J. Korean Math. Soc. **33** (1996), No. 3, pp. 657–667

CRITICAL ZEROS AND NONREAL ZEROS OF SUCCESSIVE DERIVATIVES OF REAL ENTIRE FUNCTIONS

YOUNG-ONE KIM

1. Introduction

This paper is concerned with the zeros of successive derivatives of real entire functions. In order to state the background to our results, as well as the results themselves, let us introduce some terminologies.

The *order* ρ of an entire function $f(z)$ is defined by

$$\rho = \overline{\lim_{r\to\infty}} \frac{\log\log M(r;f)}{\log r},$$

where $M(r;f)$ is the maximum modulus of $f(z)$ on the circle $|z| = r$. If an entire function $f(z)$ is of order ρ and if $0 < \rho < \infty$, then the *type* τ of $f(z)$ is defined by

$$\tau = \overline{\lim_{r\to\infty}} \frac{\log M(r;f)}{r^\rho}.$$

If $\tau = 0$, the function $f(z)$ is said to be of *minimal type*, if $0 < \tau < \infty$ of *mean type*, and if $\tau = \infty$ of *maximal type*. It is well known that order and type are unchanged by differentiation [L, p. 4, Theorem 2].

Let $\{a_j\}$ be a sequence of complex numbers with $|a_j| \to \infty$ as $j \to \infty$. The *convergence exponent* of the sequence $\{a_j\}$ is the infimum of those real numbers $\rho > 0$ such that $\sum_{a_j\neq 0} |a_j|^{-\rho} < \infty$. It is well known that the convergence exponent of the zeros of an entire function does not exceed the order of the function [L, p.16, Theorem 6].

Received November 27, 1995.

1991 AMS Subject Classification: 30D15, 30D35.

Key words: Pólya-Wiman conjecture, Laguerre-Pólya class, Fourier critical point.

This research is supported by Sejong university research fund and SNU-GARC.

The *genus* of an entire function $f(z)$ is the smallest integer p such that $f(z)$ can be represented in the form

$$f(z) = z^n e^{P(z)} \prod_j \left(1 - \frac{z}{a_j}\right) e^{\frac{z}{a_j} + \frac{1}{2}\left(\frac{z}{a_j}\right)^2 + \cdots + \frac{1}{p}\left(\frac{z}{a_j}\right)^p},$$

where $P(z)$ is a polynomial of degree $\leq p$, n is a nonnegative integer and the product converges absolutely and uniformly in compact sets in the plane. Note that if $f(z)$ is of genus p and $a_1, a_2, \cdots$ are the zeros of $f(z)$, then the convergence of the infinite product implies that $\sum_{a_j \neq 0} |a_j|^{-p-1} < \infty$. A well known theorem of Hadamard states that the order ρ and the genus p of an entire function of finite order satisfy the double inequality $p \leq \rho \leq p + 1$ [L, Chapt. 1, Sec. 10].

A *real entire function* is an entire function which assumes only real values on the real axis, and a real entire function $f(z)$ is said to be of *genus* 1^* if it can be expressed in the form

$$f(z) = e^{-\alpha z^2} g(z),$$

where $\alpha \geq 0$ and $g(z)$ is a real entire function of genus at most 1. If $f(z)$ is a real entire function, then its Maclaurin coefficients are all real, and consequently the zeros of $f(z)$ are symmetrically located with respect to the real axis. Let $\alpha \pm i\beta$, with $\alpha, \beta \in \mathbb{R}$, denote a pair of conjugate nonreal zeros of $f(z)$. The closed disk centered at the point $z = \alpha$ with radius $|\beta|$ will be called a *Jensen disk* of $f(z)$, and the union of all the Jensen disks of $f(z)$ will be denoted by $\mathcal{J}(f)$. A well known theorem of Jensen states that if $f(z)$ is a real entire function of genus 1^*, then all the nonreal zeros of $f'(z)$ are distributed in the set $\mathcal{J}(f)$. From here on, this fact will be referred as "Jensen's theorem." (For a proof of Jensen's theorem see [K2, p. 827].)

