J. Korean Math. Soc. **33** (1996), No. 3, pp. 641–650

APPROXIMATE FIBRATIONS IN TOPOLOGICAL CATEGORY AND PL CATEGORY

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0. Introduction

Let G denote an upper semicontinuous(usc) decomposition of an $(n+k)$-manifold M into closed, connected n-manifolds. What can be said about the decomposition space $B = M/G$? What regularity properties are possessed by the decomposition map $p : M \to B$? Certain forms of these questions have been addressed by D. Coram and P. Duvall [C-D].

A proper map $p : M \to B(= M/G)$ between locally compact absolute neighborhood retracts(ANRs) is called an *approximate fibration* if it has the following homotopy lifting property: Given any open cover ϵ of B, an arbitrary space X and two maps $h : X \to M$ and $F : X \times I \to B$ such that $p \circ h = F_0$, there exists a homotopy lifting map $H : X \times I \to M$ such that $H_0 = h$ and $p \circ H$ is ϵ-close to F. The latter means : to each $z \in X \times I$ there corresponds $U_z \in \epsilon$ such that $\{F(z), pH(z)\} \subset U_z$.

One of the most valuable properties of an approximate fibration $p : M \to B$ is that there is an exact sequence relating the (shape) homotopy groups of $p^{-1}(b)$, M, and B, analogue to the one for Hurewicz fibrations providing theoretically computable information about any one of these three objects when corresponding data about the other two is known;

$$\cdots \to \pi_{i+1}(B) \to \pi_i(p^{-1}b) \to \pi_i(M) \to \pi_i(B) \to \cdots$$

Since D. Coram and P. Duvall published series of papers, many mathematicians were interested in the following question.

Received November 2, 1995.

1991 AMS Subject Classification: 57N15; 55R65.

Key words: Approximate fibration; Codimension k fibrator; Hopfian group; Hyperhopfian group; Hopfian manifold; Aspherical manifold.

The second author was partially supported by KOSEF 1995, 951-0105-027-1.

Question. When is a proper map $p : M \to B$ an approximate fibration ?

For the question, we set up the following data; a specific closed n-manifold N; an $(n+k)$-manifold M; a usc decomposition G of M into copies of N (up to shape); the associated decomposition space $B = M/G$, which is presumed to be a finite dimension; and the standard decomposition map $p : M \to B$.

We will call a closed n-manifold N a *codimension k fibrator* if whenever there is a usc decomposition G of an arbitrary $(n+k)$-manifold M such that each $g \in G$ is shape(homotopy) equivalent to N and dim $B < \infty$, $p : M \to B$ is an approximate fibration.

V.T. Liem [Li] proved that any n-sphere $S^n (n > 1)$ is a codimension 1 fibrator, and R.J. Daverman [D_1] showed that if G is a decomposition of an $(n+1)$-manifold M into continua having the shape of arbitrary closed n-manifolds then M/G is a 1-dimensional manifold, furthermore, if each element of G is locally flat in M then p is an approximate fibration.

It would be an exaggeration to claim the same for codimension 2 fibrators, but reasonably general conditions are known under which a given n-manifold functions in this way. For examples, R.J. Daverman and J.J. Walsh [D-W] showed that any n-sphere S^n $(n \geq k > 1)$ is a codimension k fibrator, and R.J. Daverman showed that every simply-connected closed manifold and all closed surfaces except those of Euler characteristic zero are codimension 2 fibrators [D_3]. In addition, he showed that 3-manifolds are analyzed almost completely in [D_4], although a crucial issue still outstanding is whether all Lens spaces are codimension 2 fibrators. Recently, he [D_6] showed that higher dimensional manifolds which satisfy a certain Hopfian manifold property are codimension 2 fibrators if they have either non-zero Euler characteristics or hyperhopfian fundamental groups.

