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LINEAR TRANSFORMATIONS THAT PRESERVE THE ASSIGNMENT II

LEROY B. BEASLEY, GWANG-YEON LEE AND SANG-GU LEE

I. Introduction

Let $R = (r_1, r_2, \cdots, r_m)$ and $S = (s_1, s_2, \cdots, s_n)$ be vectors of positive integers, and let $\mathcal{U}(R, S)$ denote the class of all $m \times n$ matrices $A = [a_{ij}]$ of 0's and 1's such that

$$\sum_{k=1}^{n} a_{ik} = r_i \quad (i = 1, 2, \cdots, m),$$

$$\sum_{k=1}^{m} a_{kj} = s_j \quad (j = 1, 2, \cdots, n).$$

Thus R is the *row sum vector* and S is the *column sum vector* of each matrix in $\mathcal{U}(R, S)$. In [4] Brualdi, Hartfiel and Hwang introduced a class of functions generalizing the permanent function, which, like the permanent, are combinatorially significant as counting functions. We refer to matrices in $\mathcal{U}(R, S)$ as $(R, S)-assignments$, or as *assignments* when R and S are fixed in the discussion. For matrices $B = [b_{ij}]$ and $C = [c_{ij}]$ of the same order, write $B \leq C$ if $b_{ij} \leq c_{ij}$ for all i and j. If $X = [x_{ij}]$ is an $m \times n$ matrix of 0's and 1's, then an assignment corresponds to an $m \times n$ matrix A such that $A \in \mathcal{U}(R, S)$ and $A \leq X$. Thus, if we let

$$\text{(1.1)} \qquad \mathrm{P}_{R,S}(X) = |\{A \in \mathcal{U}(R, S) : A \leq X\}|,$$

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then $\mathrm{P}_{R,S}(X)$ counts the number of possible assignments. If we let $J_{m,n}$ be an $m \times n$ matrix whose entries are all ones, then

$$\mathrm{P}_{R,S}(J_{m,n}) = |\mathcal{U}(R,S)|. \tag{1.2}$$

We call $\mathrm{P}_{R,S}(\cdot)$ the (R,S) – *assignment function* or an *assignment function.*

A well-known special case of an assignment function occurs when $m = n$ and $R = S = (1, 1, \cdots, 1)$. In this case, $\mathrm{P}_{R,S}(X)$ counts the number of permutation matrices P with $P \leq X$ and hence $\mathrm{P}_{R,S}(X)$ is the permanent of X, per(X).

More generally, let $X = [x_{ij}]$ be an $m \times n$ matrix. We define the *support* of X to be the set $\mathrm{supp}(X) = \{(i,j) : x_{ij} \neq 0\}$. The (R,S)-assignment function $\mathrm{P}_{R,S}(\cdot)$ is defined by

$$\mathrm{P}_{R,S}(X) = \sum_{A \in \mathcal{U}(R,S)} \prod_{(i,j) \in \mathrm{supp}(A)} x_{ij}. \tag{1.3}$$

The preservers of the permanent were first determined by Marcus and May [5] and later Botta [3] gave a proof valid over any field. In this paper, we characterize the linear operators on the real matrices which preserve the value of an assignment function of each $m \times n$ matrix.

II. Results

Let $M_{m\times n}(\mathrm{R})$ be the vector space of $m \times n$ matrices. We assume throughout that $R = (r_1, \cdots, r_m)$ and $S = (s_1, s_2, \cdots, s_n)$ are vectors of positive integers with $1 \leq r_1 \leq \cdots \leq r_m < n$ and $1 \leq s_1 \leq \cdots \leq s_n < m$. If $\sum_{i=1}^{m} r_i \neq \sum_{j=1}^{n} s_j$, then $\mathcal{U}(R,S) = \emptyset$. So we assume throughout that $\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} s_j = k$ for $n + 1 \leq k \leq mn$ and $0 < r_i, s_j \leq m$ for each i, j where $m \leq n$, i.e., $\mathcal{U}(R,S) \neq \emptyset$. We have shown that for the case $m = n$ in [1]. Let $T : M_{m\times n}(\mathrm{R}) \to M_{m\times n}(\mathrm{R})$ be a linear transformation such that

$$\mathrm{P}_{R,S}(X) = \mathrm{P}_{R,S}(T(X)) \tag{2.1}$$

for any $X \in M_{m\times n}(\mathrm{R})$.