Let $f(z)$ be a nonconstant real entire function. Suppose that ξ is a real zero of $f^{(l)}(z)$ of multiplicity m but not a zero of $f^{(l-1)}(z)$. That is

$$\begin{gathered} f^{(l-1)}(\xi) \neq 0, \\ f^{(l)}(\xi) = f^{(l+1)}(\xi) = \cdots = f^{(l+m-1)}(\xi) = 0, \\ f^{(l+m)}(\xi) \neq 0. \end{gathered}$$

Put

$$k = \begin{cases} \dfrac{m}{2}, & \text{if } m \text{ is even,} \\ \dfrac{m+1}{2}, & \text{if } m \text{ is odd and} \quad f^{(l-1)}(\xi)f^{(l+m)}(\xi) > 0, \\ \dfrac{m-1}{2}, & \text{if } m \text{ is odd and} \quad f^{(l-1)}(\xi)f^{(l+m)}(\xi) < 0. \end{cases}$$

If $k > 0$, then ξ is said to be a *critical zero* of $f^{(l)}(z)$ of the multiplicity k. The critical zeros of the derivatives of $f(z)$ are the *critical points* of $f(z)$.

In 1930, G. Pólya conjectured the following three hypothetical theorems [P1].

A. *A real entire function of genus* 0 *has just as many critical points as couples of nonreal zeros.*

B. *If an entire function of genus* 1^* *has only a finite number of nonreal zeros, it has just as many critical points as couples of nonreal zeros.*

C. *If an entire function of genus* 1^* *has only a finite number of nonreal zeros, its derivatives from a certain one onward, let us say* $f^{(m)}(z), f^{(m+1)}(z), \cdots$, *have real zeros only.*

In the same paper, Pólya proved the following theorem which shows that the hypothetical theorems B and C are equivalent.

THEOREM I. *Let* $f(z)$ *be a nonconstant real entire function of genus* 1^*, *and assume that* $f(z)$ *has only a finite number of nonreal zeros. Then* $f'(z)$ *is also of genus* 1^* *and has finitely many nonreal zeros. Moreover, if* $f(z)$ *has* $2J$ *nonreal zeros and* $f'(z)$ *has* $2J'$ *nonreal zeros, then* $f'(z)$ *has exactly* $J - J'$ *critical zeros.*

The hypothetical theorem C is known as the Pólya–Wiman conjecture and it has been completely proved by T. Craven, G. Csordas, W. Smith and the author [CCS1, CCS2, K1]. (For a very simple and direct proof of the Pólya–Wiman conjecture see [K2].) On the other hand, the hypothetical theorem A remains unproved until now. Note that to prove the hypothetical theorem A it is enough to show that if a nonconstant real entire function of genus 0 has infinitely many nonreal zeros, then it has infinitely many critical points.

Recently, the author proved the following results [K3]:

THEOREM II. *Let $f(z)$ be a nonconstant real entire function of genus 1^* and suppose that a, b, with $a < b$, are real numbers which are located outside the Jensen disks of $f(z)$. Then $f(z)$ and $f'(z)$ have finitely many zeros in the region $a \leq \mathrm{Re}\, z \leq b$. Moreover, if $f(z)$ has $2J$ nonreal zeros in $a \leq \mathrm{Re}\, z \leq b$ and if $f'(z)$ has $2J'$ nonreal zeros in $a \leq \mathrm{Re}\, z \leq b$, then $f'(z)$ has exactly $J - J'$ critical zeros in the closed interval $[a, b]$.*

THEOREM III. *Let $f(z)$ be a nonconstant real entire function of genus 1^*, and assume that $f(z)$ is at most of order ρ, $0 < \rho \leq 2$, and minimal type. If there is a positive real number A such that all the zeros of $f(z)$ are distributed in the infinite strip $|\mathrm{Im}\, z| \leq A$, then for any positive constant B there is a positive integer n_1 such that $f^{(n)}(z)$ has only real zeros in $|\mathrm{Re}\, z| \leq Bn^{\frac{1}{\rho}}$ for all $n \geq n_1$.*

THEOREM IV. *Let $f(z)$ be a nonconstant real entire function of order ρ, and let ρ_C be the convergence exponent of the nonreal zeros of $f(z)$. If there is a positive real number A such that all the zeros of $f(z)$ are distributed in the infinite strip $|\mathrm{Im}\, z| \leq A$ and if $\rho + 2\rho_C < 2$, then $f(z)$ has just as many critical points as couples of nonreal zeros.*

Theorem IV is a consequence of Theorem II and Theorem III. Note that Theorem II generalizes Theorem I, and that Theorem III can be regarded as a *local* version of the Pólya–Wiman conjecture.

