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CONVERGENCE RATE FOR LOWER BOUNDS TO SELF-ADJOINT OPERATORS

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1. Introduction

Let the operator A be self-adjoint with domain, $\mathrm{Dom}(A)$, dense in $\mathcal{H}$ which is a separable Hilbert space with norm $\|\cdot\|$ and inner product $< \cdot, \cdot >$. We assume that A is bounded below so that the lower part of its spectrum consists of a finite or infinite number of isolated eigenvalues

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_\infty$$

each having finite multiplicity with λ_∞ the lowest limit point of the spectrum of A. We denote such a class of operators by $\mathcal{S}$. If A has compact resolvent, we set $\lambda_\infty = \infty$.

For the eigenvalue problem of $Au = \lambda u$, it is used to apply two complementary methods finding upper bounds and lower bounds to the eigenvalues. The most popular method for finding upper bounds is the Rayleigh–Ritz method, while that for finding lower bounds is intermediate operator methods. Another method for getting lower bounds was introduced by C. Beattie and F. Goerisch in 1991 which has an advantage of using finite element trial function. We may call this method an eigenvector free(EVF) method.

It is well known that if a sequence of bounded self-adjoint operators A_k converges to a bounded self-adjoint operator A uniformly, the eigenvalues of A_k converges to the corresponding ones of A with the convergence rates in [1,13]. In this paper we extend this result to the sequence of operators in $\mathcal{S}$. This will be applied to the sequence of operators which come from the second projection method(SPM) so that

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we derive a new result of convergence rate for the SPM as well as for the EVF method.

We denote by $\mathcal{U}$ the eigenspace of A corresponding to the eigenvalue $\lambda_i = \lambda_{i+1} = \cdots = \lambda_{i+m-1}$ with multiplicity m which is less than λ_∞^0, the lowest point of the essential spectrum of A_0. Similarly, $\mathcal{U}^{(k)}$ denotes the eigenspace of A_k corresponding to the eigenvalues $\lambda_i^{(k)}, \lambda_{i+1}^{(k)}, \ldots, \lambda_{i+m-1}^{(k)}$. We also represent the spectral projections of A and A_k onto $\mathcal{U}$ and $\mathcal{U}^{(k)}$ as E and E_k respectively.

In section 2 we review a result of Weidmann and the relevant theory for the finite element method usually used for differential eigenvalue problems. With the aid of these results, we will provide sufficient conditions for the convergence of eigenvalues and also derive the corresponding rate for a sequence of semi-bounded operators in $\mathcal{S}$. The relation between the SPM and the EVF will be briefly introduced in section 3. Section 4 deals with application of the derived results to the sequence of operator in the SPM so that we derive a convergence rate for the method.

2. Convergence Rates for Semi-bounded Operators

We present an estimate of convergence rates for the sequence, $\{A_k\}_{k=0}^{\infty}$ of operators in $\mathcal{S}$ which converges to A in $\mathcal{S}$ as well as sufficient conditions for the convergence of their eigenvalues. We introduce some convergence rates for the sequence of bounded operators whose proof may be found in [1].

THEOREM 2.1 (BABUŠKA AND OSBORN). *Let (A_k) be a sequence of bounded operators which converges to A uniformly. Then for any i and $j = i, i+1, \ldots, i+m-1$ and $u \in \mathcal{U}$, we have a sufficiently large k such that*

$$|\lambda_i - \lambda_j^{(k)}| \leq \max_{u \in \mathcal{U}, \|u\|=1} |\langle (A_k - A)u, u\rangle| + C \cdot \max_{u \in \mathcal{U}, \|u\|=1} \|(A_k - A)u\|^2$$

for a constant C independent of k.

We would like to extend this result to the case of unbounded operators in $\mathcal{S}$. The following lemma plays a crucial role in this section. The condition that $\lambda_j^{(k)}$ converges to λ_i is used to get a sufficiently large k such that a circle lies in $\rho(A_k)$ enclosing only λ_i and $\{\lambda_j^{(k)}\}_{j=i}^{i+m-1}$. The proof is almost the same as the proof of Theorem 2.1 in [1].

