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CONVOLUTION PROPERTIES FOR GENERALIZED PARTIAL SUMS

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1. Introduction

For functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ analytic in the unit disk $\Delta = \{z : |z| < 1\}$, the convolution $f * g$ is defined by $(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n$. Let S denote the family of functions $f(z) = z + \cdots$ analytic and univalent in Δ and K, St, C the subfamilies that are respectively convex, starlike, and close-to-convex.

To a finite or infinite increasing sequence of integers $\{n_k\}$ with $n_k \geq k$ and a function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ analytic in Δ we associate the function $\tilde{f}$.

$$\tilde{f}(z) = z + \sum_{k=2}^{\infty} a_{n_k} z^{n_k} = \left(z + \sum_{k=2}^{\infty} z^{n_k}\right) * f(z),$$

called a generalized partial sum of the function f. In the special case that $\{n_k\}$ is the finite sequence $n_k = k$ $(k = 2, 3, \cdots, N)$ we get the Nth section of f, $\tilde{f}(z) = f_N(z) = \left(z + \sum_{k=2}^{N} z^k\right) * f(z)$.

Szegö in [6] showed that the Nth section of functions in S, St, or K are respectively univalent, starlike, or convex in the disk $|z| < 1/4$. In each case, a function of the form $f_2(z) = z + a_2 z^2$ shows that the value $1/4$ cannot be increased. For the generalized partial sum $\tilde{f}$, the value $1/4$ may be replaced with the constant $c \approx 0.20936$, the root in $(0, 1)$ of the polynomial equation

$$(1) \qquad (1 - x^2)^3 - 4x(1 + x^2) = 0.$$

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That is, $\tilde{f}(cz)/c$ is in K or St when f is in K or St [1] and is in S when f is in S[2]. In all cases, the extremal generalized partial sum is of the form $z + \sum_{n=2}^{\infty} a_{2n} z^{2n}$.

Ruscheweyh in [5] defined the subfamily D of K consisting of functions f for which $|f''(z)| \leq \operatorname{Re} f'(z)$, $z \in \Delta$. In [3] it was conjectured for any f in D and $g, h \in S$ that

$$\text{(2)} \qquad \operatorname{Re} \frac{(f * g * h)(z)}{z} > 0, \ z \in \Delta.$$

This far reaching conjecture is stronger than the former Bieberbach Conjecture (de Branges' theorem). The conjecture was verified for $g, h \in C$. It was also established for special functions in D. In particular, for the Nth section of $z + \sum_{k=2}^{\infty} z^k$,

$$f_N(z) = 4 \sum_{k=1}^{N} \left(\frac{z}{4}\right)^k \in D \ \text{ and }$$

$$\operatorname{Re} \frac{f_N * g * h}{z} > 0 \ \text{ for } \ z \in \Delta \ \text{ and } \ g, h \in S.$$

In [2] it was shown for the generalized partial sum $\tilde{f}(z) = z + \sum_{k=2}^{\infty} z^{n_k}$ that $\tilde{f}(cz)/c \in D$. Thus, another special case of the conjecture (2) would be verified if we could show for $g, h \in S$ that

$$\text{(3)} \qquad \operatorname{Re} \frac{\left(z + \sum_{k=2}^{\infty} z^{n_k}\right) * g * h}{z} > 0, \qquad |z| < c.$$

In this note, we establish (3) in three out of the four cases needed. In the final case, we verify inequality (3) for the disk $|z| < 0.2082$ instead of $|z| < c \approx 0.20936$. The method in case 4 does not extend to $|z| < c$, but an alternate approach suggested might possibly prove fruitful.

2. Preliminaries

We will make use of the following results.

THEOREM A [4]. *If* $f \in S$, *then*

$$Re\, \frac{f(z)}{z} \geq \alpha \geq \left(\frac{e+1}{2e}\right)^2 \approx 0.468 \quad \textit{for} \quad |z| \leq \alpha^{-1/2} - 1.$$

THEOREM B [3]. *If* $g, h \in S$ *satisfy* $Re\; g(z)/z \geq \alpha$ *and* $Re\; h(z)/z \geq \alpha, z \in \Delta$, *then*

$$Re \frac{(g * h)(z)}{z} \geq 4\alpha - 2\alpha^2 - 1, \qquad z \in \Delta.$$

We now prove a lemma based on Theorems A and B.