In this paper, we will consider those real entire functions of genus 1^* whose nonreal zeros are distributed in the region

$$\mathcal{R}(k) = \{z : \mathrm{Re}\, z \geq 0, |\mathrm{Im}\, z| \leq (\mathrm{Re}\, z)^k\}$$

for some k, $0 < k < 1$, and prove the following:

THEOREM 1. *Let $f(z)$ be a nonconstant real entire function of genus 1^*. Assume that $f(z)$ is at most of order ρ, $0 < \rho < 2$, and minimal type, and that there is a constant k, $0 < k < 1$, such that the nonreal zeros of $f(z)$ are distributed in the region $\mathcal{R}(k)$. If $\rho \leq 2(1 - k)$, then for each positive constant B there is a positive integer n_1 such that for* *all* $n \geq n_1$ *the nonreal zeros of* $f^{(n)}(z)$ *are distributed in the half plane* $\operatorname{Re} z > Bn^{\frac{1}{\rho}}$.

THEOREM 2. *Let* $f(z)$ *be a nonconstant real entire function of genus* 1^*. *Assume that* $f(z)$ *is at most of order* ρ, $0 < \rho \leq 2$, *and minimal type, and that there is a constant* k, $0 < k < 1$ *such that the nonreal zeros of* $f(z)$ *are distributed in the region* $\mathcal{R}(k)$. *Let* $\{z_j\}$ *denote the sequence of distinct nonreal zeros of* $f(z)$. *If*

$$\sum_j |z_j|^{\frac{\rho}{2}+k-1} < \infty,$$

then $f(z)$ *has just as many critical points as couples of nonreal zeros.*

REMARKS. (a) Theorem 1 implies that the final set (for the definition, see [P3]) of a real entire function $f(z)$ is contained in the real axis whenever $f(z)$ satisfies the conditions of Theorem 1.

(b) Let k, $0 < k < 1$, be given. Since the convergence exponent of the zeros of an entire function does not exceed the order of the function, Theorem 2 establishes the validity of the hypothetical theorem A for functions of order less than $\frac{2}{3}(1-k)$ whose nonreal zeros are distributed in the region $\mathcal{R}(k)$.

2. Proof of the theorems

In the proof of our theorems we will use the following:

LEMMA. *Let* $z_j = \alpha_j + i\beta_j$, $\alpha_j, \beta_j \in \mathbb{R}$, $j = 0, 1, \cdots, n$, *be complex numbers, and assume that*

$$|z_{j+1} - \alpha_j| \leq \beta_j \qquad (j = 0, 1, \cdots, n-1).$$

Then we have the following inequalities:

(a) $$0 \leq \beta_n \leq \beta_{n-1} \leq \cdots \leq \beta_0.$$

(b) $$|\alpha_0 - \alpha_1| + |\alpha_1 - \alpha_2| + \cdots + |\alpha_{n-1} - \alpha_n| \leq \beta_0 - \beta_n + \sqrt{n(\beta_0{}^2 - \beta_n{}^2)},$$

and

$$\text{(c)} \qquad \frac{(\alpha_n - \alpha_0)^2}{n\beta_0^2} + \frac{\beta_n^2}{\beta_0^2} \le 1.$$

Proof. Induction on n. □

GONTCHAROFF'S FIRST ESTIMATE. *Let $f(z)$ be an analytic function in a convex domain $\mathcal{D}$. If $\sup_{z\in\mathcal{D}} |f(z)| = M < \infty$, and if $z, z_0, z_1, \cdots, z_{n-1} \in \mathcal{D}$, then*

$$\left| \int_{z_0}^{z} \int_{z_1}^{\zeta_1} \cdots \int_{z_{n-1}}^{\zeta_{n-1}} f(\zeta_n) d\zeta_n \cdots d\zeta_2 d\zeta_1 \right| \le \frac{M}{n!} \left(|z - z_0| + |z_0 - z_1| + \cdots + |z_{n-2} - z_{n-1}| \right)^n .$$

Proof. See [G, pp. 11–13]. □

GONTCHAROFF'S SECOND ESTIMATE. *Let H and σ be positive real numbers and let $f(z)$ be an entire function such that $M(r; f) < e^{Hr^\sigma}$ for all sufficiently large r. Let α be an arbitrary positive real number and let λ be the positive root of the equation*

$$H\sigma\lambda^{\sigma-1}(\lambda - \alpha) = 1.$$

Then

$$\varlimsup_{n\to\infty} n^{\frac{1}{\sigma}} \left(\frac{M(\alpha n^{\frac{1}{\sigma}}; f^{(n)})}{n!} \right)^{\frac{1}{n}} \le H\sigma\lambda^{\sigma-1} e^{H\lambda^\sigma}.$$

Proof. See [G, pp. 24–28]. □

Proof of Theorem 1. First of all, note that since $f(z)$ is assumed to be of order less than 2, all the derivatives of $f(z)$ are of order less than 2, and hence of genus 1^*, by Hadamard's theorem.