LEMMA 2.2. *Let (A_k) be a sequence of bounded operators which converges to A strongly. If for all i and $j = i, i+1, \ldots, i+m-1$, $\lambda_j^{(k)}$ converges to λ_i as k becomes large, we have a sufficiently large k such that*

$$|\lambda_i - \lambda_j^{(k)}| \le \max_{u\in\mathcal{U},\|u\|=1} |((A_k - A)u, u)| + C \cdot \max_{u\in\mathcal{U},\|u\|=1} \|(A_k - A)u\|^2$$

for a constant C independent of k.

Proof. We note that the spectral projection E associated with A and λ_i is denoted by

$$E = \frac{1}{2\pi i}\int_\Gamma R_z(A)\,dz,$$

where Γ is a circle in the complex plane centered at λ_i which lies in the resolvent set, $\rho(A)$, of A and which encloses no other points of the spectrum, $\sigma(A)$, of A and $R_z(A)$ is the resolvent operator of A at z, i.e. $R_z(A) = (z - A)^{-1}$. Since $\lambda_j^{(k)}$ converges to λ_i as k goes to ∞ for $j = i, i+1, \ldots, i+m-1$, there is a sufficiently large k such that Γ lies also in $\rho(A_k)$ enclosing only λ_i and $\{\lambda_j^{(k)}\}_{j=i}^{i+m-1}$. Thus the spectral projection E_k associated with A_k and $\{\lambda_j^{(k)}\}_{j=i}^{i+m-1}$ may be expressed as

$$E_k = \frac{1}{2\pi i}\int_\Gamma R_z(A_k)\,dz.$$

Hence for any $u \in \mathcal{U}$, we have

$$\begin{aligned}(2.1)\qquad \|(E - E_k)u\| &= \|\frac{1}{2\pi i}\int_\Gamma (R_z(A_k) - R_z(A))u\,dz\| \\ &\le \frac{1}{2\pi}\|\int_\Gamma R_z(A_k)(A - A_k)R_z(A)u\,dz\|\end{aligned}$$

$$\begin{aligned}&\leq \frac{1}{2\pi}\cdot \ell(\Gamma)\cdot \max_{z\in\Gamma}\|R_z(A_k)\|\cdot \max_{u\in\mathcal{U},\|u\|=1}\|(A_k-A)u\|\\&\quad\cdot \max_{z\in\Gamma}\|R_z(A)\|\cdot\|u\|\\&\leq C\cdot \max_{u\in\mathcal{U},\|u\|=1}\|(A_k-A)u\|\end{aligned}$$

for some C independent of k because $R_z(A_k)$ and $R_z(A)$ are uniformly bounded on Γ. Here $\ell(\Gamma)$ is the *arc length* of Γ. Since A_k converges to A strongly with $\mathcal{U}$ as a finite dimensional space, E_k converges to E on the space $\mathcal{U}$.

Let $\hat{E}_k : \mathcal{U} \longrightarrow \mathcal{U}^{(k)}$ be the restriction of E_k to the space $\mathcal{U}$. Suppose that $\hat{E}_k u = 0$ for some $u \in \mathcal{U}$. Then

$$\|u\| = \|(E-E_k)u\| \leq \max_{v\in\mathcal{U},\|v\|=1}\|(E_k-E)v\|\cdot\|u\|.$$

Since E_k converges to E on the space $\mathcal{U}$, we have that $u=0$. Since $\dim\mathcal{U} = \dim\mathcal{U}^{(k)}$, it follows that $\hat{E}_k : \mathcal{U}\to\mathcal{U}^{(k)}$ is bijective. Furthermore

$$\|\hat{E}_k^{-1}\| \leq 2$$

for k sufficiently large since for any $u\in\mathcal{U}$,

$$\|u\| - \|\hat{E}_k u\| \leq \max_{v\in\mathcal{U},\|v\|=1}\|(E_k-E)v\|\cdot\|u\| \leq \frac{1}{2}\|u\|$$

for k sufficiently large. For convenience, let $T_k = \hat{E}_k^{-1}A_k\hat{E}_k$. Then T_k is an operator from $\mathcal{U}$ onto $\mathcal{U}$ having eigenvalues which are

