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HIGHER ORDER MAXIMUM NORM CONVERGENCE OF FULLY DISCRETE SOLUTION FOR PARABOLIC PROBLEMS

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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 1$, with smooth boundary $\partial\Omega$, and let $0 < T < \infty$ be given. We consider a real-valued function $u(x,t)$ satisfying,

$$(1.1) \qquad \begin{cases} u_t = \Delta u + f(x,t), & (x,t) \in \Omega \times (0,T], \\ u = 0, & (x,t) \in \partial\Omega \times [0,T], \\ u(x,0) = u^0(x), & x \in \overline{\Omega}. \end{cases}$$

where $\Delta u = \sum_{j=1}^{d} \partial_{x_j}^2 u$ and u^0 is a given real-valued function on $\overline{\Omega}$. The initial data u^0 is assumed to be both sufficiently smooth and compatible, and $f(x,t)$ is sufficiently smooth.

Recently, the convergence of the fully discrete solution of parabolic type problem in L_2 norm are analyzed in numerous papers ([1,7,8,10]). There are also results about L_∞ norm convergence for the semidiscrete solution ([4,5,11,12,13,14,16]). Concerning to the convergence of fully discrete solution in L_∞, in ([15]), M. Wheeler adopted Crank-Nicolson method and proved $O(\Delta t^2)$ convergence in the one space dimension. In this paper, we shall approximate the solution of (1.1) using Galerkin finite element method for the spatial discretization, and implicit Runge-Kutta methods for the time stepping. And we prove the arbitrary

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higher order convergence under L_∞ norm computationally, and we'll leave the analytic proof for the future work. The remainder of this paper is organized as follows. Section 2 is devoted to some notations and preliminaries. In section 3, we develope the fully discrete approximation to the solution of initial boundary value problem. In section 4, we provide computational results which report that Galerkin Runge-Kutta methods offer arbitrarily high, optimal convergence in L_∞ norm.

2. Notations and Preliminaries

Now for $1 \le p \le \infty$, $L_p = L_p(\Omega)$ will denote the Banach space of real-valued measurable functions defined on Ω, equipped with the norm,

$$\|v\|_{L_p} = (\int_\Omega |v(x)|^p dx)^{\frac{1}{p}}, \quad 1 \le p < \infty,$$
$$\|v\|_{L_\infty} = ess_{x\in\Omega} \sup |v(x)|, \quad p = \infty.$$

(In the sequel, we use $\|\cdot\|$ instead of $\|\cdot\|_{L_2}$.)

Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d)$, $\alpha_i \ge 0$, denote a multiinteger, and let $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$. For an integer $s \ge 0$, H^s will denote the Sobolev space of measurable functions which, together with their distributional derivatives of order up to s are in L_2. In addition, let H_0^1 be the subspace of H^1 consisting of functions vanishing on $\partial\Omega$ in the sense of trace. For small h, let $\prod_h = \{\tau_i\}_{i=1}^{I(h)}$ denote a quasiuniform triangulation of Ω with

$$\max_{1\le i\le I(h)} \text{diam } \tau_i \le h$$

and let $\Omega_h \subseteq \Omega$ be the polygonal domain determined by $\prod_h$. We shall be concerned with the approximation to the solution of the problem (1.1) by means of elements in a finite-dimensional space $S_h \subset H_0^1$ which for example we take to consist of the continuous functions in Ω which are polynomials on each triangle τ_i and vanish outside Ω_h. Let Δ be extended to have domain $H^2 \cap H_0^1$. Then Δ is L_2 - selfadjoint and for every nonnegative integer s, it is bounded from $H^{s+2} \cap H_0^1$ into

H^s. Furthermore, introducing the solution operator T for the elliptic problem

$$\begin{cases} -\Delta v = w & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

as $Tw \equiv v$, it is well known ([6]) that for every nonnegative integer s, T is bounded from H^s into $H^{s+2} \cap H_0^1$. Also the solution operator is positive definite and self-adjoint on L^2. In terms of the solution operator, (1.1) can be written as:

$$\begin{cases} \partial_t T u = -u + Tf, \\ u(0) = u^0. \end{cases} \tag{2.1}$$

For a family of finite dimensional subspaces $\{S_h\}_{0<h<1}$ of H_0^1, we assume that a corresponding family of operators $\{T_h\}_{0<h<1}$ is given satisfying:

(i) $T_h : L^2 \to S_h$ is selfadjoint, positive semidefinite on L^2, and positive definite on S_h.