LEMMA. *If* $g, h \in S$ *and* $d := (\alpha^{-1/2} - 1)^2$ *for some* $\alpha \geq ((e + 1)/2e)^2$, *then*

$$Re\, \frac{(g * h)(z)}{z} \geq 4\alpha - 2\alpha^2 - 1, \qquad |z| \leq d.$$

Proof. Since $\sqrt{d} = \alpha^{-1/2} - 1$, by Theorem A we have

$$\text{Re}\, \frac{g(\sqrt{d}z)}{\sqrt{d}z} \geq \alpha \quad \text{and} \quad \text{Re}\, \frac{h(\sqrt{d}z)}{\sqrt{d}z} \geq \alpha, \qquad z \in \Delta.$$

Hence by Theorem B,

$$\text{Re}\, \left(\frac{g(\sqrt{d}z)}{\sqrt{d}z} * \frac{h(\sqrt{d}z)}{\sqrt{d}z}\right) \geq 4\alpha - 2\alpha^2 - 1, \qquad z \in \Delta.$$

This last inequality is equivalent to

$$\text{Re}\, \frac{(g * h)(z)}{z} \geq 4\alpha - 2\alpha^2 - 1, \qquad |z| \leq d,$$

and the lemma is proved.

3. Main Result

THEOREM. For $g(z)=z+\sum_{k=2}^{\infty} a_k z^k \in S$ and $h(z)=z+\sum_{k=2}^{\infty} b_k z^k \in S$, we set

$$A(z)=\frac{\left(z+\sum_{k=2}^{\infty} z^{n_k}\right)*g*h}{z}=1+\sum_{k=2}^{\infty} a_{n_k} b_{n_k} z^{n_k-1}.$$

Then $\operatorname{Re} A(z)\ge 0$ for $|z|\le 0.2082$.

Proof. In the first three cases we will prove our theorem for $|z|\le c\approx 0.20936$ defined by (1).

Case 1. $n_2\ge 3$.
Then for $|z|\le c$, $\operatorname{Re} A(z)\ge 1-\sum_{n=3}^{\infty} n^2c^{n-1}=1-\left(\frac{1+c}{(1-c)^3}-1-4c\right)>0.39$.

Case 2. $n_2=2, n_3=3$.
Since $A(z)=1+a_2b_2z+a_3b_3z^2+\sum_{k=4}^{\infty} a_{n_k}b_{n_k}z^{n_k-1}$ and

$$\frac{(g*h)(z)}{z}=1+a_2b_2z+a_3b_3z^2+\sum_{n=4}^{\infty} a_nb_nz^{n-1},$$

we see that

$$A(z)=\frac{(g*h)(z)}{z}-\sum_{\substack{j=4\\ j\ne n_k}}^{\infty} a_jb_jz^{j-1}$$

Hence for $|z|\le c$,

$$\begin{aligned}\operatorname{Re} A(z)&\ge \operatorname{Re}\frac{(g*h)(z)}{z}-\sum_{j=4}^{\infty} j^2c^{j-1}\\ &=\operatorname{Re}\frac{(g*h)(z)}{z}-\left(\frac{1+c}{(1-c)^3}-1-4c-9c^2\right)\\ &\ge \operatorname{Re}\frac{(g*h)(z)}{z}-0.22.\end{aligned}$$

When $|z|=c$, we set $d=c$ in the Lemma to get $\alpha > 0.47$. The Lemma thus yields $\operatorname{Re} A(z) \geq 0.43 - 0.22 > 0$.

We are left to consider cases where $n_2 = 2, n_3 \neq 3$, and either the first n_k after consecutive even integers is the succeeding odd integer or it is not. The next case considers when it is not.

<u>Case 3.</u> $A(z) = 1 + \sum_{n=1}^{m} a_{2n}b_{2n}z^{2n-1} + \sum_{n_k \geq 2m+3}^{\infty} a_{n_k}b_{n_k}z^{n_k-1}$ $(m \geq 1)$.

Then for $|z| \leq c$,

$$(4) \qquad \operatorname{Re} A(z) \geq 1 - \sum_{n=1}^{m}(2n)^2c^{2n-1} - \sum_{n=2m+3}^{\infty} n^2c^{n-1}.$$

In [1] an induction proof was used to show that the RHS of (4) is a decreasing function of m. Letting $m \to \infty$ we have

$$(5) \qquad \operatorname{Re} A(z) \geq 1 - \sum_{n=1}^{m}(2n)^2c^{2n-1}.$$

Writing $b(z) = \sum_{n=1}^{\infty} 2nz^{2n} = z\frac{d}{dz}\left(\sum_{n=1}^{\infty} z^{2n}\right) = \frac{2z^2}{(1-z^2)^2}$, we see that $b'(z) = \sum_{n=1}^{\infty}(2n)^2z^{2n-1} = \frac{4z(1+z^2)}{(1-z^2)^3}$. It thus follows from (5) and (1) that

$$\operatorname{Re} A(z) \geq 1 + b'(-c) = 0 \qquad (|z| \leq c).$$

Sharpness occurs in <u>Case 3</u> at $z=-c$ when $g(z) = h(z) = z/(1-z)^2$ with generalized partial sum $z + \sum_{m=1}^{\infty} z^{2m}$.