On the other hand, the problem whether the class of codimension 2 fibrators is closed under finite product is not yet settled. But Y.H. Im [Im] showed that a finite product $N = F_1 \times F_2 \times \cdots \times F_m$ of closed orientable surfaces F_i $(i = 1, 2, \cdots, m)$ with $\chi(F_i) < 0$ is a codimension 2 fibrator. Also, Y.H. Im and M.K. Kang and K.M. Woo [I-K-W_1] extended this result to the extent that any product of any n-sphere S^n $(n > 1)$ and finite closed orientable surfaces F_i $(i = 1, 2, \cdots, m)$

with $\chi(F_i) < 0$ is a codimension 2 fibrator. More generally, they [I-K-W_2] showed that a product $N = A \times F$ of any simply-connected closed manifold A and any aspherical manifold F with hyperhopfian fundamental group is a codimension 2 fibrator.

In this paper, we obtain that a bundle structure $N = F_1 \tilde{\times} F_2$ is a codimension 2 fibrator, where F_i(i=1,2) is an orientable asherical closed manifold with $\chi(F_i) \neq 0$ and its fundamental group is hophian, and in addition, a bundle structure $N = F_1 \tilde{\times} F_2$ of closed orientable surfaces $F_i (i = 1, 2)$ with $\chi(F_i) < 0$ is a codimension 2 fibrator.

Codimension k ($k \geq 3$) fibrators are not well known in topological category. In codimension $k \geq 3$ case, the decomposition spaces of manifolds need not be manifolds and their dimensions could be infinite. Therefore, the problem is very complicated. But in PL(piecewise linear) category, codimension k($k \geq 3$) fibrators are fairly understood because we can remove the obstructions in topological category.

Restriction to this PL category offers several advantages. The target spaces are standard geometric objects, obviously finite-dimensional and locally contractible, features which a priori dispel potentially troublesome issues lurking in the background of the general(non-PL) category. The chief benefit is not the simplicial structure of the image, however, but rather the potential for inductive arguments, as in classical PL topology, which apply to the restriction of p over certain links in the target and bring about lowering of fiber codimension without changing fiber character. Often this pays off in improvement of results from the general category by a minimum of one extra dimension.

A continuation of earlier investigations into proper mappings defined on $(n + k)$-manifolds and having closed manifolds as point preimages examines certain PL aspects of the subject. Its primary aim is to identify closed n-manifolds N such that, for particular values of k, any (proper) PL map defined on a PL $(n + k)$-manifold is necessarily an approximate fibration whenever all point preimages are copies of N. With any PL approximate fibration defined on a connected domain, the various point preimages are homotopy equivalent.

Actually, surprisingly many manifolds are codimension k fibrators. For example, R.J. Daverman [D_5] showed that any closed orientable surface F with $\chi(F) < 0$ is a codimension k fibrator for all $k \geq 1$, while it is proved to be a codimension 2 fibrator in topological category.

In this paper, we obtain that a Hophian n-manifold N is a codimension $m > 2$ fibrator if it is a codimension 2 fibrator, $\pi_i(N) = 0$ for $1 < i \leq m$, and $\pi_1(N)$ is normally cohophian and has no proper subgroup isomorphic to $\pi_1(N)/A$, with A an Abelian subgroup, and in addition, a product $N = S^n \times F$ of any n-sphere S^n $(n \geq 3)$ and any closed orientable surface F with $\chi(F) < 0$ is a codimension $k = (n-1)$ PL fibrator, while it is proved to be a codimension 2 fibrator in topological category [I-K-W$_1$].

1. Definitions and notations

When we use a superscripted capital letter (e.g. M^n) to denote a topological manifold, the superscript will represent the dimension of a manifold. We assume all spaces are locally compact, metrizable, absolute neighborhood retracts(ANRs), and all manifolds are finite dimensional, connected, orientable and boundaryless.

I^n denotes the n-th power of the unit interval I and the symbol χ the Euler characteristic.

Homology and cohomology groups are computed with integer coefficients unless the coefficient module is mentioned.

A manifold M is said to be *closed* if M is compact.

A manifold M is said to be ***aspherical*** if the i-th homotopy group of M, $\pi_i(M)$, is zero for all $i > 1$.

A group H is said to be ***hopfian*** if every epimorphism $\Theta : H \to H$ is necessarily an isomorphism, while a finitely presented group H is said to be ***hyperhopfian*** if every homomorphism $\Psi : H \to H$ with $\Psi(H)$ normal and $H/\Psi(H)$ cyclic is necessarily an automorphism. It is obvious that hyperhopfian groups are hopfian, by definition.