Let E_{ij} denote the (0,1)-matrix whose only nonzero entry is in the (i,j) position. A *weighted cell* is a scalar multiple of E_{ij} for some (i,j), so that the set of cells is the set $\{\alpha_{ij}E_{ij} | \alpha_{ij} \in \mathrm{R}, 1 \le j \le m$ and $1 \le i \le n\}$. We say that the two vectors R and S are *compatible* if given any two positive integers $1 \le i,j \le n$ there are two integers k,l and some $A \in \mathcal{U}(R,S)$ such that $a_{ij} = a_{kl} = 1$ and $a_{il} = a_{kj} = 0$ with $i \ne k$ $j \ne l$, and $1 \le k \le m, 1 \le l \le n$. We may have to consider the following condition;for any pair (i,j), there is some element of $\mathcal{U}(R,S)$ whose (i,j) entry is nonzero. That is,

$$\text{(2.2)} \qquad \{(i,j) : a_{ij} = 1 \text{ for some } x \in \mathcal{U}(R.S)\} \\ = \{1,2,\cdots,m\} \times \{1,2,\cdots,n\}.$$

Notice that if R and S are compatible then $r_i, s_j < n$ and condition (2.2) is satisfied. But we can easily show that (2.2) implies *compatibility* if we allow row and column permutations. This will be possible because of our final theorem allows it. That means even if we make a weaker assumptiom, this does not effect our theorem.

Therefore, throughout, we asssume that the two vectors R and S are compatible.

LEMMA 1. *T is nonsingular.*

Proof. Let $B \in \mathcal{U}(R,S)$ with $b_{pq} \ne 0$, and let $A(z) = z(B - E_{pq})$. Then, $\mathrm{P}_{R,S}(A(z)) = 0$ for all z. However, the coefficient of z^{k-1} in $\mathrm{P}_{R,S}(A(z) + X)$ is x_{pq} which is nonzero. Thus, $\mathrm{P}_{R,S}(X + A(z))$ is a nonzero polynomial in z, and hence, is nonzero for some choice of z, say z_0. But then, since T preserves $\mathrm{P}_{R,S}$,

$$\begin{aligned} 0 &= \mathrm{P}_{R,S}(A(z_0)) \\ &= \mathrm{P}_{R,S}(T(A(z_0))) \\ &= \mathrm{P}_{R,S}(T(A(z_0)) + T(X)) \\ &= \mathrm{P}_{R,S}(T(A(z_0) + X)) \\ &= \mathrm{P}_{R,S}(A(z_0) + X) \ne 0. \end{aligned}$$

This contradiction establishes the lemma. ■

Let $R_i = \{X \in M_{mn}(\mathrm{R}) : x_{kl} = 0 \text{ for all } k \ne i, \text{ for all } l\}$ and $C_j = \{X \in M_{mn}(\mathrm{R}) : x_{kl} = 0 \text{ for all } l \ne j, \text{ for all } k\}$. Suppose $r_i = 1$ and $s_j = 1$ for all $i \le q$ and $j \le p$.

LEMMA 2. *If $i \leq q$, then there exist k such that $T(R_i) \subseteq R_k$ and $k \leq q$, or $T(R_i) \subseteq C_k$ and $k \leq p$. If $j \leq p$, then there exist l such that $T(C_j) \subseteq C_l$ and $l \leq p$, or $T(C_j) \subseteq R_l$ and $l \leq q$.*

Proof. Since the column case is parallel to the row case, we consider $T(R_i)$. If $T(R_i) \not\subseteq R_k$ for all $k \leq q$ and $T(R_i) \not\subseteq C_k$ for all $k \leq p$, then there are three possible cases;