(ii) There is an integer $r \geq 2$ such that:

$$\|(T - T_h)v\| \leq Ch^s \|v\|_{s-2}, \qquad \forall v \in H^{s-2},\ 2 \leq s \leq r. \tag{2.2}$$

Now problem (2.1) has the following semidiscrete formulation. Find $u_h : [0, T] \to S_h$ such that

$$\begin{cases} \partial_t T_h u_h = -u_h + T_h f, \\ u_h(0) = u_h^0 \end{cases} \tag{2.3}$$

where $u_h^0 \in S_h$ is a suitable approximation to u^0. Define the discrete Laplacian $L_h : S_h \to S_h$ such that $(L_h w, \chi) = (\nabla w, \nabla \chi)$, $\forall \chi \in S_h$. Now problem (2.3) takes the following form. Find $u_h : [0, T] \to S_h$ satisfying:

$$\begin{cases} \partial_t u_h = -L_h u_h + P_0 f \\ u_h(0) = u_h^0 \end{cases} \tag{2.4}$$

where P_0 is orthogonal projection from L_2 onto S_h.

3. Fully Discrete approximation

For the temporal approximation of the solution to (1.1), Implicit Runge-Kutta (IRK) methods are now introduced ([3]). For $q \geq 1$ integer, a q-stage IRK method is characterized by a set of constants arranged in the following tableau form:

$$\begin{array}{ccc|c} a_{11} & \dots & a_{1q} & \tau_1 \\ \vdots & & \vdots & \vdots \\ a_{q1} & \dots & a_{qq} & \tau_q \\ \hline b_1 & \dots & b_q & \end{array}$$

It is convenient to make the following definitions:

$$A = (a_{ij})_{1 \leq i,\ j \leq q},\ b^T = (b_1, b_2, \cdots, b_q)$$
$$e^T = (1, 1, \cdots, 1) \in R^q.$$

Given the initial value problem,

$$(3.1) \qquad \begin{aligned} y' &= g(t, y), \qquad 0 \leq t \leq T, \\ y(0) &= y^0. \end{aligned}$$

IRK methods can be applied to generate approximations $\{y^n\}_{n=1}^N$ to $\{y(t^n)\}_{n=1}^N$, where $\Delta t = \frac{T}{N}$ is the temporal stepsize and $t^n = n\Delta t$, as follows. Let

$$(3.2) \qquad y^{n+1} = y^n + \Delta t \sum_{j=1}^{q} b_j g(t^{n,j}, y^{n,j}),$$

where $t^{n,j} = t^n + \tau_j \Delta t$ and the intermediate stages $y^{n,j}$ are given by the coupled system of equations

$$(3.3) \qquad y^{n,j} = y^n + \Delta t \sum_{m=1}^{q} a_{j,m} g(t^{n,m}, y^{n,m}), \qquad j = 1, 2, \cdots, q.$$

Let $\sigma(A)$ consist of the eigenvalues of A. It is assumed throughout this work that

$$(3.4) \qquad \sigma(A) \subset \{z \in C \mid Re z \geq 0\}.$$

Now we modify (3.2) and (3.3) to be applied to the semidiscrete form (2.4). Suppose that the approximations $[U_h^m]_{m=0}^n \subset S_h$ are given, where U_h^m is an approxiamtion of $u^m = u(x, t^m)$. Define $\mathcal{L}_h : [S_h]^q \to [S_h]^q$ and $\mathcal{P}_0 : [L_2]^q \to [S_h]^q$ by

$$\mathcal{L}_h = \operatorname*{diag}_{q\times q} \{L_h\} \quad \text{and} \quad \mathcal{P}_0 = \operatorname*{diag}_{q\times q} \{P_0\}.$$

Finally, let $U_h^{n+1} \cong u^{n+1}$ be given by what will thereafter be called the fully discrete scheme:

$$\begin{aligned} \overline{U}_h^n &= eU_h^n - \Delta t A \mathcal{L}_h \overline{U}_h^n + \Delta t A \mathcal{P}_0 \overline{f}^n, \\ U_h^{n+1} &= U_h^n - \Delta t b^T \mathcal{L}_h \overline{U}_h^n + \Delta t b^T \mathcal{P}_0 \overline{f}^n, \end{aligned} \tag{3.5}$$

where $\overline{f}^n = \langle f(t^{n,1}, \cdot), \ldots, f(t^{n,q}, \cdot)\rangle^T$, and $\overline{U}_h^n \in [S_h]^q$ is well-defined, provided $[I + \Delta t A \mathcal{L}_h]$ is invertible.

With regard to the initial data, it is sufficient to take as U_h^0 any element of S_h which is optimally close to u^0 in L_∞, i.e.,

$$\|u^0 - U_h^0\|_{L_\infty} \le ch^r.$$

THEOREM 1. *Assume that (3.4) holds. Then $[I + \Delta t A \mathcal{L}_h]$ is invertible. Therefore the fully discrete scheme (3.5) has a unique solution.*

Proof. For a sufficiently small Δt, the proof is trivial by spectral argument.