Suppose that z_n is a nonreal zero of $f^{(n)}(z)$ which lies in the upper half plane. From Jensen's Theorem, we can find complex numbers z_0,

$z_1, z_2, \cdots, z_{n-1}$ such that each z_j, $j = 0, 1, \cdots, n-1$, is a nonreal zero of $f^{(j)}(z)$ and that

$$(1) \qquad |z_{j+1} - \operatorname{Re} z_j| \leq \operatorname{Im} z_j \qquad (j = 0, 1, \cdots, n-1).$$

Let z be an arbitrary complex number. Since $f^{(j)}(z_j) = 0$, $j = 0, 1, \cdots, n-1$, we have

$$f(z) = \int_{z_0}^{z} \int_{z_1}^{\zeta_1} \cdots \int_{z_{n-1}}^{\zeta_{n-1}} f^{(n)}(\zeta_n) d\zeta_n \cdots d\zeta_2 d\zeta_1,$$

and hence Gontcharoff's first estimate gives

$$(2) \qquad |f(z)| \leq \frac{M}{n!} \left(|z - z_0| + |z_0 - z_1| + \cdots + |z_{n-2} - z_{n-1}|\right)^n,$$

where M is the maximum of $|f^{(n)}(\zeta)|$ on the convex hull of the set $\{z, z_0, z_1, \cdots, z_{n-1}\}$.

For $j = 1, 2, \cdots, n$, let $z_j = \alpha_j + i\beta_j$. Then (1) and the inequalities (a) and (b) of the lemma give

$$(3) \quad |z_0 - z_1| + |z_1 - z_2| + \cdots + |z_{n-1} - z_n| \leq \beta_0 - \beta_n + \sqrt{n(\beta_0^2 - \beta_n^2)}.$$

From (1) and the inequality (c) of the lemma, we have

$$\frac{(\alpha_n - \alpha_0)^2}{n\beta_0^2} + \frac{\beta_n^2}{\beta_0^2} \leq 1,$$

so that

$$\alpha_0 - |\alpha_n| \leq |\alpha_0 - \alpha_n| \leq \sqrt{n}|\beta_0| = \sqrt{n}\beta_0,$$

and consequently

$$(4) \qquad \alpha_0 - |\alpha_n| \leq \sqrt{n}\alpha_0^k,$$

since $z_0 = \alpha_0 + i\beta_0 \in \mathcal{R}(k)$. Let A be the positive real number which satisfies $A - |\alpha_n| = \sqrt{n}A^k$, then (4) implies that $\alpha_0 \leq A$, and it is clear that $|\alpha_n| \leq A$, and hence we have

$$\sqrt{n}\alpha_0^k \leq \sqrt{n}A^k \leq A = (\sqrt{n} + |\alpha_n|A^{-k})^{\frac{1}{1-k}} \leq (\sqrt{n} + |\alpha_n|^{1-k})^{\frac{1}{1-k}}.$$

It will be convenient to denote the last term of the above inequality by $A(\alpha_n)$, that is,

$$A(\alpha_n) = (\sqrt{n} + |\alpha_n|^{1-k})^{\frac{1}{1-k}}.$$

Then (3) implies that

$$\begin{aligned}
&|z_0 - z_1| + |z_1 - z_2| + \cdots + |z_{n-1} - z_n| \\
&\leq 2\sqrt{n}\beta_0 \\
&\leq 2\sqrt{n}\alpha_0^k \\
&\leq 2A(\alpha_n).
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
(5) \qquad |z_j| &\leq |z_0| + |z_0 - z_1| + \cdots + |z_{j-1} - z_j| \\
&\leq \alpha_0 + \alpha_0^k + 2A(\alpha_n) \\
&\leq 3A(\alpha_n) + \frac{1}{\sqrt{n}}A(\alpha_n) \qquad\qquad (j = 0, 1, \cdots, n-1),
\end{aligned}$$

and

$$\begin{aligned}
(6) \qquad &|z - z_0| + |z_0 - z_1| + |z_1 - z_2| + \cdots + |z_{n-2} - z_{n-1}| \\
&\leq |z| + |z_0| + |z_0 - z_1| + |z_1 - z_2| + \cdots + |z_{n-2} - z_{n-1}| \\
&\leq |z| + 5A(\alpha_n) + \frac{1}{\sqrt{n}}A(\alpha_n).
\end{aligned}$$