$$\sigma(T_k) = \{\lambda_j^{(k)}\}_{j=i}^{i+m-1}.$$

Let $w_k\in\mathcal{U}$ be defined so that $T_k w_k = \lambda_j^{(k)} w_k$ for some fixed $i\leq j\leq i+m-1$ and $\|w_k\|=1$. Then

$$\lambda_i - \lambda_j^{(k)} = \langle (A-T_k)w_k, w_k\rangle.$$

Since $\hat{E}_k^{-1}E_k$ is the identity on $\mathcal{U}$, we have for any $v\in\mathcal{U}$ with $\|v\|=1$,

$$\begin{aligned}\langle (A-T_k)v,v\rangle &= \langle \hat{E}_k^{-1}E_kAv,v\rangle - \langle \hat{E}_k^{-1}A_k\hat{E}_kv,v\rangle\\&= \langle \hat{E}_k^{-1}E_k(A-A_k)v,v\rangle\\&= \langle (I-\hat{E}_k^{-1}E_k)(A_k-A)v,v\rangle - \langle (A_k-A)v,v\rangle.\end{aligned}$$

Since $E_k\hat{E}_k^{-1} = I$ on $\mathcal{U}^{(k)}$, we have that $I - \hat{E}_k^{-1}E_k = (I - E_k)(I - \hat{E}_k^{-1}E_k)$ and $(I - E_k)v = (E - E_k)v$ for any $v \in \mathcal{U}$. It follows from (2.1) that

$$\begin{aligned}
|\langle (I - \hat{E}_k^{-1}E_k)(A_k - A)v, v\rangle| &= |\langle (I - \hat{E}_k^{-1}E_k)(A_k - A)v, (E - E_k)v\rangle| \\
&\leq \|I - \hat{E}_k^{-1}E_k\| \|(A_k - A)v\| \cdot \|(E - E_k)v\| \\
&\leq 3C \cdot \max_{u \in \mathcal{U}, \|u\|=1} \|(A_k - A)u\|^2
\end{aligned}$$

for sufficiently large k. ■

We now assume that A_k and A are bounded below such that $A_k \leq A_{k+1} \leq A$ for all $k \geq 0$.

DEFINITION. Let A_k be a sequence of self adjoint operators acting on $\mathcal{H}$. We say that A_k converges to A in the strong resolvent sence if $(A_k - z)^{-1}$converges strongly to $(A - z)^{-1}$ for some z which is bounded away from the spectra of the A_k and A.

If A_k and A are all coercive, convergence in the strong resolvent sense is equivalent to the strong convergence of A_k^{-1} to A^{-1}. We modify a result of Weidmann[14] for our problem setting.

LEMMA 2.3(WEIDMANN). *Let (A_k) be an increasing sequence of operators in $\mathcal{S}$ which converges to A in the strong resolvent sense. Let $\lambda_1^{(k)} \leq \lambda_2^{(k)} \leq \cdots \leq \lambda_\infty^{(k)}$ and $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_\infty$ be the isolated eigenvalues of A_k and A, respectively. Then for all i such that $\lambda_i < \lambda_\infty^{(0)}$, $\lambda_i^{(k)}$ converges to λ_i, where $\lambda_\infty^{(0)}$ denotes the lowest point of the essential spectrum of A_0.*

It follows from Lemmas 2.2 and 2.3 that we have the following main estimate result for a sequence of semi-bounded operators.

THEOREM 2.4. *Let (A_k) be an increasing sequence of operators in $\mathcal{S}$ which converges to A in the strong resolvent sense. Then for all i such that $\lambda_i < \lambda_\infty^{(0)}$, $\lambda_i^{(k)}$ converges to λ_i as k becomes large. Furthermore if λ_i has multiplicity m with $\lambda_i = \lambda_{i+1} = \cdots = \lambda_{i+m-1}$, we have the* *following estimate, for $j = i, i+1, \ldots, i+m-1$,*

$$\left|\frac{1}{\lambda_i} - \frac{1}{\lambda_j^{(k)}}\right| \le \max_{u\in\mathcal{U},\|u\|=1} |\langle (A_k^{-1} - A^{-1})u, u\rangle| + C \cdot \max_{u\in\mathcal{U},\|u\|=1} \|(A_k^{-1} - A^{-1})u\|^2$$

for a constant C independent of k.