Case 1. If the term rank of $T(R_i)$ is greater than or equal 2, then there is $X \in R_i$ such that the term rank of $T(X)$ is at least 2. Say $T(X) = L$ with $l_{rs} \neq 0, l_{uv} \neq 0$ and $r \neq u, s \neq v$. Choose $A \in \mathcal{U}(R, S)$ with $a_{rs} = a_{uv} = 1$ and if possible with $a_{rv} = 1$ or $a_{us} = 1$. Let $B = A - E_{rs} - E_{uv}$. Then $\mathrm{P}_{R,S}(tT(X) + B)$ is a polynomial of degree at least 2 since the coefficient of t^2 is $l_{rs}l_{uv} \neq 0$. But the polynomial $\mathrm{P}_{RS}(tX + T^{-1}(B))$ is of degree at most 1. This contradicts that T preserves $\mathrm{P}_{R,S}(\cdot)$.

Case 2. If $T(R_i) \subseteq R_k$ and $r_k > 1$ $(k > q)$. Choose $X \in R_i$ such that $T(X) = L$ has $l_{kl} \neq 0$ and $l_{ks} \neq 0$. Choose $A \in \mathcal{U}(R, S)$ with $a_{kl} = a_{ks} = 1$, and let $B = A - E_{kl} - E_{ks}$. Then $\mathrm{P}_{R,S}(tX + T^{-1}(B))$ is of degree at most 1 while $\mathrm{P}_{R,S}(tT(X) + B)$ has degree at least 2 since the coefficient of t^2 is $l_{kl}l_{ks}$. This occurs a contradiction.

Case 3. If $T(R_i) \subseteq C_k$ with $k > p$. This is parallel to the Case 2.

In any case we have arrived at a contradiction. Thus $T(R_i) \subseteq R_k$ with $k \leq q$, or $T(R_i) \subseteq C_k$ with $k \leq p$. Similarly, we have $T(C_j) \subseteq C_l$ with $l \leq p$, or $T(C_j) \subseteq R_l$ with $l \leq q$. ■

COROLLARY 1. *If $m \neq n$, then for each $i \leq q$, there is $k \leq q$ such that $T(R_i) \subseteq R_k$ and for each $j \leq p$, there is $l \leq p$ such that $T(C_j) \subseteq C_l$.*

Proof. This follows easily from the nonsingularity of T. ■

From the fact that T is nonsingular and that T, and hence T^{-1}, preserves $\mathrm{P}_{R,S}(\cdot)$, we observe:

COROLLARY 2. *For $i, j < q$. If $T(R_i) \subseteq R_k$ for some k, then there exist $l(\neq k)$ such that $T(R_j) \subseteq R_l$. If $T(R_i) \subseteq C_k$ for some k, then $m = n$, $q = p$ and there exist l such that $T(R_i) \subseteq C_l$.*

LEMMA 3. *If $i > q$ and $j > p$ then $T(E_{ij})$ has no entry in the first q rows or the first p columns.*

Proof. Without loss of generality, we may assume $r_1 = 1$ and $T(R_1) \subseteq R_k$ for some $k \leq q$. Suppose $i > q$ and $j > p$ and $T(E_{ij})$ has a nonzero entry in row k for some $k \leq q$ or in column l for some $l \leq p$. Without loss of generality, suppose $T(E_{ij} = L$ and $l_{kl} \neq 0$ for some $k \leq q$. Choose $A \in \mathcal{U}(R,S)$ with $a_{kr} = 1$, and let $B = A - E_{kr}$ then the coeficient of t in $\mathrm{P}_{R,S}(tT(E_{ij}) + B)$ is $l_{kr} \neq 0$. But $tE_{ij} + T^{-1}(B)$ has no entry in rows where $T(R_s) \subseteq R_k$ with $s \leq q$. Thus $\mathrm{P}_{R,S}(tE_{ij} + T^{-1}(B)) = 0$, a contradiction. ∎

Henceforce we assume, without loss of generality, that $T(R_i) = R_i, i \leq q$ and $T(C_j) = C_j, j \leq p$. By Lemma 3, $T^{-1}(R_i) = R_i, T^{-1}(C_j) = C_j$.