4. Computational Aspects

Consider the following problem:

$$\begin{cases} u_t = u_{xx} + f(x,t), & (x,t) \in (0.0, 1.0) \times [0, T], \\ u = 0, & (x,t) \in \{0.0, 1.0\} \times [0, T], \\ u(x, 0) = x^2 - x & x \in (0.0, 1.0), \end{cases} \tag{4.1}$$

where:

$$f(x,t) = \frac{(x-x^2)log(1+x^2)}{(1+x^2)^t} + \frac{2x(t+1)(-2x^3t+2x^3+2x^2t-x^2+2x-1)}{(1+x^2)^{t+2}} + \frac{6x^2t-4xt-6x^2+2x-2}{(1+x^2)^{t+1}}$$

The solution is given by:

$$u(x,t) = \frac{x^2-x}{(1+x^2)^t} \tag{4.2}$$

As S_h we can take, for example, the space S_h^r of smooth periodic splines of order less than or equal to $r-1$ on a uniform mesh with mesh length h.

For the time stepping procedure, we use the q stages multiply implicit Runge-Kutta method (MIRK) introduced in ([2,9]). They proved A_0-stable q stage MIRK method has maximal convergence order ν =q+1. MIRK method has been constructed to have q distinct real eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_q$ of A. Then the linear system from (3.5) can be written as

$$[I + \Delta t \wedge \mathcal{L}_h]S\Phi = S\Psi$$

which decouples to equations of the form

$$[1 + \Delta t\lambda_i L_h]\phi_i = \psi_i, \quad i = 1, \cdots, q$$

for real value λ_i. All of the experiments reported in the sequel were performed in double precision arithmetic on the Sun sparc station 2 running SunOS 4.1.2 (manufactured by the Sun Microsystems Inc.) at the Kyungsung University.

In the series of tables below, the error committed by (3.5) for the approximation of the solution to (1.1) is shown computationally to be optimal in L_∞, i.e.,

$$\max_{0\le n\le N} \|U_h^n - u^n\|_{L_\infty} \le c(h^r + (\Delta t)^\nu).$$

Now we report the outcome of a number of numerical experiments that were performed on (4.1). The measures of errors used were the L_∞-norm given by

$$E(t^n) = \|U_h^n - u(\cdot, t^n)\|_{L_\infty}$$

In Table 1 through 5, convergence rate is obtained according to the formula:

$$Rate = \frac{log(\frac{E_1(t^n)}{E_2(t^n)})}{log(\frac{h_1}{h_2})}$$

where $E_i(t^n)$ is $E(t^n)$ with $h = h_i = \Delta t_i$, i=1, 2. In Table 1 and 2, with $\Delta t = h$ we achieve optimal convergence order r=q+1.

TABLE 1 Maximum error $E(T)$ at $T = 1.0$ with r=3 and q=2

h^{-1}	N	CPU Time (sec)	E(t)	Rate
50	50	8	0.3673(-6)	
60	60	12	0.2130(-6)	2.99
70	70	15	0.1343(-6)	2.99
80	80	20	0.9010(-7)	2.99
90	90	26	0.6333(-7)	2.99
100	100	31	0.4620(-7)	2.99

TABLE 2 Maximum error $E(t)$ at $T = 0.2$ with r=4 and q=3

h^{-1}	N	CPU Time (sec)	E(t)	Rate
25	5	0.7	0.2134(-7)	
30	6	0.9	0.1009(-7)	4.11
35	7	1.2	0.5394(-8)	4.07
40	8	1.6	0.3146(-8)	4.04
45	9	2.0	0.1964(-8)	4.00

To investigate the convergence rates of the scheme in the spatial variable with various r, we choose Δt sufficiently small compared to h, so that $E(t)$ mainly contains the spatial error. In Tables 3, 4, and 5, we tested with ν=3 or ν=4, and proved ν does not affect the result.

since we choose Δt suficiently small compared to h. As shown in table 3 and 4, we achieved optimal convergence order proved by J. Nitsche, for $r \geq 3$ in [10]. In [10], he expected to have $logh$, conbining h^2 for $r = 2$, but fortunately in table 5 we proved optimal convergence order h^2 without appearing $logh$.