Now assume that there is a positive constant B and an infinite set E of positive integers such that for each $n \in E$ there is a nonreal zero $z_n = \alpha_n + i\beta_n$ of $f^{(n)}(z)$ such that $|\alpha_n| \leq Bn^{\frac{1}{\rho}}$. Since we have assumed that $\rho \leq 2(1-k)$, the set

$$\{n^{-\frac{1}{\rho}}A(\alpha_n) : n \in E\}$$

is bounded above, and hence (2), (5) and (6) imply that there is a positive constant B_1 such that

$$(7) \qquad |f(z)| \leq \frac{M(B_1 n^{\frac{1}{\rho}}; f^{(n)})}{n!}\left(B_1 n^{\frac{1}{\rho}}\right)^n \qquad (|z| \leq 1,\ n \in E).$$

For each $\epsilon > 0$ let λ_ϵ be the positive real number such that

$$\epsilon\rho\lambda_\epsilon^{\rho-1}(\lambda_\epsilon - B_1) = 1.$$

Then it is easy to see that

$$\lim_{\epsilon\to 0} \epsilon\lambda_\epsilon^{\rho-1} e^{\epsilon\lambda_\epsilon^\rho} = 0. \tag{8}$$

Since $M(r; f) = o(e^{\epsilon r^\rho})$ for all $\epsilon > 0$, Gontcharoff's second estimate gives

$$\overline{\lim_{n\to\infty}}\, B_1 n^{\frac{1}{\rho}} \left(\frac{M(B_1 n^{\frac{1}{\rho}}; f)}{n!}\right)^{\frac{1}{n}} < B_1\epsilon\rho\lambda_\epsilon^{\rho-1}e^{\epsilon\lambda_\epsilon^\rho} \qquad (\epsilon > 0),$$

and hence (7) and (8) imply that $f(z) = 0$ for $|z| \le 1$. From this contradiction, we see that for each positive constant B there is a positive constant n_1 such that for all $n \ge n_1$ the nonreal zeros of $f^{(n)}(z)$ are distributed in the region $|\mathrm{Re}\, z| > Bn^{\frac{1}{\rho}}$. Now the following observation gives the desired result: Since the nonreal zeros of $f(z)$ are distributed in the region $\mathcal{R}(k)$, Jensen's theorem and the inequality (c) of the lemma imply that for all $n = 1, 2, \cdots$, the nonreal zeros of $f^{(n)}(z)$ are distributed in the half plane

$$\mathrm{Re}\, z \ge \min_{x\ge 0}(x - \sqrt{n}x^k).$$

On the other hand, the assumption $\rho \le 2(1-k)$ implies that

$$\min_{x\ge 0}(x - \sqrt{n}x^k) = O(n^{\frac{1}{\rho}}) \qquad \text{as} \qquad n \to \infty. \quad \square$$

Proof of Theorem 2. If $f(z)$ has only a finite number of nonreal zeros, then our assertion follows from Theorem 1 (the Pólya–Wiman conjecture) of [K1] and Theorem I of section 1. So assume that $f(z)$ has infinitely many nonreal zeros, and denote the *distinct* nonreal zeros of $f(z)$ by $\alpha_j \pm i\beta_j$, $j = 1, 2, \cdots$. We can assume, without loss of generality, that $0 < \alpha_1 \le \alpha_2 \le \cdots$.

We will show that for each positive integer K, $f(z)$ has at least K critical points. Let K be an arbitrary positive integer. It will be convenient to denote the exponent $\frac{\rho}{2}+k-1$ by $-q$, so that $\sum_j |\alpha_j + i\beta_j|^{-q} < \infty$, $\rho + 2q + 2k = 2$ and $0 < q < 1$. Since $0 < k < 1$ and $|\beta_j| \leq \alpha_j^k$ for $j = 1, 2, \cdots$, we have $\sum_j \alpha_j^{-q} < \infty$, and hence

$$\overline{\lim_{j\to\infty}} \frac{\alpha_{j+1}-\alpha_j}{\alpha_{j+1}^{1-q}} = \infty. \tag{9}$$