LEMMA 4. *If $i > q$ and $j > p$, then $T(E_{ij})$ is a weighted cell.*

Proof. Since $i > q$ and $j > p$, if $T(E_{ij}) = L$ and if $l_{uv} \neq 0$ then $u > q$ and $v > p$ by Lemma 3. Suppose $l_{rs} \neq 0$ and $l_{uv} \neq 0$. Choose $A \in \mathcal{U}(R,S)$ with $a_{rs} = a_{uv} = 1$ and if possible with $a_{rs} = a_{uv} = 1$. Let $B = A - E_{rs} - E_{uv}$. Then the coefficient of t^2 in $\mathrm{P}_{R,S}(tT(E_{ij}) + B)$ is $l_{rs}l_{uv} \neq 0$. But $\mathrm{P}_{R,S}(tE_{ij} + T^{-1}(B))$ is a polynomial of degree at most $1, a$ contradiction. ∎

By the above lemmas, we may now assume that $T(E_{ij}) = E_{ij}$ if $i \leq p$ and $j \leq q, T(E_{ij})$ is a cell if $i > p$ and $j > q$ and $T(E_{ij}) = \sum_{k=q+1}^{n} \alpha_k^{(i,j)} E_{ik}$ for some $\alpha_k^{(i,j)'}s$, for $1 \leq i \leq p$ and $j > q$, and $T(E_{ij}) = \sum_{k=p+1}^{m} \beta_k^{(i,j)} E_{kj}$ for $i > p$ and $1 \leq j \leq q$.

LEMMA 5. *If $1 \leq i \leq p$ and $j > q$, then $T(E_{ij})$ is a weighted cell.*

Proof. Suppose $T^{-1}(E_{ij})$ is not a weighted cell for some $1 \leq i \leq p$ and $j > p$. By permuting we may assume $T^{-1}(E_{1,q+1}) = aE_{1r} +$

$bE_{1s} + \dots$, for some $r, s > q$, with $a, b \neq 0$, and $S_r \leq S_s$. Let

$$X = \left[\begin{array}{cccccc|ccccc} 0 & & & & & & \cdots & a & \cdots & b & \cdots \\ 1 & & & & & & & & & & \\ & \ddots & & & & & & & & & \\ & & \ddots & & & & & & & & \\ & & & 1 & & & & & & & \\ \hline & & & & 1 & & & & & & \\ & & & & & \ddots & & & & & \\ & & & & & & & & D & & \end{array}\right]$$

where a is in the $(1, r)$ position and b is in the $(1, s)$ position and

$$T\left[\begin{array}{ccc|ccccc} 0 & \cdots & 0 & \cdots & a & \cdots & b & \cdots \\ & & & & & & & \\ & & 0 & & 0 & & & \\ \hline & & 0 & & & 0 & & \end{array}\right] = E_{1q+1}.$$

We further require that $\begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}$ is a (0,1) matrix and has column sums $s_1 - 1 = 0 = \dots = 0 = s_q - 1, s_{q+1}, s_{q+2}, \dots, s_{r-1}, s_r - 1, s_{r+1}, \dots, s_n$ and row sums $r_1 - 1 = 0 = \dots = 0 = r_p - 1, r_{p+1} - 1, \dots, r_q - 1, r_{q+1} - 1, r_{q+2}, \dots, r_m$. So that $P_{R,S}(X) = a$. Let $Z = X - \sum_{i=1}^{n} x_{1i} E_{1i} + E_{1r}$, then $Z \in \mathcal{U}(R, S)$. Now,

$$T(X) = \left[\begin{array}{cccc|cccc|cccc} 0 & \cdots & & & & & & 0 & 1 & 0 & \cdots & 0 \\ 1 & & & & & & & & & & & \\ & \ddots & & & & & & & & 0 & & \\ & & & 1 & & & & & & & & \\ \hline & & & & x & y & \cdots & z & & & & \\ & 0 & & & & & & & & E & & \\ & & & & & & & & & & & \end{array}\right]$$

and since $P_{R,S}(T(X)) = P_{R,S}(X) \neq 0$, the column sums of $\begin{bmatrix} 0 & 0 \\ 0 & E \end{bmatrix}$ must be $0, \dots, 0, s_{q+1} - 1, s_{q+2}, \dots, s_n$.