In the final case, the first n_k after consecutive even integers is the succeeding odd integer.

<u>Case 4.</u> $A(z) = 1 + \sum_{n=1}^{m+1} a_{2n}b_{2n}z^{2n-1} + a_{2m+3}b_{2m+3}z^{2m+2} + \sum_{n_k \geq 2m+4} a_{n_k}b_{n_k}z^{n_k-1} \qquad (m \geq 1).$

This is equivalent to

$A(z) = \frac{(g*h)(z)}{z} - \sum_{n=1}^{m} a_{2n+1}b_{2n+1}z^{2n} - \sum_{\substack{j=2m+4 \\ j\neq n_k}} a_jb_jz^{j-1}.$

Thus,

$$(6) \quad \operatorname{Re} A(z) \geq \operatorname{Re}\frac{(g*h)(z)}{z} - \sum_{n=1}^{m}(2n+1)^2|z|^{2n} - \sum_{n=2m+4}^{\infty} n^2|z|^{n-1}.$$

As in Case 3., the RHS of (6) decreases with m when $|z| \le c$, and hence

$$\text{(7)} \qquad \text{Re } A(z) \ge \text{Re } \frac{(g*h)(z)}{z} - \sum_{n=1}^{\infty}(2n+1)^2|z|^{2n}$$
$$= \text{Re } \frac{(g*h)(z)}{z} - \frac{|z|^2(|z|^4 - 2|z|^2 + 9)}{(1-|z|^2)^3}.$$

When $|z| = d = 0.2082$, it follows from the Lemma that $\alpha = (1+\sqrt{d})^{-2} \approx 0.471525$ and Re $\frac{(g*h)(z)}{z} \ge 4\alpha - 2\alpha^2 - 1 \ge 0.44142$. Since $d^2(d^4 - 2d^2 + 9)/(1-d^2)^3 < 0.4414$, we see from (7) that Re $A(z) > 0$ for $|z| \le d = 0.2082$. This completes the proof.

4. Concluding Remarks

The method in Case 4. does not directly extend to $|z| = c \approx 0.20936$, which we strongly believe to be the sharp result. When $|z| = c$, the method gives Re $\frac{(g*h)(z)}{z} \approx 0.439683$ and $\frac{c^2(c^4-2c^2+9)}{(1-c^2)^3} \approx 0.44916$.

So the best we can get from (7) when $|z| \le c$, is roughly Re $A(z) > -0.0095$.

Perhaps some refinement of inequality (6) could lead to the sharp result. The method of Case 2. shows that (6) is valid for $|z| \le c$ when $m = 1 (n_2 = 2, n_3 = 4, n_4 = 5)$, but does not extend to $m \ge 2$.

An alternate approach is to consider the following problem, of independent interest.

Find the largest r for which

$$\inf_{|z|=r} \text{Re } \frac{(g*h)(z)}{z} = \inf_{|z|=r} \text{Re } g'$$

taken over all $g, h \in S$.

In other words, when is the Koebe function the extremal h over all $g \in S$?

If we should find that $r \ge c$, then we would have

$$\text{Re } \frac{(g*h)(z)}{z} \ge \frac{1-c}{(1+c)^3} = \frac{c^2(c^4 - 2c^2 + 9)}{(1-c^2)^3} \qquad (|z| \le c),$$

and Case 4. would then extend to $|z| \le c$.

References

1. R. Fournier and H. Silverman, *Radii problems for generalized sections of convex functions*, Proc. Amer. Math. Soc. **112** (1991), 101-107.
2. ———, *On generalized sections of univalent functions*, Complex Variables **17** (1992), 141-147.
3. V. Gruenberg, F. Ronning and St. Ruscheweyh, *On a multiplier conjecture for univalent functions*, Trans. Amer. Math. Soc. **322** (1990), 377-393.
4. M. Reade and H. Silverman, *Radii problems for linear properties of univalent functions*, Houston J. Math. **17** (1991), 227-235.
5. St. Ruscheweyh, *Extension of Szegö's theorem on the sections of univalent functions*, SIAM J. Math. Anal. **19** (1988), 1442-1449.
6. G. Szegö, *Zur Theorie der schlichten Abbildungen*, Math. Ann. **100** (1928), 188-211.

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