A closed manifold N is called a ***Hopfian manifold*** if every degree one map $R : N \to N$ which induces a π_1-automorphism is a homotopy equivalence. G.A. Swarup [Sw] has established this Hopfian feature for closed n-manifolds N with $\pi_i(N) = 0$ for $1 < i < n-1$. Whether $\pi_1(N)$ a hopfian group necessarily makes N a Hopfian manifold is part of a significant, old unsolved problem, due to Hopf and recently reexamined by J.C. Hausmann [Ha].

As a matter of fact, the essential point whether or not a proper map is an approximate fibration depends on the fact that any retraction

$R : p^{-1}U \to p^{-1}b$ restricts to homotopy equivalences $p^{-1}c \to p^{-1}b$ for all points $c \in B$ sufficiently close to each $b \in B$. Thus the concepts of a Hopfian manifold, a hopfian group and a hyperhopfian group aid to convert a homology equivalence into a homotopy equivalence.

A group G is said to be *residually finite* if for each $e_G \neq g \in G$, there exists a finite group H and a homomorphism $\phi : G \to H$ with $\phi(g) \neq e_H$.

For simplicity, we will assume each element g of an usc decomposition of a manifold M be an ANR having the homotopy type of N^n.

Throughout PL category of this paper, we fix once and for all the setting and notation to be used throughout : M is a connected, orientable PL $(n+k)$-manifold, B is a polyhedron, and $p : M \to B$ is a PL map such that each $p^{-1}b$ has the homotopy type of a closed, connected, orientable n-manifold.

When the symbol $\cong$ appears between two algebraic objects, it indicates they are isomorphic; when it appears between two polyhedra, it indicates they are PL homeomorphic.

Let $f : X \to Y$ be a closed map, Γ a commutative ring with identity, and $m \in Z$. The symbol $H^m[f; \Gamma]$ denotes the m-th cohomology sheaf of f with coefficients in Γ.

We use $B^{(j)}$ to denote the j-skeleton of B.

When N is a fixed n-manifold, such a (PL) map $p : M \to B$ is said to be N-*like* if each $p^{-1}b$ collapses to an n-complex homotopy equivalent to N (this PL tameness feature imposes significant homotopy-theoretic relationships between N and preimages of links in B).

We call N a *codimension k PL fibrator* if, for all $(n+k)$-manifolds M and N-like PL maps $p : M \to B$, p is an approximate fibration. If N has this property for all $k > 0$, call N simply a *PL fibrator*.

2. The Main result in topological category

The aim of this chapter is to describe a closed n-manifold N which forces a map $p : M \to B$ to be approximate fibrations, when M is an $(n+2)$-manifold and each $p^{-1}b$ has the homotopy type of N.

Y.H. Im [Im] showed that a product $N = F_1 \times F_2$ of two closed orientable surfaces $F_i (i = 1, 2)$ with $\chi(F_i) < 0$ is a codimension 2 fibrator. In this chapter, we obtain the more generalized fact that a bundle structure $N = F_1 \tilde{\times} F_2$ of two closed orientable surfaces $F_i (i = 1, 2)$ with $\chi(F_i) < 0$ is a codimension 2 fibrator.

Before verifying Main Theorem 2.5, we needs Definition 2.1, Theorem 2.2 and the following lemmas.

DEFINITION 2.1. A group G is a *semidirect product of K by H* if G contains subgroups K and H such that:

(1) K is a normal subgroup of G,
(2) $KH = G$, and
(3) $K \cap H = \{1\}$.