Since $s_r \leq s_s$, and $Z \in \mathcal{U}(R,S)$, there must be some $k > p$ such that $x_{kr} = 0$ and $x_{ks} = 1$. Let $Y = X - E_{ks} + E_{kr}$. Then $\mathrm{P}_{R,S}(Y) = b \neq 0$, so $\mathrm{P}_{R,S}(T(Y))$ must nonzero. Further

$$T(Y) = \left[\begin{array}{cccc|c|cccc|cccc}
0 & \cdots & & & & & & & 0 & 1 & 0 & \cdots & 0 \\
1 & & & & & & & & & & & & \\
 & & \ddots & & & & & & & & 0 & & \\
 & & & & 1 & & & & & & & & \\
\hline
 & & & & & x & y & \cdots & z & & & & \\
 & & 0 & & & & & & & & H & & \\
 & & & & & & & & & & & &
\end{array}\right]$$

where $\begin{bmatrix} 0 & 0 \\ 0 & H \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & E \end{bmatrix} - F + G$ when $F = T(E_{ks})$ and $G = T(E_{kr})$ are weighted cells. Now the number of nonzero entries in each columns of $\begin{bmatrix} 0 & 0 \\ 0 & H \end{bmatrix}$ must be the same as those of $\begin{bmatrix} 0 & 0 \\ 0 & E \end{bmatrix}$ in order that $\mathrm{P}_{R,S}(T(Y))$ be nonzero. Now, since $Z \in \mathcal{U}(R,S), \mathrm{P}_{R,S}(T(Z)) = \mathrm{P}_{R,S}(Z) = 1$ and

$$T(Z) = \left[\begin{array}{cccc|c|cccc|cccc}
0 & \cdots & & & & & & & 0 & \alpha & \beta & \cdots & \gamma \\
1 & & & & & & & & & & & & \\
 & & \ddots & & & & & & & & 0 & & \\
 & & & & 1 & & & & . & & & & \\
\hline
 & & & & & x & y & \cdots & z & & & & \\
 & & 0 & & & & & & & & E & & \\
 & & & & & & & & & & & &
\end{array}\right]$$

where $T(E_{1,q+1}) = \begin{bmatrix} 0 & \cdots & 0 & \alpha & \beta & \cdots & \gamma \\ & 0 & & & 0 & & \end{bmatrix}$.

Thus $\alpha \neq 0$ since $1 = \mathrm{P}_{R,S}(T(Z)) = \alpha \mathrm{P}_{R,S}(T(X)) = \alpha a$. Now, let

$W = Z - E_{ks} + E_{kr}$ then

$$T(W) = T(Z) - F + G$$

$$= \left[\begin{array}{ccc|cccc|ccc}
0 & \cdots & & & & & 0 & \alpha & \beta & \cdots \\
1 & & & & & & & & & \\
& \ddots & & & & & & 0 & & \\
& & 1 & & & & & & & \\
\hline
& & & x & y & \cdots & z & & & \\
& 0 & & & & & & & H & \\
& & & & & & & & &
\end{array}\right],$$

so that $\mathrm{P}_{R,S}(T(W)) = \alpha \mathrm{P}_{R,S}(T(Y)) = \alpha \mathrm{P}_{R,S}(Y) = \alpha b \neq 0$. But, the $s-th$ column of W has $s_s - 1$ $1's$ and hence $\mathrm{P}_{R,S}(W) = 0$, a contradiction to the fact that T preserves $\mathrm{P}_{R,S}(\cdot)$. It follows from this contradiction that $T^{-1}(E_{1,q+1})$ is a weighted cell, and hence that $T(E_{ij})$ is a weighted cell for all $1 \leq i \leq p$ and $j > q$. ■

LEMMA 6. *If $i > p$ and $1 \leq j \leq q$ then $T(E_{ij})$ is a weighted cell.*

Proof. The proof is identical to that of lemma 5 with the roles of the rows and columns exchanged ■

We will now show that T preserves the term rank of any matrix.