TABLE 3 Pure spatial maximum error $E(t)$ at $T = 0.1$ with r=3, N=10, and q=3

h^{-1}	CPU Time (sec)	E(t)	Rate
25	1.0	0.3143(-6)	
30	1.1	0.1817(-6)	3.01
35	1.3	0.1143(-6)	3.00
40	1.4	0.7654(-7)	3.00
45	1.7	0.5373(-7)	3.00
50	1.8	0.3914(-7)	3.01

TABLE 4 Pure spatial maximum error $E(t)$ at $T = 0.1$ with r=4, N=10, and q=2

h^{-1}	CPU Time (sec)	E(t)	Rate
5	0.2	0.6224(-5)	
10	0.4	0.4316(-6)	3.85
15	0.5	0.8519(-7)	4.00
20	0.7	0.2638(-7)	4.08
25	0.9	0.1061(-7)	4.08
30	1.0	0.5082(-8)	4.04

TABLE 5 Pure spatial maximum error $E(t)$ at $T = 0.1$ with r=2, N=10, and q=2.

h^{-1}	CPU Time (sec)	E(t)	Rate
25	0.5	0.3495(-3)	
30	0.5	0.2429(-3)	2.00
35	0.6	0.1785(-3)	2.00
40	0.7	0.1367(-3)	2.00
45	0.7	0.1080(-3)	2.00
50	0.9	0.8744(-4)	2.00

To determine the convergence order ν=q+1 of the pure temporal accuracy, instead of using very small h, we used the following very efficient device. For a fixed h, we make a reference calculation with a sufficiently small $\Delta t = \Delta t_{ref}$.

For the same value h, we then defined a modified error associated to values of Δt that are larger than Δt_{ref}, namely,

$$E^*(t) = \|U^n_{h,\Delta t}(\cdot,t) - U^m_{h,\Delta t_{ref}}(\cdot,t)\|_{L_\infty},$$

where $U^n_{h,\Delta t}$ and $U^m_{h,\Delta t_{ref}}$ are approximations of $u(\cdot,t)$ in S^r_h and $t = n\Delta t = m\Delta t_{ref}$.

$E^*(t)$ can be considered as a pure temporal error because subtracting $U^m_{h,\Delta t_{ref}}$ from $U^n_{h,\Delta t}$ essentially cancels the spatial error inherent in the latter approximation.

The results of these comparisons are shown in tables 6, 7, and 8 which refer to cubic splines of order $r = 4$. The expected temporal rate of convergence, ν=3 and 4 respectively, emerges clearly from these experiments for the 2 and 3 stage MIRK method, respectively. In Table 6,7, and 8, convergence order of the temporal error computed in the following formula:

$$Rate = \frac{log(\frac{E_1^*(t^n)}{E_2^*(t^n)})}{log(\frac{\Delta t_1}{\Delta t_2})}$$

where $\Delta t_i = \frac{T}{N_i}$, and $E_i^*(t)$ is $E^*(t)$ with $\Delta t = \Delta t_i$.

TABLE 6 Pure temporal maximum error $E^*(t)$ at $T = 0.5$ with r=3, q=2, and h=$\frac{1}{50}$.

N	CPU Time(sec)	E(t)	E*(t)	Rate
25	3.0	0.1836(-6)	0.1457(-7)	
30	3.5	0.1836(-6)	0.8543(-8)	2.93
35	4.2	0.1836(-6)	0.5437(-8)	2.93
40	4.7	0.1836(-6)	0.3675(-8)	2.94
45	5.3	0.1836(-6)	0.2601(-8)	2.94
50	6.0	0.1836(-6)	0.1909(-8)	2.94
500(Ref)	58.6	0.1836(-6)		

TABLE 7 Pure temporal maximum error $E^*(t)$ at $T = 0.5$ with r=4, q=2, and h=$\frac{1}{50}$.

N	CPU Time(sec)	E(t)	E*(t)	Rate
25	4.0	0.1528(-7)	0.1457(-7)	
30	4.8	0.9265(-8)	0.8543(-8)	2.93
35	5.6	0.6165(-8)	0.5438(-8)	2.93
40	6.4	0.4407(-8)	0.3676(-8)	2.93
45	7.1	0.3335(-8)	0.2601(-8)	2.94
50	8.0	0.3330(-8)	0.1909(-8)	2.94
500(Ref)	78.9	0.3338(-8)		

TABLE 8 Pure temporal maximum error $E^*(t)$ at $T = 0.2$ with r=4, q=3, and h=$\frac{1}{30}$.

N	CPU Time(sec)	E(t)	E*(t)	Rate
5	0.8	0.1010(-7)	0.3447(-8)	
6	0.9	0.1009(-7)	0.1700(-8)	3.88
7	1.0	0.1009(-7)	0.9322(-9)	3.90
8	1.2	0.1009(-7)	0.5532(-9)	3.91
9	1.4	0.1009(-7)	0.3460(-9)	3.98
10	1.5	0.1009(-7)	0.2290(-9)	3.92
300(Ref)	43.2	0.1008(-7)		

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