THEOREM 2.2 [D_6]. *If N is an aspherical closed manifold with hopfian fundamental group and $\chi(N) \neq 0$, then N is a codimension 2 fibrator.*

LEMMA 2.3 [HE, 15.16 LEMMA (2)]. *A finitely generated group G is residually finite if and only if $\cap\{H < G \mid [G; H] < \infty\} = \{1\}$.*

LEMMA 2.4 [HE, 15.17 LEMMA]. *If G is a finitely generated, residually finite group, then G is a hopfian group.*

MAIN THEOREM 2.5. *A bundle structure $N^2 = F_1 \tilde{\times} F_2$, where F_i(i=1,2) is an orientable aspherical closed manifold with $\chi(F_i) \neq 0$ and its fundamental group is hophian, is a codimension 2 fibrator.*

Proof. Let ζ be a usc decomposition on a $(n+2)$-manifold M into copies of $N = F_1 \tilde{\times} F_2$ and $p : M^{n+2} \to B^2$ be a proper map. Due to Theorem 2.2, it suffices to show that $N = F_1 \tilde{\times} F_2$ is an aspherical manifold with a hopfian fundamental group and $\chi(N) \neq 0$.

Consider the following homotopy sequence:

$$\cdots \to \pi_{i+1}(F_1) \to \pi_i(F_2) \to \pi_i(F_1 \tilde{\times} F_2) \to \pi_i(F_1) \to \cdots.$$

Then since the i-th homotopy groups of F_1 and F_2 are trivial for $i \geq 2$, the i-th homotopy group of $N = F_1 \tilde{\times} F_2$ is trivial for $i \geq 2$. Hence N is an aspherical manifold.

In order to see that N has a hopfian fundamental group, consider the following homotopy exact sequence:

$$\cdots \to \pi_2(F_1)(= 1) \to \pi_1(F_2) \to \pi_1(F_1 \tilde{\times} F_2) \to \pi_1(F_1) \to \pi_0(F_2)(= 1)$$

where $\pi_1(F_2) = K$, $\pi_1(F_1 \tilde{\times} F_2) = G$, $\pi_1(F_1) = H$, $\alpha : K \to G$ and $\beta : G \to H$ are homomorphisms.

Then G is a semidirect product of K by H. Let G' be any subgroup of G. We are going to show that $\cap\{G' < G \mid [G; G'] < \infty\} = \{1\}$.

Consider the following commutative diagram :

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K & \xrightarrow{\alpha} & G & \xrightarrow{\beta} & H & \longrightarrow & 1 \\ \uparrow{\scriptstyle e} & & \uparrow & & \uparrow{\scriptstyle i} & & \uparrow & & \uparrow{\scriptstyle e} \\ 1 & \longrightarrow & K' & \longrightarrow & G' & \longrightarrow & H' & \longrightarrow & 1 \end{array}$$

where $e : 1 \to 1$ is the identity map and $i : G' \to G$ is the inclusion map and H' is the image of $\beta \circ i$ and K' is the inverse image of α of kernel of $\beta \circ i$.

Then since H' is a subgroup of G' and K' is a normal subgroup of G', G' is a semidirect product of K' by H'. Let G_{ij} be a semidirect product of K_i by H_j such that $[K; K_i] = n$ and $[H; H_j] = m$. Then $[G; G_{ij}] \leq nm$. Since K and H are residually finite, $\cap_i\{K_i < K \mid [K; K_i] < \infty\} = \{1\}$ and $\cap_j\{H_j < H \mid [H; H_j] < \infty\} = \{1\}$ by Lemma 2.3, so that $\cap_{ij}\{G_{ij} < G \mid [G; G_{ij}] < \infty\} = \{1\}$. Hence $\cap\{G' < G \mid [G; G'] < \infty\} = \{1\}$ since $\cap\{G' < G \mid [G; G'] < \infty\}$ is a subset of $\cap_{ij}\{G_{ij} < G \mid [G; G_{ij}] < \infty\}$, and then G is residually finite by Lemma 2.3. Due to Lemma 2.4, $G = \pi_1(F_1 \tilde{\times} F_2)(= \pi_1(N))$ is a hopfian group.

Finally, the Euler characteristic of $N = F_1 \tilde{\times} F_2$ is non-zero because $\chi(N) = \chi(F_1)\chi(F_2)$.

COROLLARY 2.6. *A bundle structure $N = F_1 \tilde{\times} F_2$ of two orientable closed surfaces $F_i (i = 1, 2)$ with $\chi(F_i) < 0$ is a codimension 2 fibrator.*

Proof. Every orientable closed surface F with $\chi(F) < 0$ is an aspherical manifold and its fundamental group is hophian.

