of a matroid, then for any $v \in X$ the number N of elements which cover v satisfies $2 \leq N \leq |B|$. On the other hand, if $P = ADP(B)$ for some base B of a matroid, then for any $v \in X$ the number of elements which cover v is 2.

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