3. On the methods of Second Projection and EVF

In this section we describe briefly the SPM and the EVF method. For more details on the EVF method, one should refer to [5] and for the second projection method, refer to [3,16].

Let A be an opertor in $\mathcal{S}$ and let $a(\cdot)$ be the quadratic form which is the closure of $\langle A\cdot, \cdot\rangle$. We assume that a self adjoint operator A_0 in $\mathcal{S}$ is taken to be such that $A_0 \le A$, and the isolated eigenvalues of A_0

$$\lambda_1^{(0)} \le \lambda_2^{(0)} \le \cdots \le \lambda_\infty^{(0)}$$

are known. We assume that the quadratic form $a(u)$ is decomposed as

$$a(u) = a_0(u) + \|Tu\|_*^2$$

where T is a closed operator on $\mathcal{H}$ to another Hilbert space $\mathcal{H}_*$.

Let T^* be the adjoint operator of T. We take a sequence of finite dimensional spaces $\{\mathcal{P}_k\}$ such that

$$\mathcal{P}_1 \subset \cdots \subset \mathcal{P}_k \subset \mathcal{P}_{k+1} \subset \cdots \subset \mathrm{Dom}(T^*) \subset \mathcal{H}_*$$

and let $P_k : \mathcal{H}_* \longrightarrow \mathcal{P}_k$ be the projection that is orthogonal with respect to the inner product $\langle \cdot, \cdot \rangle_*$. We construct the intermediate quadratic forms $a_k(u)$ as

$$a_k(u) = a_0(u) + \|P_k T u\|_*^2$$

for all $u \in \mathrm{Dom}(a_k) = \mathrm{Dom}(a_0) \cap \mathrm{Dom}(T)$, which may be associated with a self-adjoint operator given by

$$A_k = A_0 + T^* P_k T$$

with $\mathrm{Dom}(A_k) = \mathrm{Dom}(A_0)$. Then

$$A_0 \leq A_k \leq A_{k+1} \leq A.$$

For any positive constant θ, the operator A_k may be rewritten by

$$A_k = (A_0 - \theta) + (T^* P_k T + \theta).$$

Let $B_k^\theta = T^* P_k T + \theta$ for each k. The operator B_k^θ produces a new inner product $\langle B_k^\theta \cdot, \cdot \rangle \equiv \langle \cdot, \cdot \rangle_{b_k^\theta}$ on the Hilbert space $\mathcal{H}$. Let a sequence of finite dimensional subspaces $\{\hat{\mathcal{P}}_k\}$ be given such that

$$\hat{\mathcal{P}}_1 \subset \cdots \subset \hat{\mathcal{P}}_n \subset \hat{\mathcal{P}}_{n+1} \subset \cdots \subset \mathcal{H}$$

and let $\hat{P}_n : \mathcal{H} \longrightarrow \hat{\mathcal{P}}_n$ be the projection that is orthogonal with respect to the inner product $\langle \cdot, \cdot \rangle_{b_k^\theta}$. We form the intermediate operators as

$$A_{k,n} = (A_0 - \theta) + B_k^\theta \hat{P}_n.$$

Then

$$A - \theta \leq A_{k,n}^\theta \leq \begin{pmatrix} A_{k+1,n}^\theta \\ A_{k,n+1}^\theta \end{pmatrix} \leq A.$$

The eigenvalues of $A_{k,n}^\theta$ converge from below to the corresponding ones of A as k and n go to ∞ under some conditions [2,16]. The associated $n \times n$ Weinstein and Aronszajn(W-A) matrix of the operator $A_{k,n}^\theta$ is given by

$$W_{k,n}(\lambda) = [\langle \hat{p}_i + R_{\lambda+\theta}^0 B_k^\theta \hat{p}_i, B_k^\theta \hat{p}_j \rangle] \tag{3.1}$$

for $i, j = 1, \ldots, n$, where R_μ^0 is the resolvent operator, $(A_0 - \mu)^{-1}$, of A_0 at μ. If we let $\mu = \lambda + \theta$ and introduce the change of variable $q_i = R_\mu^0 B_k^\theta \hat{p}_i$ into the W-A matrix (3.1), we get