THEOREM 1. *The operator T is bijective on the set of weighted cells.*

LEMMA 7. *Suppose that $1 \leq r_i, s_j < n$ for all r_i and s_j. If $A \in \mathcal{U}(R,S)$ and $a_{pq} = a_{uv} = 1$ then $A' = A - E_{pq} - E_{uv} + E_{ij} + E_{rs} \in \mathcal{U}(R,S)$ if and only if*

i) $(i,j) = (p,q)$ *and* $(r,s) = (u,v)$;
ii) $(i,j) = (u,v)$ *and* $(r,s) = (p,q)$;
iii) $(i,j) = (p,v)$, $(r,s) = (u,q)$, *and* $a_{ij} = a_{rs} = 0$ *or*
iv) $(i,j) = (u,q)$, $(r,s) = (p,v)$, *and* $a_{ij} = a_{rs} = 0$.

Proof. Note that in cases *i)* and *ii)*, $A' = A$.

The sufficiency is easily checked. For the necessity, the only way to make $A - E_{pq} - E_{uv}$ into a member of $\mathcal{U}(R,S)$ by adding two cells is if those two cells have ones in rows p and u and in columns q and v(or two ones in row p if $p = u$, etc). It then follows that $i = p$, or

$i = u$, $r = p$ or $r = u$, $j = p$ or $j = v$, and $s = q$ or $s = v$. Further, if $i = p$ then we must have that $r = u$, and visa versa. Likewise, if $j = q$ then we must have that $s = v$, and visa versa. Finally, if a_{ij} or a_{rs} were nonzero then A' would nat be a (0 1) matrix. These facts establish the necessity. ■

LEMMA 8. *Suppose that* $1 \leq r_i$, $s_j < n$ *for all* r_i *and* s_j, *and that* R *and* S *are compatible. If* T *preserves the assignment function* $P_{R,S}$ *then* T *preserves the set of matrices of term rank 1.*

Proof. Suppose that some matrix of term rank 1 is not mapped into a matrix of term rank 1. Then, since T is bijective on the cells, there is some pair of cells of term rank 1 whose images are not term rank 1. Without loss of generality, assume that $T(E_{pq}) = xE_{ij}$ and $T(E_{pv}) = yE_{rs}$. Now, choose $A \in \mathcal{U}(R, S)$ with $a_{pq} = a_{bv} = 1$, and $a_{pv} = a_{bp} = 0$. This is always possible since R and S are compatible.

Now, let $A' = A - E_{pq} - E_{bv} + E_{pv} + E_{bp}$. By lemma 3, $\mathrm{P}_{R,S}(A') = 1$. Thus $\mathrm{P}_{R,S}(T(A'))$ must be 1. Since T is bijective on the cells, we must have that the pattern $\overline{T(A')}$ of $T(A')$ is in $\mathcal{U}(R, S)$. But the pattern $\overline{T(A)}$ of $T(A)$ differs from that of $T(A')$ only by changing two ones to zeros and two zeros to ones. That is,

$$\begin{aligned}
\overline{T(A')} &= \overline{T(A - E_{pq} - E_{bv} + E_{pv} + E_{bp})} \\
&= \overline{T(A) - T(E_{pq}) - T(E_{bv}) + T(E_{pv}) + T(E_{bq})} \\
&= \overline{T(A)} - \overline{T(E_{pq})} - \overline{T(E_{bv})} + \overline{T(E_{pv})} + \overline{T(E_{bq})} \\
&= \overline{T(A)} - E_{ij} - E_{gh} + E_{rs} + E_{kl}
\end{aligned}$$

for some (g, h) and (k, l).