3. The Main result in PL category

Throughout the rest of this paper, in the presence of a PL map $p : M^{n+k} \to B$, v will denote a vertex of B, L =Link (v, B), S =star $(v, B) = v * L$, $L' = p^{-1}L$, $S' = p^{-1}S$ and $R : S' \to p^{-1}v$ will denote a collapse map.

DEFINITION 3.1. A usc decomposition G of a PL $(n+k)$-manifold M is an N*-like* if dim $B < \infty$ and each $p^{-1}v$ $(v \in B)$ is shape equivalent to a closed n-manifold N.

DEFINITION 3.2. An N^n-like decomposition G of a PL $(n+k)$-manifold M has *Property* $R_* \cong$(resp. *Property* $R \cong$) if, for each $p^{-1}v$ $(v \in B)$, a collapse map $R : S' \to p^{-1}v$ induces H_1-isomorphisms $(R|p^{-1}c)_* : H_1(p^{-1}c) \to H_1(p^{-1}v)$(resp. π_1-isomorphisms $(R|p^{-1}c)_\# : \pi_1(p^{-1}c) \to \pi_1(p^{-1}v)$) for all $p^{-1}c$ sufficiently close to $p^{-1}v$, where c is in a link $L \subset B$.

THEOREM 3.3 [D$_7$]. *If N is a closed Hopfian n-manifold with hopfian fundamental group and $p : M^{n+k} \to B$ is an N-like PL map such that $H^n[p; Z]$ is locally constant, then p is an approximate fibration.*

LEMMA 3.4 [D$_8$]. *If X is a CW complex such that $\pi_i(X) = 0$ for $1 < i \leq k$ and if the map $f : X \to X$ induce an isomorphism $\pi_1(X) \to \pi_1(X)$, then f also induces isomorphisms $f_* : H_i(X) \to H_i(X)$ and $f^* : H^i(X) \to H^i(X)$ $(i \leq k)$.*

PROPOSITION 3.5. *Suppose N^n is a Hopfian n-manifold and m, k are integers, $1 < m \leq k$, such that $\pi_i(N^n) = 0$ for $1 < i \leq m$ and $H_i(N^n) = 0$ for $m < i \leq k$, and suppose $p : M^{n+k} \to B$ is an N^n-like PL map. Then p is an approximate fibration if and only if p has Property $R \cong$.*

Proof. Due to Theorem 3.3, it suffice to show that $H^n[p]$ is locally constant. If $R : p^{-1}c \to p^{-1}v$ induces an isomorphism at the fundamental group level, then it also does so for i-th cohomology groups, $0 \leq i \leq k$, by Lemma 3.4 and a standard universal coefficient theorem. Hence, the i-th cohomology sheaf, $H^i[p]$, is locally constant in the same range. According to [D-S, Theorem 3.6], $H^n[p]$ is locally constant.

Now, we obtain the following more extensive result, while it is proved to be a codimension 2 fibrator in topological category [I-K-W$_1$].

MAIN THEOREM 3.6. *A Hophian n-manifold N is a codimension $m > 2$ fibrator if it is a codimension 2 fibrator, $\pi_i(N) = 0$ for $1 < i \leq m$, and $\pi_1(N)$ is normally cohophian and has no proper subgroup isomorphic to $\pi_1(N)/A$, with A an Abelian subgroup.*

Proof. According to [D_7,Lemma 4.2], any N-like map $p : M^{n+m} \to B^m$ has Property $R \cong$. Then such a manifold N is a codimension m fibrator by Proposition 3.5.

COROLLARY 3.7. *A product $N = S^n \times F$ of any n-sphere $S^n (n \geq 3)$ and a closed orientable surface F with $\chi(F) < 0$ is a codimension $(n-1)$ PL fibrator.*

Proof. It is known that N is a Hopfian manifold and a codimension 2 fibrator[I-K-W_1]. Since the fundamental group of $N = S^n \times F$ is essentially as same as the fundamental group of F, $\pi_1(N)$ is normally cohophian and contains no Abelian subgroup. Also N satisfies the group conditions of Main Theorem 3.6. Thus, by Main Theorem 3.6, N is a codimension (n-1) fibrator.

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