$$W_{k,n}(\lambda) = [\langle B_k^{\theta^{-1}} (A_0 - \mu) q_i, (A_0 - \mu) q_j \rangle + \langle q_i, (A_0 - \mu) q_j \rangle]$$

which is more simplified with a formula for $B_k^{\theta^{-1}}$ (see [2]) to get

$$W_{k,n}(\lambda) =[\langle q_i, (A_0 - \mu)q_j\rangle + \frac{1}{\mu-\lambda}\{\langle (A_0 - \mu)q_i, (A_0 - \mu)q_j\rangle \\ - \sum_{l,m=1}^{k} \langle (A_0 - \mu)q_i, T^* p_l\rangle c_{lm} \langle T^* p_m, (A_0 - \mu)q_j\rangle\}].$$

If we define the matrices as

$$F_1 = [\langle q_i, (A_0 - \mu)q_j\rangle], \quad F_2 = [\langle p_i, p_j\rangle_*], \quad H = [\langle (A_0 - \mu)q_i, T^* p_j\rangle] \\ G_1 = [\langle (A_o - \mu)q_i, (A_0 - \mu)q_j\rangle], \quad G_2 = [\langle T^* p_i, T^* p_j\rangle],$$

then the W-A matrix is compactly expressed as

$$W_{k,n}(\lambda) = F_1 + \frac{1}{\mu-\lambda}\{G_1 - H[(\mu-\lambda)F_2 + G_2]^{-1}H^*\}. \tag{3.2}$$

Based on this W–A matrix, Beattie and Goerisch introduced the EVF method. For more general case, refer to [5].

THEOREM 3.1(BEATTIE AND GOERISCH). *Let μ and r be chosen so that $\lambda^0_{r-1} < \mu \le \lambda^0_r$. Suppose that $\{p_i\}_{i=1}^k \subset Dom(T^*)$ and $\{q_i\}_{i=1}^n \subset Dom(A_0)$ such that $\{(A_0 - \mu)q_i\}_{i=1}^n$ and $\{T^* p_i\}_{i=1}^k$ are jointly linearly independent. If the generalized matrix eigenvalue problem*

$$\begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \xi \begin{bmatrix} G_1 & H \\ H^* & G_2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{3.3}$$

has discrete finite eigenvalues ordered as

$$\xi_1 \le \xi_2 \le \cdots \le \xi_l < 0 \le \xi_{l+1} \le \cdots,$$

then for each eigenvalue ξ_p with $p \le l$ we have a corresponding lower bound to an eigenvalue of A;

$$\mu + \frac{1}{\xi_p} \le \lambda_{r-p}.$$

We note here that if $\{q_i\}_{i=1}^n$ and $\{p_j\}_{j=1}^k$ are chosen to have local support as with finite-element trial functions, the resulting matrices will be sparse and the matrix eigenvalue problem may be efficiently handled using sparse techniques, even for quite large values of n and k. The following theorem which shows the relation between eigenvalues of the SPM and the EVF was proved in [11].

THEOREM 3.2. *For any p with $1 \le p \le l$, $\mu + \dfrac{1}{\xi_p}$ is equal to the $(r-p)$th eigenvalue, $\lambda_{r-p}^{(k,n,-\frac{1}{\xi_p})}$ of $A_{k,n}^{-\frac{1}{\xi_p}}$.*

This theorem means that the bound $\mu + \frac{1}{\xi_p}$ of the EVF method corresponding to the $(r-p)$th eigenvalue of A is the $(r-p)$th eigenvalue of the intermediate operator $A_{k,n}^{\theta}$ with $\theta = -\frac{1}{\xi_p}$ which comes from the SPM. Hence the convergence rate for the EVF method is equivalent to that of the SPM. It remains to get the convergence rate for the SPM.