By lemma 7, and the fact that T is bijective on the set of cells we have that $r = i$ or $s = j$, a contradiction. Thus T preserves term rank 1. ■

We now obtain some of the structure of assignment preserves from the following lemma.

LEMMA 9. [2, Beasley and Pullman, Corollary 3.1.2] *Suppose that* T *is a nonsingular linear operator on* $M_{mxn}(R)$. *The linear operator*

T preserves the set of matrices of term rank 1 if and only if T is one of or a composition of some of the following operators:

(i) $X \to X^t$
(ii) $X \to PXQ$ *for fixed but arbitrary $n \times n$ permutation matrices P and Q.*
(iii) $X \to X \circ A$ *for some fixed but arbitrary matrix A with no zero entries.*

In order to complete our characterization of operators which preserve assignment functions when $1 \le r_i,\ s_j < n$ for all r_i and s_j we show that the the three types of operators in Lemma 9 which also preserve the assignment function are the types specified in the theorem.

LEMMA 10. *Let P and Q are permutation matrices. Then*

$$P_{R,S}(X) = P_{R,S}(PXQ) \tag{2.3}$$

if and only if $PR^t = R^t$ and $SQ = S$.

Proof. For each $A \in \mathcal{U}(R,S)$, $PAQ \in \mathcal{U}(R,S)$ only if $PR^t = R^t$ and $SQ = S$. This establishes the necessity. Now, suppose that $PR^t = R^t$ and $SQ = S$. Then,

$$\begin{aligned} \mathrm{P}_{R,S}(X) &= \sum_{A\in\mathcal{U}(R,S)} \prod_{(i,j)\in\mathrm{supp}(A)} x_{ij} \\ &= \sum_{PAQ\in\mathcal{U}(R,S)} \prod_{(i,j)\in\mathrm{supp}(PAQ)} x_{ij} \\ &= \sum_{A\in\mathcal{U}(R,S)} \prod_{(i,j)\in\mathrm{supp}(A)} (PXQ)_{ij} \\ &= \mathrm{P}_{R,S}(PXQ). \quad \blacksquare \end{aligned}$$

REMARK. We note that the assignment is not invariant under permutations of rows and columns and under transposition. For example, if $R = (2,2,2)$ and $S = (3,2,1)$, then $\mathcal{U}(R,S) = \{A_1, A_2, A_3\}$ where

$$A_1 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},\ A_2 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix},\ A_3 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

Then $\mathrm{P}_{R,S}(A_1) = 1$. But $\mathrm{P}_{R,S}(A_1^t) = 0$. And let

$$Q = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

then

$$A_1 Q = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

So, $\mathrm{P}_{R,S}(A_1) = 1$ and $\mathrm{P}_{R,S}(A_1 Q) = 0$.

Note that if $R = S$ then $\mathcal{U}(R,S) = \mathcal{U}(S,R) = \mathcal{U}(R,R)$ and hence, in this case, $\mathrm{P}_{R,S}(A) = \mathrm{P}_{R,S}(A^t)$ for all A. We have thus established the following lemma.

LEMMA 11. *The transpose operator preserves the assignment function $P_{R,S}$ if and only if $R = S$.*

LEMMA 12. *Suppose $1 \leq r_1, s_j < n$ for all r_i and s_j, and that R and S are compatible. If T preserves the assignment function $P_{R,S}$ and if $T(X) = X \circ M$, then there exist diagonal matrices D_1 and D_2 such that $M = D_1 J D_2$, where J is the matrix of all ones;thus, $T(X) = D_1 X D_2$.*