Since $q > 0$, we have $\rho < 2(1-k) < 2$, and hence we can apply Theorem 1 to obtain a positive integer n_1 such that for all $n \geq n_1$, $f^{(n)}(z)$ has only real zeros in the region $\operatorname{Re} z \leq n^{\frac{1}{\rho}}$. From (9), we can find a positive integer J such that

$$K \leq J, \tag{10}$$

$$2\sqrt{2}\alpha_{J+1}^{1-q} < \alpha_{J+1} - \alpha_J, \quad \text{and} \tag{11}$$

$$n_1^{\frac{1}{\rho}} < \alpha_{J+1}. \tag{12}$$

From (12), we have $1 < \alpha_{J+1}^{\rho}$, and hence there is a positive integer N such that

$$\alpha_{J+1}^{\rho} < N < 2\alpha_{J+1}^{\rho}. \tag{13}$$

Since $\rho + 2q + 2k = 2$, (11) and (13) imply that
(14)
$$(|\beta_J| + |\beta_{J+1}|)\sqrt{N} < 2\alpha_{J+1}^{k}\sqrt{N} < 2\sqrt{2}\alpha_{J+1}^{\frac{\rho}{2}+k} = 2\sqrt{2}\alpha_{J+1}^{1-q} < \alpha_{J+1} - \alpha_J.$$

From Jensen's theorem and the inequality (c) of the lemma, we obtain

$$\mathcal{J}(f^{(n)}) \subset \bigcup_j \{z \mid \alpha_j - |\beta_j|\sqrt{n+1} \leq \operatorname{Re} z \leq \alpha_j + |\beta_j|\sqrt{n+1}\} \qquad (n = 0, 1, 2, \cdots).$$

Hence (14) implies that there is a positive real number B such that

$$B \notin \mathcal{J}(f) \cup \mathcal{J}(f') \cup \cdots \cup \mathcal{J}(f^{(N-1)}), \quad \text{and} \tag{15}$$

$$\alpha_J < B < \alpha_{J+1}. \tag{16}$$

From (16), $f(z)$ has at least $2J$ nonreal zeros in the region $\mathrm{Re}\, z \leq B$. From (12) and (13), we have $N \geq n_1$, and hence $f^{(N)}(z)$ has only real zeros in the region $\mathrm{Re}\, z \leq N^{\frac{1}{\rho}}$. Since $B < \alpha_{J+1} < N^{\frac{1}{\rho}}$, (15) and Theorem II of section 1 imply that the sum of the multiplicities of the critical zeros of $f'(z), f''(z), \cdots, f^{(N)}(z)$ in the interval $(-\infty, B]$ is at least J. Since $K \leq J$, $f(z)$ has at least K critical points in the interval $(-\infty, B]$. □

References

[CCS1] T. Craven, G. Csordas and W. Smith, *The zeros of derivatives of entire functions and the PólyaWiman conjecture*, Ann. of Math. (2) **125** (1987), 405-431.

[CCS2] ———, *Zeros of derivatives of entire functions*, Proc. Amer. Math. Soc. **101** (1987), 323-326.

[G] W. Gontcharoff, *Recherches sur le dériveés des fonctions analytiques*, Ann. École Norm. **47** (1930), 1-78.

[K1] Y. O. Kim, *A proof of the Pólya-Wiman conjecture*, Proc. Amer. Math. Soc. **109** (1990), 1045-1052.

[K2] ———, *On a generalization of the Pólya-Wiman conjecture*, Comm. Korean Math. Soc. **9** (1994), 825-830.

[K3] ———, *Critical points of real entire functions and a conjecture of Pólya*, Proc. Amer. Math. Soc. **124** (1996), 819-830.

[L] B. Ja. Levin, *Distribution of Zeros of Entire Functions*, Transl. Math. Mono., vol 5, Amer. Math. Soc., Providence, R.I., 1964.

[P1] G. Pólya, *Some problems connected with Fourier's work on transcendental equations*, Quart. J. Math. Oxford Ser. 1 (1930), 21-34.

[P2] ———, *Über die Realität der Nullstellen fast aller Ableitungen gewisser ganzer Funktionen*, Math. Ann. **114** (1937), 622-634.

[P3] ———, *On the zeros of derivatives of a function and its analytic character*, Bull. Amer. Math. Soc. **49** (1943), 178-191.

Department of Mathematics
College of Natural Sciences
Sejong University
Seoul 133-747, Korea