4. Convergence Rate for the SPM

We represent in this section the inner product and norm of the difference between the operators ${A_{k,n}^{\theta}}^{-1}$ and A^{-1} as the sum of norms of projections $I - P_k$ and $I - \hat{P}_n^{\theta}$. Since

$$
\begin{aligned}
{A_{k,n}^{\theta}}^{-1} - A^{-1} &= {A_{k,n}^{\theta}}^{-1}(A - A_{k,n}^{\theta})A^{-1} \\
&= {A_{k,n}^{\theta}}^{-1}[T^*(I-P_k)T + B_k^{\theta}(I - \hat{P}_n^{\theta})]A^{-1},
\end{aligned}
$$

we have

$$
\begin{aligned}
\left\langle ({A_{k,n}^{\theta}}^{-1} - A^{-1})u, u \right\rangle &= \left\langle {A_{k,n}^{\theta}}^{-1} T^*(I - P_k)TA^{-1}u, u \right\rangle \\
&\quad + \left\langle {A_{k,n}^{\theta}}^{-1} B_k^{\theta}(I - \hat{P}_n^{\theta})A^{-1}u, u \right\rangle .
\end{aligned}
$$

The right-hand sides are expressed as following:

$$
\begin{aligned}
&\left| \left\langle {A_{k,n}^{\theta}}^{-1} T^*(I-P_k)TA^{-1}u, u \right\rangle \right| \\
&\qquad = \left| \left\langle (I-P_k)TA^{-1}u, (I-P_k)T{A_{k,n}^{\theta}}^{-1}u \right\rangle_* \right| \\
&\qquad \le ||(I-P_k)TA^{-1}u||_* ||(I-P_k)T{A_{k,n}^{\theta}}^{-1}u||_*
\end{aligned}
$$

where

$$
\begin{aligned}
&||(I-P_k)T{A_{k,n}^{\theta}}^{-1}u||_* \\
&\qquad \le ||(I-P_k)TA^{-1}u||_* + ||(I-P_k)T({A_{k,n}^{\theta}}^{-1} - A^{-1})u||_* \\
&\qquad \le ||(I-P_k)TA^{-1}u||_* + ||T({A_{k,n}^{\theta}}^{-1} - A^{-1})u||_*
\end{aligned}
$$

and

$$
\begin{aligned}
&||T({A^{\theta}_{k,n}}^{-1} - A^{-1})u||_* \\
&\leq ||T{A^{\theta}_{k,n}}^{-1}T^*(I-P_k)TA^{-1}u||_* + ||T{A^{\theta}_{k,n}}^{-1}B^{\theta}_k(I-\hat{P}^{\theta}_n)A^{-1}u||_* \\
&\leq ||T{A^{\theta}_{k,n}}^{-\frac{1}{2}}||_*^2||(I-P_k)TA^{-1}u||_* + ||T{A^{\theta}_{k,n}}^{-1}B^{\theta}_k(I-\hat{P}^{\theta}_n)A^{-1}u||_* \\
&\leq ||T{A^{\theta}_{k,n}}^{-\frac{1}{2}}||_*^2||(I-P_k)TA^{-1}u||_* \\
&\quad + ||T{A^{\theta}_{k,n}}^{-\frac{1}{2}}||_*||{A^{\theta}_{k,n}}^{-\frac{1}{2}}{B^{\theta}_k}^{\frac{1}{2}}||_*||(I-\hat{P}^{\theta}_n)A^{-1}u||_{b^{\theta}_k}.
\end{aligned}
$$

We have thus

$$
\begin{aligned}
&|\left\langle {A^{\theta}_{k,n}}^{-1}T^*(I-P_k)TA^{-1}u, u\right\rangle| \\
&\quad = (1 + ||T{A^{\theta}_{k,n}}^{-\frac{1}{2}}||_*^2)||(I-P_k)TA^{-1}u||_*^2 \\
&\qquad + ||T{A^{\theta}_{k,n}}^{-\frac{1}{2}}||_*||{A^{\theta}_{k,n}}^{-\frac{1}{2}}{B^{\theta}_k}^{\frac{1}{2}}||_*||(I-P_k)TA^{-1}u||_* \\
&\qquad ||(I-\hat{P}^{\theta}_n)A^{-1}u||_{b^{\theta}_k}.
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
|\left\langle {A^{\theta}_{k,n}}^{-1}B^{\theta}_k(I-\hat{P}^{\theta}_n)A^{-1}u, u\right\rangle| &= |\left\langle B^{\theta}_k(I-\hat{P}^{\theta}_n)A^{-1}u, {A^{\theta}_{k,n}}^{-1}u\right\rangle| \\
&= |\left\langle (I-\hat{P}^{\theta}_n)A^{-1}u, (I-\hat{P}^{\theta}_n){A^{\theta}_{k,n}}^{-1}u\right\rangle|_{b^{\theta}_k} \\
&\leq ||(I-\hat{P}^{\theta}_n)A^{-1}u||_{b^{\theta}_k}||(I-\hat{P}^{\theta}_n){A^{\theta}_{k,n}}^{-1}u||_{b^{\theta}_k}
\end{aligned}
$$