Proof. Let

$$D_1 = \operatorname{diag}\{m_{11}, m_{21}, \cdots, m_{n1}\}$$

and

$$D_2 = \operatorname{diag}\{1, m_{12} m_{11}^{-1}, \cdots, m_{1n} m_{11}^{-1}\},$$

and let $N = D_1^{-1} M D_2^{-1}$. Let $2 \leq i, j l2n$ be fixed, and choose $A \in \mathcal{U}(R,S)$ with $a_{11} = a_{ij} = 1$ and $a_{i1} = a_{1j} = 0$. Such an element always exist since $1 \leq r_i, s_j < n$ for all r_i and s_j. Let $B = A - E_{11} - E_{ij} + E_{i1} + E_{1j}$ so $B \in \mathcal{U}(R,S)$. Now, $\mathrm{P}_{R,S}(D_1^{-1} A D_2^{-1}) = \prod_{i=1}^{n} m_{i1}^{-r_i} \cdot \prod_{j=2}^{n} (m_{1j} m_{11}^{-1})^{-s_j} = \mathrm{P}_{R,S}(D_1^{-1} B D_2^{-1})$. Thus $\mathrm{P}_{R,S}(D_1^{-1} A D_2^{-1}) = \mathrm{P}_{R,S}(D_1^{-1} B D_2^{-1})$, and hence $\mathrm{P}_{R,S}((D_1^{-1} A D_2^{-1}) \circ M) = \mathrm{P}_{R,S}((D_1^{-1} B D_2^{-1}) \circ M)$ since T preserves $\mathrm{P}_{R,S}$. Since for diagonal matrices D and E, $DXE \circ M = D(X \circ M)E = X \circ DME$, and since T preserves $\mathrm{P}_{R,S}$ we have that $\mathrm{P}_{R,S}(A \circ N) = \mathrm{P}_{R,S}(B \circ N)$. Now, $\mathrm{P}_{R,S}(A \circ N) = n_{11} \cdot n_{ij} \cdot \beta$ and $\mathrm{P}_{R,S}(B \circ N) = n_{i1} \cdot n_{1j} \cdot \beta$ where β is $\prod_{(k,l) \in \operatorname{supp}(A) \setminus \{(1,1),(i,j)\}} n_{kl}$.

It now follows that $n_{ij} = 1$ since $n_{11} = n_{i1} = n_{1j} = 1$. Since i and j were chosen arbitrarily, we have that $N = J$, and hence $T(X) = X \circ M = D_1 X D_2$. ■

We now only have to describe the allowable diagonal equivalence operators.

LEMMA 13. *If $T(X) = DXL$ for some diagonal matrices*

$$D = diag\{d_1, d_2, \cdots, d_m\}$$

and

$$L = diag\{l_1, l_2, \cdots, l_n\}$$

in $M_{m\times n}(R)$, then $\prod_{i=1}^m d_i^{r_i} \cdot \prod_{j=1}^n l_j^{s_j} = 1$.

Proof. Let $A \in \mathcal{U}(R,S)$, then $P_{R,S}(A) = 1$, and hence $P_{R,S}(T(A)) = 1$. That is, $P_{R,S}(DAL) = 1$. But $P_{R,S}(DAL) = \prod_{i=1}^m d_i^{r_i} \cdot \prod_{j=1}^n l_j^{s_j}$. ■

An immediate consequence of the above lemmas is the following theorem.

THEOREM 2. *If T is a linear operator on $M_{m\times n}(R)$ and $1 \le r_i, s_j < n$ for all r_i and s_j, and R and S are compatible, then T preserves the assignment function $P_{R,S}$ if and only if*

$$T(X) = PDXLQ \text{ for all } X \in M_{m\times n}(R),$$

or

$$T(X) = PDX^tLQ \text{ and } R = S \text{ for all } X \in M_n(R),$$

where P and Q are permutation matrices such that $PR^t = R^t$ and $SQ = S$ and $D = diag\{d_1, d_2, \cdots, d_m\}$ and $L = diag\{l_1, l_2, \cdots, l_n\}$ are diagonal matrices such that $\prod_{i=1}^m d_i^{r_i} \cdot \prod_{j=1}^n l_j^{s_j} = 1$.

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LeRoy B. Beasley
Department of Mathematics
Utah State University
Logan, Utah 84322-3900 U.S.A

Gwang-Yeon Lee
Department of mathematics
Hanseo University
Seosan 356-820, Korea

Sang-Gu Lee
Department of Mathematics
SungKyunKwan University
Suwon 440-746, Korea