where

$$
\begin{aligned}
&||(I-\hat{P}^{\theta}_n){A^{\theta}_{k,n}}^{-1}u||_{b^{\theta}_k} \\
&\qquad \leq ||(I-\hat{P}^{\theta}_n)A^{-1}u||_{b^{\theta}_k} + ||(I-\hat{P}^{\theta}_n)({A^{\theta}_{k,n}}^{-1} - A^{-1})u||_{b^{\theta}_k} \\
&\qquad \leq ||(I-\hat{P}^{\theta}_n)A^{-1}u||_{b^{\theta}_k} + ||({A^{\theta}_{k,n}}^{-1} - A^{-1})u||_{b^{\theta}_k}
\end{aligned}
$$

and

$$
\begin{aligned}
&||({A^{\theta}_{k,n}}^{-1} - A^{-1})u||_{b^{\theta}_k} = ||{B^{\theta}_k}^{\frac{1}{2}}({A^{\theta}_{k,n}}^{-1} - A^{-1})u|| \\
&\leq ||{B^{\theta}_k}^{\frac{1}{2}}{A^{\theta}_{k,n}}^{-1}T^*(I - P_k)TA^{-1}u|| \\
&\quad + ||{B^{\theta}_k}^{\frac{1}{2}}{A^{\theta}_{k,n}}^{-1}B^{\theta}_k(I - \hat{P}^{\theta}_n)A^{-1}u|| \\
&\leq ||{B^{\theta}_k}^{\frac{1}{2}}{A^{\theta}_{k,n}}^{-\frac{1}{2}}|| \cdot ||{A^{\theta}_{k,n}}^{-\frac{1}{2}}T^*|| \cdot ||(I - P_k)TA^{-1}u||_* \\
&\quad + ||{B^{\theta}_k}^{\frac{1}{2}}{A^{\theta}_{k,n}}^{-\frac{1}{2}}||^2||{B^{\theta}_k}^{\frac{1}{2}}(I - \hat{P}^{\theta}_n)A^{-1}u||_* \\
&= ||{B^{\theta}_k}^{\frac{1}{2}}{A^{\theta}_{k,n}}^{-\frac{1}{2}}|| \cdot ||{A^{\theta}_{k,n}}^{-\frac{1}{2}}T^*|| \cdot ||(I - P_k)TA^{-1}u||_* \\
&\quad + ||{B^{\theta}_k}^{\frac{1}{2}}{A^{\theta}_{k,n}}^{-\frac{1}{2}}||^2||(I - \hat{P}^{\theta}_n)A^{-1}u||_{b^{\theta}_k}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&|\left\langle {A^{\theta}_{k,n}}^{-1}B^{\theta}_k(I - \hat{P}^{\theta}_n)A^{-1}u, u\right\rangle| \\
&\leq (1 + ||{B^{\theta}_k}^{\frac{1}{2}}{A^{\theta}_{k,n}}^{-\frac{1}{2}}||^2)||(I - \hat{P}^{\theta}_n)A^{-1}u||^2_{b^{\theta}_k} \\
&\quad + ||{B^{\theta}_k}^{\frac{1}{2}}{A^{\theta}_{k,n}}^{-\frac{1}{2}}||^2||{A^{\theta}_{k,n}}^{-\frac{1}{2}}T^*|| \cdot ||(I - \hat{P}^{\theta}_n)A^{-1}u||_{b^{\theta}_k} \\
&\cdot ||(I - P_k)TA^{-1}u||_*.
\end{aligned}
$$

If we let $||T{A^{\theta}_{k,n}}^{-\frac{1}{2}}||_* = \gamma$ and $||{A^{\theta}_{k,n}}^{-\frac{1}{2}}{B^{\theta}_k}^{\frac{1}{2}}|| = \mu$, then we have

$$
\begin{aligned}
|\left\langle ({A^{\theta}_{k,n}}^{-1} - A^{-1})u, u\right\rangle| \leq{}& \frac{1+\gamma^2}{\lambda_i^2}||(I - P_k)Tu||^2_* \\
&+ \frac{2\gamma\mu}{\lambda_i^2}||(I - P_k)Tu||_*||(I - \hat{P}^{\theta}_n)u||_{b^{\theta}_k} \\
&+ \frac{1+\mu^2}{\lambda_i^2}||(I - \hat{P}^{\theta}_n)u||^2_{b^{\theta}_k}
\end{aligned}
$$

because $A^{-1}u = \frac{1}{\lambda_i}u$. We note also that

$$
\begin{aligned}
&||({A^{\theta}_{k,n}}^{-1} - A^{-1})u|| \\
&\leq ||{A^{\theta}_{k,n}}^{-1}T^*(I - P_k)TA^{-1}u|| + ||{A^{\theta}_{k,n}}^{-1}B^{\theta}_k(I - \hat{P}^{\theta}_n)A^{-1}u|| \\
&\leq \frac{\gamma}{\lambda_i}||{A^{\theta}_{k,n}}^{-\frac{1}{2}}|| \cdot ||(I - P_k)Tu||_* + \frac{\mu}{\lambda_i}||{A^{\theta}_{k,n}}^{-\frac{1}{2}}|| \cdot ||(I - \hat{P}^{\theta}_n)u||_{b^{\theta}_k}.
\end{aligned}
$$

Finally it remains to show that the γ and μ are bounded by some constants which are independent of k and n. Since $B_k^\theta \leq T^*T + \theta$ and $A_0 - \theta \leq A_{k,n}^\theta$, it follows that

$$\gamma \leq \|T(A_0 - \theta)^{-\frac{1}{2}}\| \text{ and } \mu \leq \|(T^*T + \theta)^{\frac{1}{2}}(A_0 - \theta)^{-\frac{1}{2}}\|.$$

If we assume that T^*T is relatively bounded to A_0, then T^*T and $T^*T + \theta$ are also relatively bounded to $A_0 - \theta$. Thus $\|T(A_0 - \theta)^{-\frac{1}{2}}\|_*$ and $\|(T^*T + \theta)^{\frac{1}{2}}(A_0 - \theta)^{-\frac{1}{2}}\|$ are bounded by some constants. This implies that the γ and μ are bounded by some constants which are independent of k and n.

LEMMA 4.4. *If T^*T is relatively bounded with respect to A_0. then γ and μ are bounded by some constants which are independent of k and n.*

Combing together, we get the following result from Theorem 2.4 and Lemma 4.4.

THEOREM 4.5. *If T^*T is relatively bounded with respect to A_0, then*

$$\left|\frac{1}{\lambda_{r-p}} - \frac{1}{\lambda_{r-p}^{(k,n)}}\right| \leq C \cdot \|(I - P_k)Tu\|_*^2 + D \cdot \|(I - \hat{P}_n^\theta)u\|_{b_k^\theta}^2$$
$$+ E \cdot \|(I - P_k)Tu\|_* \cdot \|(I - \hat{P}_n^\theta)u\|_{b_k^\theta}$$

for some constants C, D and E which are independent of k and n where $\lambda_{r-p}^{(k,n)} = \mu + \dfrac{1}{\xi_p}$ which is corresponding to the k and n.

Let k and n increase simultaneously. That is, let $k = n$. If we assume that

$$\|(I - P_n)Tu\|_* = O(n^{-\alpha}) \text{ and } \|(I - \hat{P}_n^\theta)u\|_{b_n^\theta} = O(n^{-\beta}),$$

we have then

$$\left|\frac{1}{\lambda_{r-p}} - \frac{1}{\lambda_{r-p}^{(n,n)}}\right| = O(n^{-2\delta})$$

where $\delta = \min(\alpha, \beta)$.

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